2. (Classical Mechanics)

Consider the classical field theory in one space dimension, parameterized by the coordinate x, with a single real scalar field $\phi(t,x)$, governed by the follwoing action,

$$S[\phi] = S_0 \int dt\, dx \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{c^2}{2}(\partial_x \phi)^2 - \omega^2 (1 - \cos\phi) \right)$$

Here, ω and c are real constants, respectively with dimensions of frequency and velocity. The overall constant S_0 has dimensions of angular momentum divided by velocity.

(a) Use the variational principle to obtain the Euler-Lagrange equation for $\phi(t,x)$, and give the expression for the total energy E of a general field configuration.

(b) Consider solutions to the Euler-Lagrange equation of (a) of the form,

$$\phi(t,x) = f(y) \qquad y = \gamma(v)\,(x - vt)$$

for arbitrary constant velocity v. Show that it is possible to choose $\gamma(v)$ such that f (as a function) is governed by an equation which is independent of v; determine this $\gamma(v)$, and the corresponding solution(s) f such that $\cos(f(\pm\infty)) = 1$, and $f(+\infty) \neq f(-\infty)$.

(c) Derive the relation between the total energy E of the solution and its velocity v, and show that this relation is the relativistic one. Derive the mass of the soliton.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) The problem specifically asks us to use the variational principle to find the Euler-Lagrange equation. Consider a small variation $\delta\phi$ of the field ϕ. Then, we can write

$$S\left[\phi+\delta\phi\right] = S_0 \int dt\, dx \left(\frac{1}{2}(\partial_t(\phi+\delta\phi))^2 - \frac{c^2}{2}(\partial_x(\phi+\delta\phi))^2 - \omega^2(1-\cos(\phi+\delta\phi))\right)$$

Expanding each term in the integrand to first order in $\delta\phi$, we get

$$\begin{aligned}\frac{1}{2}\left(\partial_t(\phi+\delta\phi)\right)^2 &= \frac{1}{2}\left(\partial_t\phi + \partial_t\left(\delta\phi\right)\right)^2 \\ &= \frac{1}{2}\left(\partial_t\phi\right)^2 + \left(\partial_t\phi\right)\,\left(\partial_t(\delta\phi)\right) + \mathcal{O}(\delta\phi)^2\end{aligned}$$

$$\begin{aligned}-\frac{c^2}{2}\left(\partial_x(\phi+\delta\phi)\right)^2 &= -\frac{c^2}{2}\left(\partial_x\phi + \partial_x\left(\delta\phi\right)\right)^2 \\ &= -\frac{c^2}{2}\left(\partial_x\phi\right)^2 - c^2\left(\partial_x\phi\right)\,\left(\partial_x(\delta\phi)\right) + \mathcal{O}(\delta\phi)^2\end{aligned}$$

$$-\omega^2(1-\cos(\phi+\delta\phi)) = -\omega^2\left(1-\cos\phi + (\delta\phi)\sin\phi + \mathcal{O}(\delta\phi)^2\right)$$

Putting everything back in the integrand and sorting the terms by powers of the small parameter $\delta\phi$, we get (to first order in $\delta\phi$)

$$\begin{aligned}S\left[\phi+\delta\phi\right] = &\underbrace{S_0\int dt\,dx\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \omega^2(1-\cos\phi)\right)}_{S[\phi]} \\ &+ S_0\int dt\,dx\Big(\left(\partial_t\phi\right)\,\left(\partial_t(\delta\phi)\right) - c^2\left(\partial_x\phi\right)\,\left(\partial_x(\delta\phi)\right) - \omega^2(\delta\phi)\sin\phi\Big) \qquad (38)\end{aligned}$$

The variation in the action is defined by

$$\delta S \equiv S\left[\phi+\delta\phi\right] - S\left[\phi\right] \qquad (39)$$

Therefore, in this case,

$$\delta S = S_0\int dt\,dx\Big(\left(\partial_t\phi\right)\,\left(\partial_t(\delta\phi)\right) - c^2\left(\partial_x\phi\right)\,\left(\partial_x(\delta\phi)\right) - \omega^2(\delta\phi)\sin\phi\Big) \qquad (40)$$

We need to separate the $\delta\phi$ factor from the rest of the integrand. To do so, integrate the first two terms by parts (and don't forget to change the sign of each term when doing so). As is usual for these kinds of problems, we assume that ϕ decays quickly enough at infinity that we can ignore the boundary term. This gives us

$$\begin{aligned}\delta S &= S_0\int dt\,dx\Big(-\left(\partial_t^2\phi\right)\,(\delta\phi) + c^2\left(\partial_x^2\phi\right)\,(\delta\phi) - \omega^2(\delta\phi)\sin\phi\Big) \\ &= S_0\int dt\,dx\left[-\left(\partial_t^2\phi\right) + c^2\left(\partial_x^2\phi\right) - \omega^2\sin\phi\right]\,(\delta\phi) \qquad (41)\end{aligned}$$

If ϕ satisfies the Euler-Lagrange equation, then the variation of the action δS is equal to zero for *any* variation of the field $\delta\phi$. Given this definition, we are justified in setting the term in brackets equal to zero to get

$$-\left(\partial_t^2\phi\right) + c^2\left(\partial_x^2\phi\right) - \omega^2\sin\phi = 0$$

Multiplying through by a minus sign, we get the Euler-Lagrange equation

$$\boxed{\partial_t^2\phi - c^2\partial_x^2\phi + \omega^2\sin\phi = 0} \tag{42}$$

This is known as the *sine-Gordon equation.* It resembles the massless Klein-Gordon equation, which can be written $\partial_t^2\phi - c^2\partial_x^2\phi = 0$, with the addition of the sine term $\omega^2\sin\phi$. (It is customary to remark on the quality of the pun in the name "sine-Gordon." Nicholas Wheeler has called it a "weak but inevitable pun," while Michael Gutperle has called it a "bad pun.")

To find the energy of the field configuration, we use a generalization of a result valid for standard classical mechanics:

If the Lagrangian has no *explicit* time-dependence, meaning $\frac{\partial L}{\partial t} = 0$, then the energy $E \equiv \sum_a \frac{\partial L}{\partial \dot{q}_a}\dot{q}_a - L$ is conserved.

Here, we have a field ϕ instead of a generalized coordinate q_a, but the logic is the same. Since $S = \int L\,dt$, we can write the Lagrangian

$$L = S_0\int dx\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \omega^2(1-\cos\phi)\right) \tag{43}$$

(Note the difference between the Lagrangian and the Lagrangian *density*; in this case, the Lagrangian density is the integrand of the Lagrangian above.)

In the case of classical field theory with a single field ϕ, the generalization of the equation $E \equiv \sum_a \frac{\partial L}{\partial \dot{q}_a}\dot{q}_a - L$ is

$$E \equiv \frac{\partial L}{\partial(\partial_t\phi)}\partial_t\phi - L \tag{44}$$

For this Lagrangian (43), we have

$$\frac{\partial L}{\partial(\partial_t\phi)} = S_0\int dx\ (\partial_t\phi) \tag{45}$$

so

$$E = S_0\int dx\ (\partial_t\phi)^2 - S_0\int dx\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{c^2}{2}(\partial_x\phi)^2 - \omega^2(1-\cos\phi)\right)$$

$$\boxed{E = S_0\int_{-\infty}^{\infty} dx\left(\frac{1}{2}(\partial_t\phi)^2 + \frac{c^2}{2}(\partial_x\phi)^2 + \omega^2(1-\cos\phi)\right)} \tag{46}$$

where we have inserted the limits of the integral, $-\infty \leq x \leq \infty$.

(b) We are asked to consider solutions to the Euler-Lagrange equation for which

$$\phi(t,x) = f(y) \quad \text{and} \quad y = \gamma(v)\,(x - vt) \tag{47}$$

for v an arbitrary constant. Before plugging this ansatz into the Euler-Lagrange equation, let's calculate the partial derivatives of $\phi(t,x)$. Using the chain rule on y, we get

$$\begin{aligned}\partial_t\phi &= f'(y)\,\frac{dy}{dt} \quad \text{where } f'(y) \equiv \frac{df}{dy}\\ &= -v\,\gamma(v)\,f'(y) \quad \text{since } y = \gamma(v)\,(x-vt)\end{aligned} \tag{48}$$

$$\begin{aligned}\partial_t^2\phi &= \partial_t\left(-v\,\gamma(v)\,f'(y)\right) \quad \text{since } \partial_t\phi = -v\,\gamma(v)\,f'(y)\\ &= -v\,\gamma(v)\,f''(y)\,\frac{dy}{dt} \quad \text{using the chain rule}\\ &= v^2\left(\gamma(v)\right)^2 f''(y) \quad \text{since } y = \gamma(v)\,(x-vt) \qquad (49)\end{aligned}$$

$$\partial_x\phi = f'(y)\,\frac{dy}{dx} \quad \text{where } f'(y) \equiv \frac{df}{dy} \qquad (50)$$

$$= \gamma(v)\,f'(y) \quad \text{since } y = \gamma(v)\,(x-vt) \qquad (51)$$

$$\begin{aligned}\partial_x^2\phi &= \partial_x\left(\gamma(v)\,f'(y)\right) \quad \text{since } \partial_x\phi = \gamma(v)\,f'(y)\\ &= \gamma(v)\,f''(y)\,\frac{dy}{dt} \quad \text{using the chain rule}\\ &= \left(\gamma(v)\right)^2 f''(y) \quad \text{since } y = \gamma(v)\,(x-vt) \qquad (52)\end{aligned}$$

Plugging (49) and (52) into the Euler-Lagrange equation (42) and using the fact that $\phi = f(y)$, we get

$$\begin{aligned}0 &= v^2\left(\gamma(v)\right)^2 f''(y) - c^2\left(\gamma(v)\right)^2 f''(y) + \omega^2\sin(f(y))\\ 0 &= \left(\gamma(v)\right)^2\left(v^2-c^2\right) f''(y) + \omega^2\sin(f(y)) \qquad (53)\end{aligned}$$

This equation will be independent of the parameter v if $(\gamma(v))^2$ is proportional to $(v^2-c^2)^{-1}$; in other words, if

$$\left(\gamma(v)\right)^2 \propto \frac{1}{v^2-c^2} \qquad (54)$$

You might now remember that the letter γ represents the Lorentz factor in special relativity. To heighten the analogy with special relativity, let's choose the constant of proportionality so that

$$\left(\gamma(v)\right)^2 = \frac{1}{1-\frac{v^2}{c^2}} \qquad (55)$$

Then, we have $\gamma(v)$ exactly equal to the Lorentz factor. (Since we can choose v arbitrarily now, we will choose it to be less than c.)

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (56)$$

We could define $\gamma(v)$ differently, but any such difference could be absorbed into our definition of f. With this definition of $\gamma(v)$, (53) becomes

$$\begin{aligned}0 &= \frac{1}{1-\frac{v^2}{c^2}}\left(v^2-c^2\right) f''(y) + \omega^2\sin(f(y))\\ 0 &= -c^2 f''(y) + \omega^2\sin(f(y))\end{aligned}$$

or

$$f''(y) = \frac{\omega^2}{c^2}\sin(f(y)) \qquad (57)$$

In order to solve this differential equation, we need to use a trick. First, multiply both sides of the equation by $f'(y)$.

$$f''(y)\,f'(y) = \frac{\omega^2}{c^2}\sin(f(y))\,f'(y) \qquad (58)$$

The left-hand side of this equation can be written as a total derivative using the chain rule:

$$\left((f'(y))^2\right)' = 2f'(y)\,f''(y), \quad \text{so} \quad f''(y)\,f'(y) = \left(\frac{1}{2}(f'(y))^2\right)' \tag{59}$$

Similarly, the right-hand side can be written as a total derivative:

$$(\cos(f(y)))' = -\sin(f(y))\,f'(y), \quad \text{so} \quad \frac{\omega^2}{c^2}\sin(f(y))\,f'(y) = \left(-\frac{\omega^2}{c^2}\cos(f(y))\right)' \tag{60}$$

Plugging into (58), we get

$$\left(\frac{1}{2}(f'(y))^2\right)' = \left(-\frac{\omega^2}{c^2}\cos(f(y))\right)'$$

Taking the antiderivative (and remembering the constant of integration), we get

$$\frac{1}{2}(f'(y))^2 = -\frac{\omega^2}{c^2}\cos(f(y)) + C \quad \text{for a constant } C \tag{61}$$

To find the constant of integration C, we need to find a boundary condition for $f'(y)$. The problem statement doesn't explicitly state one, but it does state that $\cos(f(\pm\infty)) = 1$. This implies that $f(y)$ has a well-defined limit as $y \to \pm\infty$. For physics purposes, this is enough to imply that $f'(y)$ approaches 0 as $y \to \pm\infty$. Plugging in the boundary conditions, we can solve for C:

$$\begin{aligned} 0 &= -\frac{\omega^2}{c^2} + C \quad \text{since } f'(\pm\infty) = 0 \text{ and } \cos(f(\pm\infty)) = 1 \\ \implies C &= \frac{\omega^2}{c^2} \end{aligned} \tag{62}$$

This gives us an ordinary differential equation for $f'(y)$:

$$\frac{1}{2}(f'(y))^2 = \frac{\omega^2}{c^2}\left[1 - \cos(f(y))\right] \tag{63}$$

This differential equation can be solved by direct integration. Before integrating, though, it is useful to simplify the right-hand side by using the power-reducing formula

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

This gives us

$$\frac{1}{2}(f'(y))^2 = \frac{\omega^2}{c^2}\left[2\sin^2\left(\frac{f(y)}{2}\right)\right] \tag{64}$$

Solving for $f'(y)$, we get

$$\begin{aligned} (f'(y))^2 &= \frac{4\omega^2}{c^2}\sin^2\left(\frac{f(y)}{2}\right) \\ f'(y) &= \pm\frac{2\omega}{c}\sin\left(\frac{f(y)}{2}\right) \end{aligned} \tag{65}$$

Note the use of the $\pm$ sign to account for both solutions of the quadratic equation. Now, we are ready to set up the antiderivatives that we will use to solve this equation:

$$\begin{aligned} \frac{df}{dy} &= \pm\frac{2\omega}{c}\sin\left(\frac{f}{2}\right) \\ \int \frac{df}{\sin(f/2)} &= \pm\frac{2\omega}{c}\int dy \end{aligned} \tag{66}$$

Making a change of variables $x \equiv f/2$, the integral on the left-hand side becomes

$$\int \frac{df}{\sin(f/2)} = 2\int \frac{dx}{\sin x} \tag{67}$$

This trig integral is best evaluated by knowing the answer already:

$$\int \frac{dx}{\sin x} = \ln\left[\tan\left(\frac{x}{2}\right)\right] + \text{ constant} \tag{68}$$

If you don't know the answer already, this integral is a good example of solving trig integrals using the "sledgehammer method":

Sledgehammer method for solving trig integrals:
For a general trig integral that cannot be solved by another method, consider the substitution

$$t = \tan\left(\frac{x}{2}\right) \tag{69}$$

Using the double-angle formulas

$$\sin(2x) = 2\sin x\cos x \quad \text{and} \quad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sin(2x)}{2\cos^2 x} = \frac{\sin(2x)}{\cos(2x)+1}$$

so

$$t = \tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x} \tag{70}$$

Using the Pythagorean identity $1+\tan^2 x = \sec^2 x$ and the power-reducing identity $\cos^2 x = \frac{1+\cos(2x)}{2}$, we can also get

$$1+t^2 = 1+\tan^2\left(\frac{x}{2}\right) = \sec^2\left(\frac{x}{2}\right) = \frac{2}{1+\cos x} \tag{71}$$

Using (70) and (71), we can solve for $\sin x$ and $\cos x$ in terms of t:

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2} \tag{72}$$

We can also use the derivative identity $(\tan x)' = \sec^2 x$ and (71) to differentiate (69) and solve for dx in terms of dt:

$$dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = \frac{1+t^2}{2}\,dx \qquad \Longrightarrow \qquad dx = \frac{2}{1+t^2}\,dt \tag{73}$$

After using (72) and (73) to perform the substitution, the resulting integral in t can be solved by the method of partial fractions.

For this integral, the substitution yields

$$\begin{aligned}\int \frac{dx}{\sin x} &= \int \left(\frac{1+t^2}{2t}\right)\left(\frac{2}{1+t^2}\,dt\right) \quad \text{for} \quad t \equiv \tan\left(\frac{x}{2}\right) \\ &= \int \frac{1}{t}\,dt \end{aligned} \tag{74}$$

This integral is elementary:

$$\begin{aligned}\int \frac{dx}{\sin x} &= \ln t + \text{ constant} \\ &= \ln\left[\tan\left(\frac{x}{2}\right)\right] + \text{ constant}\end{aligned} \tag{75}$$

Earlier, we made the change of variables $x \equiv f/2$, so writing (67) in terms of f, we get

$$\begin{aligned}\int \frac{df}{\sin(f/2)} &= 2\int \frac{dx}{\sin x} \\ &= 2\ln\left[\tan\left(\frac{x}{2}\right)\right] + \text{ constant} \\ &= 2\ln\left[\tan\left(\frac{f}{4}\right)\right] + \text{ constant} \quad \text{since } x \equiv \frac{f}{2}\end{aligned} \tag{76}$$

Now that we have this antiderivative, we can now return to the differential equation in (66):

$$\begin{aligned}\int \frac{df}{\sin(f/2)} &= \pm\frac{2\omega}{c}\int dy \\ 2\ln\left[\tan\left(\frac{f}{4}\right)\right] &= \pm\frac{2\omega}{c}\,y + \text{ constant} \\ \ln\left[\tan\left(\frac{f}{4}\right)\right] &= \pm\frac{\omega}{c}\,y + \delta \quad \text{for a constant } \delta\end{aligned} \tag{77}$$

Solving for f, we get

$$\begin{aligned}\tan\left(\frac{f}{4}\right) &= e^{\pm\omega y/c+\delta} \\ f(y) &= 4\arctan\left[e^{\pm\omega y/c+\delta}\right] \quad \text{for a constant } \delta\end{aligned} \tag{78}$$

Since the value of δ does not have any effect on the behavior of $f(y)$ as $y \to \pm\infty$, the boundary conditions are not sufficient to define δ. (The constant δ is basically a phase shift, and it can be removed by redefining the origin of time.) So we will leave it as is. All in all,

$$\boxed{f(y) = 4\arctan\left[e^{\pm\omega y/c+\delta}\right] \quad \text{for a constant } \delta} \tag{79}$$

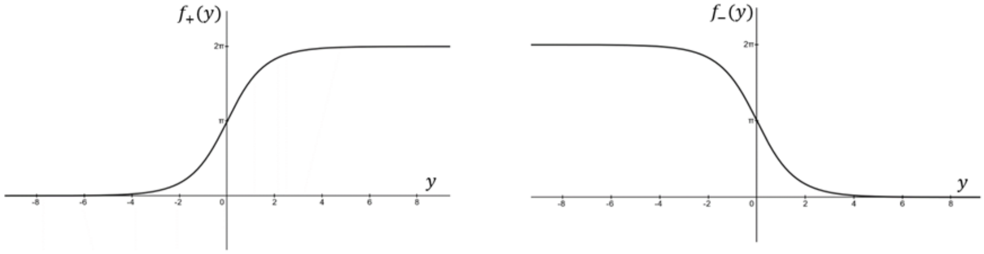
We can break this up into two possible options, one for each of the $\pm$ options:

$$f_\pm(y) \equiv 4\arctan\left[e^{\pm\omega y/c+\delta}\right] \tag{80}$$

Note that $\arctan(\infty) = \pi/2$ and $\arctan(0) = 0$. For that reason, we have

$$\begin{aligned} f_+(-\infty) = 0 \quad &\text{and} \quad f_+(+\infty) = 4\left(\frac{\pi}{2}\right) = 2\pi \\ f_-(-\infty) = 4\left(\frac{\pi}{2}\right) = 2\pi \quad &\text{and} \quad f_-(+\infty) = 0\end{aligned} \tag{81}$$

These solutions for $f(y)$ interpolate between 0 and 2π (or vice-versa), as shown in the graphs below:



(c) In part (a), we found the total energy of a field configuration (46)

$$E = S_0 \int_{-\infty}^{\infty} dx \left(\frac{1}{2}(\partial_t \phi)^2 + \frac{c^2}{2}(\partial_x \phi)^2 + \omega^2(1 - \cos\phi) \right)$$

Using the definition of y, $y \equiv \gamma\,(x - vt)$, we have $dy = \gamma\, dx$. In part (b), we calculated in (48) and (51) that

$$\partial_t \phi = -v\,\gamma\, f'(y) \quad \text{and} \quad \partial_x \phi = \gamma\, f'(y)$$

Plugging all this in, we get

$$\begin{aligned}
E &= S_0 \int_{-\infty}^{\infty} \frac{dy}{\gamma} \left[\frac{1}{2}\left(-v\,\gamma\, f'(y)\right)^2 + \frac{c^2}{2}(\gamma\, f'(y))^2 + \omega^2\left[1 - \cos(f(y))\right] \right] \\
&= S_0 \int_{-\infty}^{\infty} \frac{dy}{\gamma} \left[\frac{c^2}{2}(f'(y))^2 \gamma^2 \left(1 + \frac{v^2}{c^2}\right) + \omega^2\left[1 - \cos(f(y))\right] \right] \\
E &= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy \left[\frac{1}{2}(f'(y))^2 \gamma^2 \left(1 + \frac{v^2}{c^2}\right) + \frac{\omega^2}{c^2}\left[1 - \cos(f(y))\right] \right] \qquad (82)
\end{aligned}$$

Before proceeding further, note that the terms in the integrand looks similar to parts of a differential equation for f' that we encountered in part (b) ((63) and (65)):

$$\begin{aligned}
\frac{1}{2}(f'(y))^2 &= \frac{\omega^2}{c^2}\left[1 - \cos(f(y))\right] \\
\Longrightarrow \qquad f'(y) &= \pm \frac{2\omega}{c} \sin\left(\frac{f}{2}\right) \qquad (83)
\end{aligned}$$

Let's use this equation to substitute for the second term in the integrand of (82):

$$\begin{aligned}
E &= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy \left[\frac{1}{2}(f'(y))^2 \gamma^2 \left(1 + \frac{v^2}{c^2}\right) + \frac{1}{2}(f'(y))^2 \right] \\
&= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy\, \frac{1}{2}(f'(y))^2 \left[\gamma^2 \left(1 + \frac{v^2}{c^2}\right) + 1 \right] \\
&= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy\, \frac{1}{2}(f'(y))^2 \left[\frac{1 + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} + 1 \right] \quad \text{since } \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \\
&= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy\, \frac{1}{2}(f'(y))^2 \left[\frac{2}{1 - \frac{v^2}{c^2}} \right] \\
&= \frac{S_0 c^2}{\gamma} \int_{-\infty}^{\infty} dy\, \frac{1}{2}(f'(y))^2 \left[2\gamma^2 \right] \quad \text{since } \gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \\
E &= \gamma S_0 c^2 \int_{-\infty}^{\infty} dy\, (f'(y))^2
\end{aligned}$$

To take this integral, change variables to $f(y)$, where $df = f'(y)dy$:

$$E = \gamma S_0 c^2 \int_{f(-\infty)}^{f(\infty)} \frac{df}{f'(y)} \left(f'(y)\right)^2$$
$$E = \gamma S_0 c^2 \int_{f(-\infty)}^{f(\infty)} df\, f'(y) \tag{84}$$

Applying (83) again, we can write the integrand in terms of f:

$$E = \gamma S_0 c^2 \int_{f(-\infty)}^{f(\infty)} df \left[\pm \frac{2\omega}{c} \sin\left(\frac{f}{2}\right)\right]$$
$$E = \pm 2\gamma S_0 \omega c \int_{f(-\infty)}^{f(\infty)} df\, \sin\left(\frac{f}{2}\right) \tag{85}$$

Taking the antiderivative with respect to f, we get

$$E = \pm 2\gamma S_0 \omega c \left[-2\cos\left(\frac{f}{2}\right)\right]_{f(-\infty)}^{f(+\infty)}$$
$$E_\pm = \pm 2\gamma S_0 \omega c \left[-2\cos\left(\frac{f_\pm(\infty)}{2}\right) + 2\cos\left(\frac{f_\pm(-\infty)}{2}\right)\right] \tag{86}$$

Here, $f_\pm$ is the solution for f corresponding to the $\pm$ sign in (83). The limits for $f_\pm(\pm\infty)$ were given in (81): f_+ interpolates between 0 and 2π, and f_- does the opposite. Using these limits, we get

$$\begin{aligned}
E_+ &= +2\gamma S_0 \omega c \left[-2\cos\left(\frac{f_+(\infty)}{2}\right) + 2\cos\left(\frac{f_+(-\infty)}{2}\right)\right] \\
&= 2\gamma S_0 \omega c \left[-2\cos\left(\frac{2\pi}{2}\right) + 2\cos\left(\frac{0}{2}\right)\right] \\
&= 2\gamma S_0 \omega c \left[-2\cos\pi + 2\cos 0\right] \\
&= 2\gamma S_0 \omega c \left[-2(-1) + 2(1)\right] \\
&= 8\gamma S_0 \omega c
\end{aligned}$$

$$\begin{aligned}
E_- &= +2\gamma S_0 \omega c \left[-2\cos\left(\frac{f_-(\infty)}{2}\right) + 2\cos\left(\frac{f_-(-\infty)}{2}\right)\right] \\
&= -2\gamma S_0 \omega c \left[-2\cos\left(\frac{0}{2}\right) + 2\cos\left(\frac{2\pi}{2}\right)\right] \\
&= -2\gamma S_0 \omega c \left[-2\cos 0 + 2\cos\pi\right] \\
&= -2\gamma S_0 \omega c \left[-2(1) + 2(-1)\right] \\
&= 8\gamma S_0 \omega c
\end{aligned}$$

In both cases, the energy is the same, so we can write

$$E = 8\gamma S_0 \omega c \tag{87}$$

The only dependence of E on the velocity v is through the Lorentz factor γ:

$$E = (\text{constant})\,\gamma(v) = (\text{constant})\,\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{88}$$

This is exactly what we would expect from special relativity, where the constant in this equation is related to the mass of a particle:

$$E = \gamma m c^2 \quad \text{for} \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{89}$$

Rewriting (87) to match this form, we can derive the mass of the field configuration identified in (b) (which is one example of something called a *soliton*)

$$E = \gamma \underbrace{\left(\frac{8 S_0 \omega}{c} \right)}_{m} c^2$$

where the mass of the soliton is defined by

$$\boxed{m = \frac{8 S_0 \omega}{c}} \tag{90}$$