6. (Classical Mechanics)

Consider a walled rectangular enclosure of length L that is open at the end $z = L$. The cross section is $a \times b$ running from 0 to a along the $x-$axis and 0 to b along the $y-$axis. The walls at $z = 0,\ y = 0, x = a,$ and $y = b$ are sealed (so that there is no flow across them). The cavity is filled with a fluid that has a density ρ_0 and speed of sound c. The open end is exposed to the atmosphere with the ambient pressure p_0. (Gravity acts to hold the fluid in the cavity but otherwise can be neglected for this problem.) Now an external force causes the pressure at the open end $z = L$ to oscillate (around p_0) by an amount $p^{'}e^{i\omega t}$. Find the most general solution for the pressure in the fluid in the small amplitude approximation. This is the limit in which the physical equations can be linearized, so the terms higher than first order in p' can be neglected. Viscosity and damping (such as thermal conduction) are zero and the motion is irrotational. Note the solution includes a driven response as well as a homogeneous response.

Solution: *Solution by Gabrielle Guttormsen (gabiguttorm@g.ucla.edu)*

For some context, this question is the first fluids question ever given on the comp exam. As fluids aren't generally taught in physics undergrad, and since approachable resources on fluid mechanics can be difficult to come by, I will include a little general primer on fluids after the solution.

1 Introduction - Eliminating terms in Navier-Stokes

This problem gives a lot of information, in somewhat of a confusing order, the problem boils down to solving the wave equation for the homogeneous (non-driven) and inhomogenous (driven) velocity (and pressure) perturbations. If you want to skip to just solving the wave equation, look for the first green box. For a viscous, isotropic, Newtonian fluid in a gravity field, the mass (continuity equation), momentum (Navier-Stokes equation), and (internal) energy conservation equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{194}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\, \vec{v} \right) = \rho \vec{g} - \nabla P + \nabla \cdot \left[\mu \left(\nabla \vec{v} + (\nabla \vec{v})^T \right) - \frac{2}{3} \mu \, (\nabla \cdot \vec{v})\, \mathbb{I} \right] \tag{195}$$

$$\rho \left(\frac{\partial \mathcal{E}}{\partial t} + (\vec{v} \cdot \nabla)\, \mathcal{E} \right) = -P \ \nabla \cdot \vec{v} + \chi + \nabla \cdot (\kappa \nabla T) \tag{196}$$

1.1 Assumptions from the problem statement

The list below outlines what assumptions/simplifications to make to Equations 194, 195, 196, based on the problem wording

1. Boundary conditions: "no flow across" solid sides $\longrightarrow \vec{v} \cdot \hat{n} = 0$ where v is the fluid velocity, and $\hat{n}$ is unit vector normal to the surface

2. External forces: "gravity...can be neglected" - hydrostatic equilibrium with constant equilibrium pressure and density ($\nabla P_0 = 0,\ \nabla \rho_0 = 0$)

3. Shear forces: "viscosity ... [is] zero" - inviscid flow, neglect terms with μ, including viscous heating

4. Thermal conduction: "damping (such as thermal conduction) [is] zero" - no heat transfer, so isentropic flow, can treat as adiabatic ideal gas

5. Irrotational velocity: can (and probably should) use velocity potential to solve conservation equations

6. Sound speed: by including the speed of sound c_s, this indicates that the fluid is compressible, as sound speed (at constant entropy s) is

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho} \right)_s} \tag{197}$$

   Since our fluid is isentropic, it is confirmed that sound speed is a constant for small density fluctuations, as implied in the wording

7. Linearization: Solve for small (time dependent) amplitude oscillations around equilibrium of the form $f(\vec{x}, t) = f_0(\vec{x}) + f_1(\vec{x}, t)$

1.2 Euler Equations

Already we can eliminate thermal conduction, viscosity and gravity terms, and take advantage of the convective derivative $\left(\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)$ to get the Euler equations for adiabatic and inviscid flow

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v} \tag{198}$$
$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla P \tag{199}$$
$$\frac{D\mathcal{E}}{Dt} = -\frac{P}{\rho}\ \nabla\cdot\vec{v} \tag{200}$$

There are now 4 unknowns ($\vec{v},\ \rho,\ P,\ \mathcal{E}$) and 3 equations, therefore it is necessary to either eliminate $\mathcal{E}$ or include a thermodynamic equation of state to solve the system. Firstly one could recognize that since the fluid is adiabatic, it obeys the adiabatic equation of state: $P/\rho^\gamma = $ const, meaning the fluid is barotropic (pressure only depends on density, not temperature, energy or entropy). This is analogous to energy conservation, and now there are 3 equations and 3 unknowns ($\vec{v},\ \rho,\ P$).

Another method to reach the same conclusion is to use the specific internal energy of an ideal gas, $\mathcal{E} = c_v T$ and the pressure equation of state $P = \rho RT$. Knowing that the specific heat at constant pressure (c_P) and volume (c_V) are related to each other by the constant $\gamma = c_P/c_V$, and to the gas constant $R = c_P - c_V$, gives the specific internal energy in terms of the pressure and density $\mathcal{E} = \frac{P}{\rho(\gamma-1)}$. After eliminating gravity as well, we get the equations we can solve for this system,

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\vec{v} \tag{201}$$
$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla P \tag{202}$$
$$\frac{D}{Dt}\left(\frac{P}{\rho^\gamma}\right) = 0 \tag{203}$$

Main solution (this is where to start on an exam) - solving the wave equation

2 Solution to the wave equation

2.1 Deriving wave equation - linearizing Euler's equation

Taking the Euler equations for adiabatic and inviscid flow, and ignoring the contribution of gravity we have mass conservation,

$$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\nabla)\,\rho = -\rho\nabla\cdot\vec{v} \tag{204}$$

and momentum conservation,

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\rho}\nabla P \tag{205}$$

From here we linearize (expand to first order) the density, velocity, and pressure, with constant equilibrium density and pressure, and take the velocity to only be pertrubative/first order

$$\rho(\vec{x},t) = \rho_0 + \rho_1(\vec{x},t) \tag{206}$$
$$\vec{v}(\vec{x},t) = \vec{v}_1(\vec{x},t) \tag{207}$$
$$P(\vec{x},t) = P_0 + P_1(\vec{x},t) \tag{208}$$

There are two (4 if counting the 3 velocity components) equations and 3 (5) unknowns, so often in acoustics, one will explicitly assume barotropicity, that the pressure only depends on the density (this will come in handy later). Applying these linearized quantities to the equations 204 and 205 from above, then only keeping 1st order terms (one one f_1 term), gives

$$\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\,\rho = -\rho\nabla\cdot\vec{v}$$

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1(\vec{x},t)) + (\vec{v}_1\cdot\nabla)\,(\rho_0 + \rho_1(\vec{x},t)) = -(\rho_0+\rho_1)\nabla\cdot\vec{v}_1(\vec{x},t)$$

$$\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho_1}{\partial t} + \cancelto{0}{(\vec{v}_1\cdot\nabla)\,\rho_0} + \cancel{(\vec{v}_1\cdot\nabla)\,\rho_1} = -\rho_0\nabla\cdot\vec{v}_1 + \cancel{\rho_1\nabla\cdot\vec{v}_1} \tag{209}$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0\nabla\cdot\vec{v}_1 \tag{210}$$

and,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial}{\partial t}\vec{v}_1(\vec{x},t) + (\vec{v}_1\cdot\nabla)\,\vec{v}_1(\vec{x},t) = -\frac{1}{\rho_0+\rho_1}\nabla(P_0 + P_1(\vec{x},t))$$

$$\rho_0\frac{\partial}{\partial t}\vec{v}_1 + \cancel{\rho_1\frac{\partial}{\partial t}\vec{v}_1} + \cancel{(\vec{v}_1\cdot\nabla)\,\vec{v}_1} = -\cancelto{0}{\nabla P_0} - \nabla P_1$$

$$\rho_0\frac{\partial \vec{v}_1}{\partial t} = -\nabla P_1 \tag{211}$$

giving our two linearized conservation equations

$$\frac{\partial \rho_1}{\partial t} = -\rho_0\nabla\cdot\vec{v}_1 \tag{212}$$

$$\rho_0\frac{\partial \vec{v}_1}{\partial t} = -\nabla P_1 \tag{213}$$

Here we can eliminate velocity by taking a time derivative of equation 212 and then plug in equation 213 for the velocity,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial \rho_1}{\partial t} = -\rho_0\nabla\cdot\vec{v}_1\right)\\ \frac{\partial^2 \rho_1}{\partial t^2} = -\nabla\cdot\left(\rho_0\frac{\partial \vec{v}_1}{\partial t}\right)\\ \frac{\partial^2 \rho_1}{\partial t^2} = +\nabla\cdot\nabla P_1 = \nabla^2 P_1\end{aligned} \tag{214}$$

Next to relate pressure to density, we the expand the pressure $P(\rho)$ around small perturbations in the density $\rho - \rho_0$ [1].

$$P = P_0 + \left.\frac{\partial P}{\partial \rho}\right|_{\rho_0}(\rho-\rho_0) + \frac{1}{2}\left.\frac{\partial^2 P}{\partial \rho^2}\right|_{\rho_0}(\rho-\rho_0)^2 + ... \tag{215}$$

This is valid in the adiabatic ideal gas regime, where dynamics happen too quickly for thermal conduction (as indicated by neglecting damping). Plugging this into our linearized pressure equation (208), and keeping only first order terms, we get the adiabatic compressibility

$$P_1(\vec{x},t) = P(\vec{x},t) - P_0 \approx \left.\frac{\partial P}{\partial \rho}\right|_{\rho_0}\rho_1(\vec{x},t) \tag{216}$$

where $\rho_1 = \rho - \rho_0$, and the derivative

$$c_s = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{\rho_0}} \tag{217}$$

is the speed of sound at equilibrium density, (and constant entropy), and as the wave propagation is adiabatic, the entropy is uniformly constant at all times. We could also come to the same conclusion by using the adiabatic energy conservation equation $P/\rho^\gamma =$ const.,

$$\begin{aligned}\frac{D}{Dt}\left(\frac{P}{\rho_0}\right) = 0 &= \frac{\partial}{\partial t}\left(\frac{P}{\rho^\gamma}\right) + (\vec{v}\cdot\nabla)\left(\frac{P}{\rho^\gamma}\right) \\ &= \frac{1}{\rho^\gamma}\left(\frac{\partial P}{\partial t} + (\vec{v}\cdot\nabla)\,P\right) - \frac{\gamma P}{\rho^{\gamma-1}}\left(\frac{\partial \rho}{\partial t} + (\vec{v}\cdot\nabla)\,\rho\right)\end{aligned} \tag{218}$$

Linearizing this equation gives us

$$\begin{aligned}0 = &\frac{1}{(\rho_0+\rho_1)^\gamma}\left(\cancelto{0}{\frac{\partial P_0}{\partial t}} + \frac{\partial P_1}{\partial t} + \cancelto{0}{(\vec{v_1}\cdot\nabla)\,P_0} + \cancel{(\vec{v_1}\cdot\nabla)\,P_1}\right) \\ &- \frac{\gamma(P_0+P_1)}{(\rho_0+\rho_1)^{\gamma-1}}\left(\cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho_1}{\partial t} + \cancelto{0}{(\vec{v_1}\cdot\nabla)\,\rho_0} + \cancel{(\vec{v_1}\cdot\nabla)\,\rho_1}\right) \end{aligned}\tag{219}$$

$$= \frac{1}{(\rho_0+\cancel{\rho_1})^\gamma}\frac{\partial P_1}{\partial t} - \frac{\gamma(P_0+\cancel{P_1})}{(\rho_0+\cancel{\rho_1})^{\gamma-1}}\frac{\partial \rho_1}{\partial t} \tag{220}$$

$$= \frac{1}{\rho_0^\gamma}\frac{\partial P_1}{\partial t} - \frac{\gamma P_0}{\rho_0^{\gamma-1}}\frac{\partial \rho_1}{\partial t} \tag{221}$$

And now we have the time derivative of P_1 in terms of the time derivative of ρ_1,

$$\frac{\partial P_1}{\partial t} = \frac{\gamma P_0}{\rho_0}\frac{\partial \rho_1}{\partial t} = \left.\frac{\partial P}{\partial \rho}\right|_{\rho_0}\frac{\partial \rho_1}{\partial t} = c_s^2\frac{\partial \rho_1}{\partial t} \tag{222}$$

Where the sound speed comes from $c_s^2 = \frac{\partial P}{\partial \rho} \propto \gamma\rho^{\gamma-1} = \frac{\gamma P}{\rho}$.

Now the wave equation for the pressure is

$$\frac{1}{c_s^2}\frac{\partial^2 P_1}{\partial t^2} - \nabla^2 P_1 = 0 \tag{223}$$

and similarly for density,

$$\frac{1}{c_s^2}\frac{\partial^2 \rho_1}{\partial t^2} - \nabla^2 \rho_1 = 0 \tag{224}$$

For velocity, we use that the fluid is irrotational to now incorporate the velocity potential, $\vec{v} = -\nabla\phi$, so equation 212 and 213 become

$$-\rho_0\frac{\partial}{\partial t}\nabla\phi = -\nabla P_1 \qquad\qquad \frac{\partial \rho_1}{\partial t} = +\rho_0\nabla\cdot\nabla\phi \tag{225}$$

$$\tag{226}$$

Here we can pull out the gradient to the pressure perturbation in terms of ϕ (up to an arbitrary constant)

$$\nabla\left(\rho_0 \frac{\partial \phi}{\partial t} = P_1\right) \tag{227}$$

$$\rho_0 \frac{\partial \phi}{\partial t} = P_1 + \text{const.} \tag{228}$$

and then plug in the adiabatic compressibility, (which also defines the density perturbation in terms of ϕ), to get the velocity potential wave equation

$$\rho_0 \nabla^2 \phi = \frac{\partial \rho_1}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{c_s^2} P_1\right) \tag{229}$$

$$\rho_0 \nabla^2 \phi = \frac{\partial}{\partial t}\left(\frac{1}{c_s^2} \rho_0 \frac{\partial \phi}{\partial t}\right) \tag{230}$$

$$\nabla^2 \phi = \frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} \tag{231}$$

$$\tag{232}$$

Giving us the three equations to solve to define the system (up to an arbitrary constant):

$$\frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi \tag{233}$$

$$P_1 = \rho_0 \frac{\partial \phi}{\partial t} \tag{234}$$

$$\rho_1 = \frac{\rho_0}{c_s^2} \frac{\partial \phi}{\partial t} \tag{235}$$

$$\tag{236}$$

2.2 General solution to the wave equation

To solve the wave equation we use separation of variables, $\phi(\vec{x}, t) = X(x)Y(y)Z(z)T(t)$, and solve the corresponding eigenvalue equations

$$\frac{1}{c_s^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0 \tag{237}$$

$$\frac{1}{c_s^2} X(x)Y(y)Z(z) \frac{d^2 T}{dt^2} - T(t)\left(Y(y)Z(z)\frac{d^2 X}{dx^2} + X(x)Z(z)\frac{d^2 Y}{dy^2} + X(x)Y(y)\frac{d^2 Z}{dz^2}\right) = 0 \tag{238}$$

$$\frac{1}{c_s^2 T} \frac{d^2 T}{dt^2} - \left(\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2}\right) = 0 \tag{239}$$

$$\tag{240}$$

To parameterize these equations we use the wave vector $\vec{k} = (k_x, k_y, k_z)$, taking advantage of the independence of the separated variables

$$\frac{1}{c_s^2 T} \frac{d^2 T}{dt^2} = \left(\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2}\right) = k_x^2 + k_y^2 + k_z^2 = |\vec{k}|^2 = k^2 \tag{241}$$

$$\tag{242}$$

creating four ODE equations which can be solved in terms of the wave vector,

$$\frac{d^2T}{dt^2} = c_s^2|\vec{k}|^2T \longrightarrow T(t) = \begin{cases} A_t e^{c_s kt} + B_t e^{-c_s kt}, & k \neq 0 \\ C_t + D_t t, & k = 0 \end{cases} \tag{243}$$

$$\frac{d^2X}{dx^2} = k_x^2 X \longrightarrow X(x) = \begin{cases} A_x e^{k_x x} + B_x e^{-k_x x}, & k_x \neq 0 \\ C_x + D_x x, & k_x = 0 \end{cases} \tag{244}$$

$$\frac{d^2Y}{dy^2} = k_y^2 Y \longrightarrow Y(y) = \begin{cases} A_y e^{k_y y} + B_y e^{-k_y y}, & k_y \neq 0 \\ C_y + D_y y, & k_y = 0 \end{cases} \tag{245}$$

$$\frac{d^2Z}{dz^2} = k_z^2 Z \longrightarrow Z(z) = \begin{cases} A_z e^{k_z z} + B_z e^{-k_z z}, & k_z \neq 0 \\ C_z + D_z z, & k_z = 0 \end{cases} \tag{246}$$

$$\tag{247}$$

Where the A_is are all constants that depend on the wave-number, not the independent variables.

2.3 Homogeneous Boundary conditions

Now we incorporate the von Neumann boundary conditions, as we have the conditions on $\vec{v} = -\nabla\phi$. Explicitly in terms of our separated variables, the velocity is

$$\vec{v} = -\nabla\phi = -T(t)Y(y)Z(z)\frac{dX}{dx}\hat{x} - T(t)X(x)Z(z)\frac{dY}{dy}\hat{y} - T(t)X(x)Y(y)\frac{dZ}{dz}\hat{z} \tag{248}$$

No flow across solid sides (no slip boundary conditions) means $\hat{n}\cdot\vec{v} = -\hat{n}\cdot\nabla\phi = 0$, at the bottom ($z = 0$, $\hat{n} = -\hat{z}$) boundary,

$$\hat{n}\cdot\vec{v}\Big|_{z=0} = \hat{z}\cdot(\nabla\phi)_{z=0} = T(t)X(x)Y(y)\frac{dZ}{dz}\Big|_{z=0} = 0 \longrightarrow \frac{dZ}{dz}\Big|_{z=0} = 0 \tag{249}$$

and the sides ($x = 0 \to \hat{n} = -\hat{x};\ y = 0, \to \hat{n} = -\hat{y};\ x = a, \to \hat{n} = \hat{x};\ y = b \to \hat{n} = \hat{y}$) boundaries,

$$\hat{n}\cdot\vec{v}\Big|_{x=0} = \hat{x}\cdot(\nabla\phi)_{x=0} = T(t)Y(y)Z(z)\frac{dX}{dx}\Big|_{x=0} = 0 \longrightarrow \frac{dX}{dx}\Big|_{x=0} = 0 \tag{250}$$

$$\hat{n}\cdot\vec{v}\Big|_{x=a} = -\hat{x}\cdot(\nabla\phi)_{x=a} = -T(t)Y(y)Z(z)\frac{dX}{dx}\Big|_{x=a} = 0 \longrightarrow \frac{dX}{dx}\Big|_{x=a} = 0 \tag{251}$$

$$\hat{n}\cdot\vec{v}\Big|_{y=0} = \hat{y}\cdot(\nabla\phi)_{y=0} = T(t)X(x)Z(z)\frac{dY}{dy}\Big|_{y=0} = 0 \longrightarrow \frac{dY}{dy}\Big|_{y=0} = 0 \tag{252}$$

$$\hat{n}\cdot\vec{v}\Big|_{y=0} = \hat{y}\cdot(\nabla\phi)_{y=0} = T(t)X(x)Z(z)\frac{dY}{dy}\Big|_{y=0} = 0 \longrightarrow \frac{dY}{dy}\Big|_{y=b} = 0 \tag{253}$$

Where we ignore the trivial cases where the components of the separated equation are zero. For $X(x)$, (and similarly $Y(y)$), taking the derivative gives

$$\frac{dX}{dx} = 0 = \begin{cases} k_x\left(A_x e^{k_x x} - B_x e^{-k_x x}\right), & k_x \neq 0 \\ D_x, & k_x = 0 \end{cases} \tag{254}$$

Already we know that $D_x = 0$, so any constant C_x is a solution when the wave-number $k_x = 0$. For $k_x \neq 0$, first plugging in $x = 0$ gives,

$$\left.\frac{dX}{dx}\right|_{x=0} = k_x (A_x - B_x) = 0 \qquad \longrightarrow \qquad A_x = B_x \tag{255}$$

then $x = a$ gives

$$\left.\frac{dX}{dx}\right|_{x=a} = k_x A_x \left(e^{k_x a} - e^{-k_x a}\right) = 0 \tag{256}$$

which non-trivially gives $e^{k_x a} - e^{-k_x a} = 0$. As $k_x \neq 0$, the only solutions to this equation are for purely imaginary k_x of the form,

$$e^{2k_x a} = 1 \quad \longrightarrow \quad 2k_x a = 2\pi i n_x \quad \rightarrow \quad k_x = i\frac{\pi n_x}{a} \tag{257}$$

This gives $X(x)$, and (with the same procedure) $Y(y)$,

$$X(x) = A_x \left(e^{i(\pi n_x/a)x} + e^{-i(\pi n_x/a)x}\right) + C_x = 2A_x \cos\left(\frac{\pi n_x}{a} x\right) \tag{258}$$

$$Y(y) = A_y \left(e^{i(\pi n_y/b)y} + e^{-i(\pi n_y/b)y}\right) + C_y = 2A_y \cos\left(\frac{\pi n_y}{b} y\right) \tag{259}$$

Where the constants $(C_x,\ C_y)$ have been set equal to zero as they are not relevant for the velocity. For the $Z(z)$ component the bottom boundary gives the same constraints as $x = 0,\ y = 0$,

$$\left.\frac{dZ}{dz}\right|_{z=0} = k_z \left(A_z e^{k_z(0)} - B_z e^{-k_z(0)}\right) = k_z (A_z - B_z) = 0 \qquad \longrightarrow \qquad A_z = B_z \tag{260}$$

2.3.1 Free surface boundary

The top boundary requires a little intuition about the surface oscillations of the fluid. Gravity is acting only to confine the fluid in the container, at some constant atomospheric pressure (in the homogeneous case). This type of fluid surface boundary is called a free surface, meaning that the the fluid cannot move through the z boundary and "mix" with the air (which can be modelled as a fluid with different physical properties), and forces acting on the free surface are in equilibrium, so momentum is conserved, and pressure is constant. These two conditions are (respectively) called the kinematic and dynamic free surface boundary conditions. This boundary, given that there are no non-linear effects like wave breaking, can be defined as $z = \eta(x, y, t)$, and satisfies the kinematic condition: $\frac{D}{Dt}(z - \eta(x, y, t)) = 0$, or

$$\frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta} \frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta} \frac{\partial \eta}{\partial y} = \left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} \tag{261}$$

which is independent of z at the surface [2, 3, 4, 5, 6, 7]. Therefore we assume the velocity at $z = L$ is oscillating independent of z, with an arbitrary maximum amplitude we call A_v, so $v_1(z = L) = A_v e^{if(x,y,t)}$. Plugging this into our equation for $Z(z)$ gives,

$$\left.\frac{dZ}{dz}\right|_{z=L} = k_z A_z \left(e^{k_z L} - e^{-k_z L}\right) = A_v \tag{262}$$

As we know we have an oscillating solution that varies from $-A_v$ to A_v, and k_z is purely imaginary, we can solve for $k_z L = i\pi\left(n_z + \frac{1}{2}\right)$. The $Z(z)$ equation is then

$$Z(z) = A_z \left(e^{i(\pi(n_z+1/2)/L)z} + e^{-i(n_z+1/2)/L)z}\right) = 2A_z \cos\left(\frac{\pi(n_z + 1/2)}{L} z\right) \tag{263}$$

Now for the time dependence, we have already eliminated the case where $k_x = 0$, $k_y = 0$, and $k_z = 0$, and know that k_x, k_y, and k_z are all purely imaginary. The only additional condition we can set is that the velocity is real, and therefore the velocity potential must be real. For this to be true, then

$$T(t) = T(t)^* \tag{264}$$
$$A_t e^{ic_s|k|t} + B_t e^{-ic_s|k|t} = A_t^* e^{-ic_s|k|t} + B_t^* e^{+ic_s|k|t} \tag{265}$$
$$B_t = A_t^* \tag{266}$$

and finally

$$T(t) = A_t e^{ic_s|k|t} + A_t^* e^{-ic_s|k|t} = A_t e^{i\omega_k t} + A_t^* e^{-i\omega_k t} \tag{267}$$

where the frequency ω_k is a function of wavenumber,

$$\omega_k^2 = c_s^2 \pi^2 \left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{\left(n_z + \frac{1}{2}\right)^2}{L^2} \right) \tag{268}$$

2.3.2 Homogeneous solution

The full, general, solution to the homogeneous velocity potential is the sum of all the individual Fourier modes,

$$\phi(x, y, z, t) = \sum_{n_x, n_y, n_z \neq 0} \left(A_n e^{i\omega_k t} + A_n^* e^{-i\omega_k t}\right) \cos\left(\frac{\pi n_x}{a} x\right) \cos\left(\frac{\pi n_y}{b} y\right) \cos\left(\frac{\pi(n_z + 1/2)}{L} z\right) \tag{269}$$

Where the Fourier coefficients have all been absorbed into $A_n = A_{n_x, n_y, n_z}$ and its complex conjugate.

2.4 In-homogeneous solution

For the in-homogeneous component, ϕ_p, we have the same equation to solve for the velocity potential (equation 233), but different boundary conditions. The top surface of the liquid at $z = L$ is no longer a free surface, instead there is a condition on the time derivative due to the pressure perturbation. Using the same separation of variables $\phi_p = X(x)Y(y)Z(z)T(t)$,

$$P_1(x, y, z = L, t) = \rho_0 \left.\frac{\partial \phi_p}{\partial t}\right|_{z=L} = \rho_0 X(x)Y(y)Z(L) \left.\frac{dT}{dt}\right|_{z=L} = P' e^{i\omega t} \tag{270}$$

As P' is a constant, the $X(x)$ and $Y(y)$ must be constant for all $x \in [0, a]$ and $y \in [0, b]$, and ϕ_p is only a function of z and t. The wave equation simplifies to

$$\frac{1}{c_s^2}\frac{\partial^2 \phi_p}{\partial t^2} - \frac{\partial^2 \phi_p}{\partial z^2} = \frac{Z(z)}{c_s^2}\frac{d^2 T}{dt^2} - T(t)\frac{d^2 Z}{dz^2} = 0 \tag{271}$$

$$\frac{1}{c_s^2 T}\frac{d^2 T}{dt^2} = \frac{1}{Z}\frac{d^2 Z}{dz^2} = k_p^2 \tag{272}$$

And we have the same form solutions as before, (eliminating the $k_p = 0$/constant solutions)

$$\frac{d^2 T}{dt^2} = c_s^2 k_p^2 T \quad \longrightarrow \quad T(t) = A_t e^{c_s k_p t} + B_t e^{-c_s k_p t} \tag{273}$$
$$\frac{d^2 Z}{dz^2} = k_p^2 Z \quad \longrightarrow \quad Z(z) = A_z e^{k_p z} + B_z e^{-k_p z} \tag{274}$$
$$\tag{275}$$

For the time dependent component, from the pressure perturbation

$$P_1(z=L,t) = \rho_0 \frac{\partial \phi_p}{\partial t}\bigg|_{z=L} = \rho_0 Z(L) \frac{dT}{dt}\bigg|_{z=L} = P' e^{i\omega t} \tag{276}$$

$$\frac{dT}{dt}\bigg|_{z=L} = c_s k_p Z(L)\left(A_t e^{c_s k_p t} - B_t e^{-c_s k_p t}\right) = \frac{P'}{\rho_0} e^{i\omega t} \quad \longrightarrow \quad c_s k_p = i\omega, \quad \text{and} \quad B_t = 0 \tag{277}$$

Plugging in $k_p = i\omega/c_s$ gives $T(t) = A_t e^{i\omega t}$. For the $Z(z)$ component,

$$\frac{\partial \phi_p}{\partial z}\bigg|_{z=0} = T(t)\frac{dZ}{dz}\bigg|_{z=0} = 0 \quad \longrightarrow \quad \frac{dZ}{dz}\bigg|_{z=0} = 0 \tag{278}$$

We eliminate a Fourier coefficient like before

$$z = 0 \quad \longrightarrow \quad k_p A_z + k_p B_z = 0, \qquad A_z = B_z \tag{279}$$

and at $z = L$, we have already solved for k_p using the pressure perturbation, so

$$Z(z) = A_z\left(e^{i(\omega/c_s)z} + e^{-i(\omega/c_s)z}\right) = 2A_z \cos\left(\frac{\omega}{c_s}z\right) \tag{280}$$

Putting this together we have

$$\phi_p(z,t) = Z(z)T(t) = 2A_z A_t \cos\left(\frac{\omega}{c_s}z\right)e^{i\omega t} = A_\omega \cos\left(\frac{\omega}{c_s}z\right)e^{i\omega t} \tag{281}$$

To solve for A_ω, we plug ϕ_p back into the pressure perturbation,

$$\frac{\partial \phi_p}{\partial t}\bigg|_{z=L} = \rho_0 Z(L)\frac{dT}{dt}\bigg|_{z=L} = i\omega A_\omega \cos\left(\frac{\omega}{c_s}L\right)e^{i\omega t} = \frac{P'}{\rho_0}e^{i\omega t} \tag{282}$$

$$A_\omega = -\frac{iP'}{\rho_0 \omega \cos\left(\frac{\omega}{c_s}L\right)} \tag{283}$$

And we get the in-homogeneous solution,

$$\phi_p(z,t) = -\frac{iP'}{\rho_0 \omega \cos\left(\frac{\omega}{c_s}L\right)} \cos\left(\frac{\omega}{c_s}z\right)e^{i\omega t} \tag{284}$$

2.5 Full Solution for the pressure

Putting adding the homogeneous and the inhomogeneous solutions together gives the general solution for the velocity potential,

$$\begin{aligned}\phi(x,y,z,t) = &-\frac{iP'}{\rho_0 \omega \cos\left(\frac{\omega}{c_s}L\right)} \cos\left(\frac{\omega}{c_s}z\right)e^{i\omega t} \\ &+ \sum_{n_x,n_y,n_z \neq 0} \left(A_n e^{i\omega_k t} + A_n^* e^{-i\omega_k t}\right)\cos\left(\frac{\pi n_x}{a}x\right)\cos\left(\frac{\pi n_y}{b}y\right)\cos\left(\frac{\pi(n_z+1/2)}{L}z\right)\end{aligned} \tag{285}$$

And the pressure is found by taking the time derivative $P(\vec{x},t) = P_0 + P_1 = P_0 + \rho_0 \frac{\partial \phi}{\partial t}$,

$$\boxed{\begin{aligned}P(\vec{x},t) = P_0 + &\frac{P'}{\rho_0 \cos\left(\frac{\omega}{c_s}L\right)} \cos\left(\frac{\omega}{c_s}z\right)e^{i\omega t} \\ + \sum_{n_x,n_y,n_z \neq 0} i\omega_k &\left(A_n e^{i\omega_k t} - A_n^* e^{-i\omega_k t}\right)\cos\left(\frac{\pi n_x}{a}x\right)\cos\left(\frac{\pi n_y}{b}y\right)\cos\left(\frac{\pi(n_z+1/2)}{L}z\right)\end{aligned}}$$

3 Short guide to fluid mechanics equations

In addition to treating fluid elements as a continuous medium, there are three main assumptions about the nature of fluids that will be key to nearly every possible fluid mechanics problem,

1. Fluids are isotropic - mechanical properties (speed of sound, permeability, etc.) are the same in all directions
2. Fluids are Newtonian - viscosity is constant (there is a linear relationship between local shear stress and local rate of strain)
3. Fluids are classical - macroscopic description given by Newtonian dynamics

Like any Newtonian system, first step to solving a system is to understanding the forces acting on it. Generally, the total force on a fluid can be split into volume forces (long range forces, like gravity) and surface forces (short range, acting on a small fluid surface element, modelled as momentum transport through a fluid). The i^{th} component of the total force on a fluid element

$$f_i = \int_V F_i dV + \oint_S \sigma_{ij} dS_j \tag{286}$$

The first component are the integrated volume forces acting on the fluid volume V, and the second component the describes the surface forces on the enclosed surface S. The stress tensor, $\sigma_{ij}(\vec{r}, t)$, given by

$$\sigma_{ij} = -P\delta_{ij} + 2\mu(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}) \quad e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{287}$$

is the i^{th} component of the force per unit area exerted at position $\vec{r}$ at time t across a plane surface normal to the j direction ($d\vec{S} = \hat{n}dS$). The first component is isotropic, describing normal (static) forces, and the second is non-isotropic, describing shear forces (where v_i is the i^{th} velocity component). $P(\vec{r}, t)$ is the static/hydrostatic/volumetric fluid pressure, and $\mu(\vec{r}, t)$ is the scalar viscosity, which describes the resistance to shear flow of two parallel planes slipping past one another.

In most situations, the evolution of fluids are described by three conservation equations (amount leaving = amount created - amount convected out/flux through surface): conservation of mass, conservation of momentum, and conservation of energy.

1. Mass conservation (continuity equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{288}$$

ρ is the mass density of the fluid, and $\vec{v}$ is the velocity of the fluid elements. In the case where the flow is incompressible ($\nabla \cdot \vec{v} = 0$), this can be simplified using the convective derivative $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla)$

$$\frac{\partial \rho}{\partial t} + (\vec{v} \cdot \nabla)\, \rho = \frac{D\rho}{Dt} = 0 \tag{289}$$

2. Momentum conservation (other names include Euler equation, Navier-Stokes, Cauchy momentum equation etc.) The rate of momentum change $\frac{dp_i}{dt} = \int_V \rho v_i dV$ plus the momentum flux through the surface $\Phi_i = \oint_S \rho v_i v_j dS_j$ is equal to the force on a fluid element (286), which can be expanded to give the Cauchy momentum equation,

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i \tag{290}$$

where $\vec{F}$ is an external force, like gravity. To get the Navier-Stokes equation, which describes the motion of an isotropic, viscous, Newtonian fluid

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right)=\vec{F}-\nabla P+\nabla\cdot\left[\mu\left(\nabla\vec{v}+(\nabla\vec{v})^T\right)-\frac{2}{3}\mu\,(\nabla\cdot\vec{v})\,\mathbb{I}\right] \tag{291}$$

One simplification is in the case where there are no strong temperature gradients in the fluid, so the viscosity is spatially uniform,

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right)=\vec{F}-\nabla P+\mu\left(\nabla^2\vec{v}+\frac{1}{3}\nabla\,(\nabla\cdot\vec{v})\right) \tag{292}$$

3. Energy conservation The change total energy E, given by the sum of the change in internal $\left(\int_V \rho\varepsilon dV\right)$ plus kinetic $\left(\int_V \frac{1}{2}\rho v_i v_i dV\right)$ energy, plus the energy flux is balanced by the gains from work, $\dot{W}=\int_v v_i F_i dV+\oint_S v_i\sigma_{ij}dS_j$, minus heat flux (thermal losses) from temperature gradients $\dot{Q}=\oint_S \kappa\frac{\partial T}{\partial x_i}dS_i$. For an isotropic, Newtonian fluid this comes together to give,

$$\rho\left(\frac{\partial E}{\partial t}+(\vec{v}\cdot\nabla)\,E\right)=-P\;\nabla\cdot\vec{v}+\chi+\nabla\cdot(\kappa\nabla T) \tag{293}$$

The new physical quantities include χ, the rate of heat generation per unit volume due to viscosity; κ, the thermal conductivity; and T, the temperature. To summarize, the evolution of a co-moving fluid element is governed by the work done by volumetric pressure, $-P\;\nabla\cdot\vec{v}$; the heat conducted, $\nabla\cdot(\kappa\nabla T)$; and the viscous heat generated by shear flows, χ.

3.1 Common fluid regimes

1. Incompressible flow Incompressible flow means the rate of mass density change is zero, so

$$\frac{D\rho}{Dt}=0 \qquad\longrightarrow\qquad \nabla\cdot\vec{v}=0 \tag{294}$$

and the velocity field must be divergence free, and the volume of the comoving fluid is a constant of motion. When the initial density is constant, if the fluid is incompressible, the density will remain constant throughout the motion. A constant viscosity, incompressible fluid in a gravity field is described by

$$\nabla\cdot\vec{v}=0\frac{D\vec{v}}{Dt} \qquad\qquad =\frac{\nabla P}{\rho}+\vec{g}+\nu\nabla^2\vec{v} \tag{295}$$

where $\nu=\mu/\rho$ the kinematic viscosity. Here one solves for velocity $\vec{v}$ and pressure P, where all the other terms are known constants.

2. Compressible flow As discussed in previously in section 1.2, in compressible flow it is necessary to include an equation of state to describe the internal energy in terms of density and pressure. Summarizing from section 1.2, for an ideal gas, $P=\rho RT$, and $\mathcal{E}=c_vT$ so $\mathcal{E}=\frac{P}{\rho(\gamma-1)}$. The complete set of compressible equations for a gas with uniform viscosity is

$$\frac{D\rho}{Dt}=-\rho\nabla\cdot\vec{v} \tag{296}$$

$$\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\,\vec{v}\right)=\vec{g}-\frac{1}{\rho}\nabla P+\frac{\mu}{\rho}\left(\nabla^2\vec{v}+\frac{1}{3}\nabla\,(\nabla\cdot\vec{v})\right) \tag{297}$$

$$\frac{1}{\gamma-1}\left(\frac{DP}{Dt}-\frac{\gamma P}{\rho}\frac{D\rho}{Dt}\right)=\chi+\frac{\kappa}{R}\nabla^2\left(\frac{P}{\rho}\right) \tag{298}$$

that have velocity $\vec{v}$, pressure P, and density ρ as unknowns.

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