5. (Classical Mechanics)

An electron (charge $e = -|e|$ and rest mass m) with mechanical momentum

$$\mathbf{p} = p_0 \, (\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta)$$

enters into a static magnetic field region ($x > 0$) from a region of free space (zero magnetic field and zero vector potential) at $x < 0$.

The magnetic field has no y-component. It is due to a vector potential which has only a y-component A_y with (x, z) dependence, i.e.

$$\mathbf{A} = \hat{\mathbf{y}} \, A_y(x, z)$$

In addition, at $z = 0$ the magnetic field is perpendicular to the x-y plane.

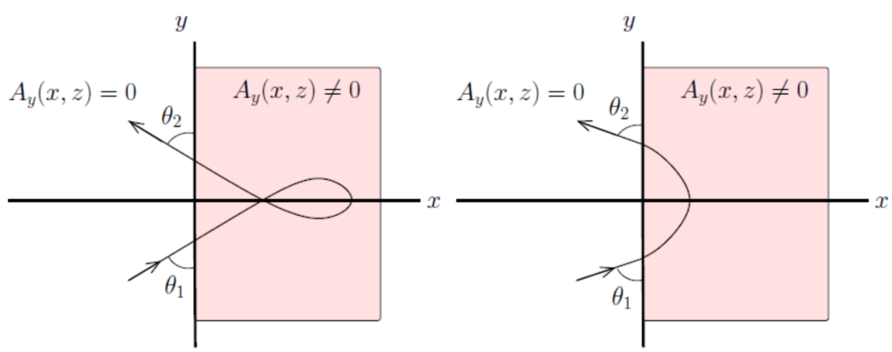
Note: for this problem, you can assume the electron to be non-relativistic or fully relativistic, just make that clear in your answers.

(a) Starting from the Lagrangian of a charged particle in external electromagnetic fields, construct the relativistic Hamiltonian of the system and the canonical momentum of the particle.

(b) Show that the trajectory of an electron located at $z = 0$ with its momentum in the x-y plane will stay in the x-y plane.

(c) Obtain two conserved quantities for the problem above and show, assuming that the electron eventually leaves the static magnetic field region, that this system is indeed a mirror for trajectories in the x-y plane, namely an electron with initial momentum $\mathbf{p}$ is reflected such that the angles that the incoming and outgoing trajectories make with the y-axis are equal in magnitude and opposite in sign (i.e. $\theta_1 = \theta_2$ in the picture below).

(d) Find an equation for the depth of the penetration (the furthest the electron reaches into the magnetic field region) and solve the resulting equation for the particular case of field $\mathbf{B} = G \, (\hat{\mathbf{x}} \, z - \hat{\mathbf{z}} \, x)$. Which sign of G corresponds to the trajectories shown in each figure of the figures below?



Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

Unusually, this problem gives us a choice. We can assume the electron to be either relativistic or nonrelativistic. To make things easier for ourselves, we will solve this problem assuming that the electron is nonrelativistic. We will work in SI units throughout.

(a) The Lagrangian for an nonrelativistic electron of mass m and charge e starts with the nonrelativistic kinetic energy of a point particle:

$$K = \frac{1}{2}m\dot{\mathbf{r}}^2 \tag{65}$$

All that remains is to account for the magnetic field, which comes from the vector potential given in the problem:

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \hat{\mathbf{y}}\,A_y(x,z) & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{66}$$

We need to know how to incorporate this vector potential in the Lagrangian. Here is the relevant information:

Incorporating a vector potential into the Lagrangian:
SI units: Add $+q\dot{\mathbf{r}}\cdot\mathbf{A}$.
Gaussian units: Add $+\frac{q\dot{\mathbf{r}}\cdot\mathbf{A}}{c}$.
Incorporating a vector potential into the Hamiltonian:
SI units: Replace $\mathbf{p}$ with $\mathbf{p} - q\mathbf{A}$.
Gaussian units: Replace $\mathbf{p}$ with $\mathbf{p} - \frac{q\mathbf{A}}{c}$.

Part (a) essentially asks us to confirm the second half of this box, given the first half of this box. The Lagrangian is SI units is therefore

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\,\dot{\mathbf{r}}\cdot\mathbf{A} \tag{67}$$

The canonical momentum is defined by

$$\mathbf{p}_C = \frac{\partial L}{\partial \dot{\mathbf{r}}} \tag{68}$$

Applying this formula to the Lagrangian for this problem (67), we get the canonical momentum

$$\boxed{\mathbf{p}_C = m\dot{\mathbf{r}} + e\mathbf{A}} \tag{69}$$

Note that for systems with vector potentials, the *canonical* momentum is not the same as the *physical* momentum! Indeed, from basic mechanics, the physical momentum of a nonrelativistic particle is equal to

$$\mathbf{p} = m\dot{\mathbf{r}} \tag{70}$$

Note that the problem statement calls $\mathbf{p}$ to define the *physical* momentum. We will use $\mathbf{p}_C$ to define the *canonical* momentum. (69) tells us that

$$\mathbf{p}_C = \mathbf{p} + e\mathbf{A} \tag{71}$$

The Hamiltonian comes from the Lagrangian via a Legendre transform:

$$H(\mathbf{p}_C, \mathbf{r}) = \mathbf{p}_C\cdot\dot{\mathbf{r}} - L \tag{72}$$

Using this formula, we get

$$\begin{aligned} H &= (m\dot{\mathbf{r}} + e\mathbf{A}) \cdot \dot{\mathbf{r}} - \left(\frac{1}{2}m\dot{\mathbf{r}}^2 + e\,\dot{\mathbf{r}} \cdot \mathbf{A}\right) \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 \\ &= \frac{1}{2}m\left(\frac{\mathbf{p}_C - e\mathbf{A}}{m}\right)^2 \quad \text{writing } H \text{ in terms of } \mathbf{p}_C \text{ and } \mathbf{r} \text{ using (68)} \\ &= \frac{(\mathbf{p}_C - e\mathbf{A})^2}{2m} \end{aligned} \tag{73}$$

Using (71), we can write this in terms of the physical momentum as well:

$$\boxed{H = \frac{(\mathbf{p}_C - e\mathbf{A})^2}{2m} = \frac{\mathbf{p}^2}{2m} \quad \text{where } \mathbf{p} = p_0\,(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)} \tag{74}$$

(b) In order to determine the trajectory of a particle, we need to find the equation of motion of the particle. The equation of motion of a particle of mass m, charge e, and velocity $\mathbf{v} = \dot{\mathbf{r}}$ in a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is given by the Lorentz force law:

$$m\ddot{\mathbf{r}} = e\,\dot{\mathbf{r}} \times \mathbf{B} \quad \Longrightarrow \quad \ddot{\mathbf{r}} = \frac{e}{m}\,\dot{\mathbf{r}} \times \mathbf{B} \tag{75}$$

The problem asks us to consider a particle that starts in the xy-plane (at $z = 0$) with momentum in the xy-plane. This condition can be expressed as

$$z(0) = 0 \quad \text{and} \quad \dot{z}(0) = 0 \tag{76}$$

To establish that the particle stays with these initial conditions stays in the xy-plane, we need to establish that $z(t) = 0$ for all t. For that, we should examine the z-component of (75):

$$\ddot{z} = \frac{e}{m}\dot{z}\,\hat{\mathbf{z}} \times \mathbf{B} \tag{77}$$

The problem tells us that at $z = 0$, the magnetic field is perpendicular to the xy-plane, so $\mathbf{B}$ has only a z-component at $z = 0$. Since $\hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$, (77) implies that

$$\ddot{z} = 0 \quad \text{if } z = 0 \tag{78}$$

One assumption of the Lagrangian method is that the first and second derivatives of the generalized coordinates are sufficient to specify the motion of the particle. For that reason, since $z = \dot{z} = \ddot{z} = 0$ at $t = 0$, we can infer that $z = \dot{z} = 0$ at a time $t = dt$ shortly after $t = 0$. But since $z = \dot{z} = 0$ at $t = dt$, (78) means that $\ddot{z} = 0$ at $t = dt$.

We can repeat the same argument to get that $z = \dot{z} = 0$ at time $t = 2dt$. Repeating the same argument, we can find that $z = 0$ at all times after $t = 0$. Therefore, if the electron starts at $z = 0$ with momentum in the xy-plane, it will stay in the xy-plane.

(c) The first step is to determine the two conserved quantities mentioned in the problem. For this step, we need to know some basic conservation laws:

Basic conservation laws from the Lagrangian:
If the Lagrangian has no *explicit* time-dependence, meaning $\frac{\partial L}{\partial t} = 0$, then the energy $E \equiv \sum_a \frac{\partial L}{\partial \dot{q}_a}\dot{q}_a - L$ is conserved.
If the Lagrangian does not depend on a generalized coordinate q_a, the canonical momentum associated with that coordinate, $p_a \equiv \frac{\partial L}{\partial \dot{q}_a}$ is conserved.

Here's a quick proof of each fact (using the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0$):

$$\begin{aligned}\frac{dE}{dt} &= \sum_a \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right)\dot{q}_a + \frac{\partial L}{\partial \dot{q}_a}\ddot{q}_a\right] - \frac{dL}{dt} \\ &= \sum_a \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right)\dot{q}_a + \frac{\partial L}{\partial \dot{q}_a}\ddot{q}_a\right] - \sum_a \left[\frac{\partial L}{\partial \dot{q}_a}\ddot{q}_a + \frac{\partial L}{\partial q_a}\dot{q}_a\right] \quad \text{since } \frac{\partial L}{\partial t} = 0 \\ &= \sum_a \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a}\right]\dot{q}_a \\ &= 0 \quad \text{by the Euler-Lagrange equation}\end{aligned}$$

$$\begin{aligned}\frac{dp_a}{dt} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) \\ &= \frac{\partial L}{\partial q_a} \quad \text{by the Euler-Lagrange equation} \\ &= 0 \quad \text{since } L \text{ does not depend on } q_a\end{aligned}$$

Note that since the Lagrangian is defined only up to a total derivative, there are other circumstances in which the energy or canonical momentum is conserved.

For this problem, plugging in the specific vector potential (66) into the generic Lagrangian (67), we get

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\,\dot{\mathbf{r}}\cdot\mathbf{A} = \begin{cases}\frac{1}{2}m\dot{\mathbf{r}}^2 + e\,\dot{y}\,A_y(x,z) & \text{for} \quad x > 0 \\ \frac{1}{2}m\dot{\mathbf{r}}^2 & \text{for} \quad x < 0\end{cases} \tag{79}$$

The Lagrangian has no explicit time dependence, so one conserved quantity is the energy. Using the formula above, we get

$$\begin{aligned}E &= \frac{\partial L}{\partial \dot{\mathbf{r}}}\cdot\dot{\mathbf{r}} - L \\ &= (m\dot{\mathbf{r}} + e\mathbf{A})\cdot\dot{\mathbf{r}} - \left(\frac{1}{2}m\dot{\mathbf{r}}^2 + e\,\dot{\mathbf{r}}\cdot\mathbf{A}\right) \\ &= \frac{1}{2}m\dot{\mathbf{r}}^2 \\ &= \frac{\mathbf{p}^2}{2m} \quad \text{since } \mathbf{p} = m\dot{\mathbf{r}}\end{aligned} \tag{80}$$

The Lagrangian (79) has no dependence on y (note that a dependence on $\dot{y}$ does not count as a dependence on y), so the y component of the canonical momentum $\mathbf{p}_C$ is conserved. Using the canonical momentum for this setup (71), we get the conserved quantity

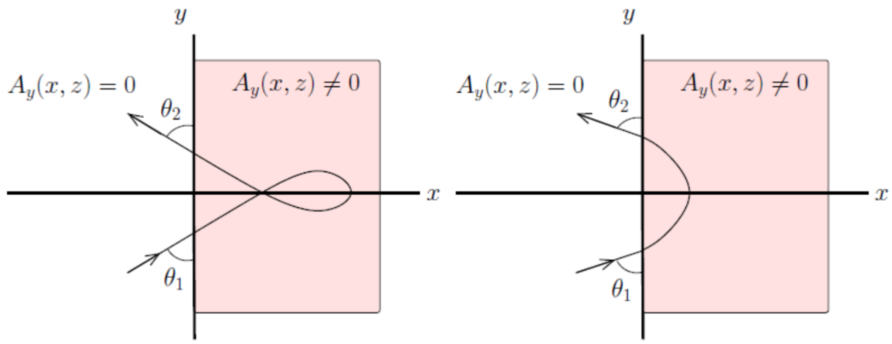
$$p_{C,y} = p_y + e\,\hat{\mathbf{y}}\cdot\mathbf{A} = \begin{cases}p_y + e\,A_y(x,z) & \text{for} \quad x > 0 \\ p_y & \text{for} \quad x < 0\end{cases} \tag{81}$$

Therefore, the two conserved quantities for this problem are

$$\boxed{E = \frac{\mathbf{p}^2}{2m} \quad \text{and} \quad p_{C,y} = \begin{cases}p_y + e\,A_y(x,z) & \text{for} \quad x > 0 \\ p_y & \text{for} \quad x < 0\end{cases}} \tag{82}$$

Now to show that the system is a mirror for trajectories in the xy-plane. The problem statement clarifies that the initial physical momentum is what gets mirrored. Consider two points 1 and

2, where point 1 is right before the electron enters the region of magnetic field and point 2 is right after the electron leaves the region of magnetic field (both locations are at $x = 0^-$, so the vector potential is zero at both locations).



The diagrams given in the problem statement (above) represent trajectories in the xy-plane. We can use them to write the physical momenta $\mathbf{p}_1$ (right before the electron enters the magnetic field) and $\mathbf{p}_2$ (right after the electron leaves the magnetic field) in terms of the angles θ_1 and θ_2:

$$\mathbf{p}_1 = |\mathbf{p}_1|\,(\sin\theta_1\,\hat{\mathbf{x}} + \cos\theta_1\,\hat{\mathbf{y}}) \quad \text{and} \quad \mathbf{p}_2 = |\mathbf{p}_2|\,(-\sin\theta_2\,\hat{\mathbf{x}} + \cos\theta_2\,\hat{\mathbf{y}}) \tag{83}$$

Then, we can use the conserved quantities (82) to relate the physical momenta at these points, $\mathbf{p}_1$ and $\mathbf{p}_2$:

$$E \text{ conserved} \implies |\mathbf{p}_1| = |\mathbf{p}_2| \tag{84}$$

$$p_{C,y} \text{ conserved} \implies p_{1,y} = p_{2,y} \tag{85}$$

Since $p_{1,y} = p_{2,y}$, we have

$$\begin{aligned} |\mathbf{p}_1|\,\cos\theta_1 &= |\mathbf{p}_2|\,\cos\theta_2 \\ \cos\theta_1 &= \cos\theta_2 \quad \text{since } |\mathbf{p}_1| = |\mathbf{p}_2| \\ \theta_1 &= \theta_2 \quad \text{since } \theta_1 \text{ and } \theta_2 \text{ are both between 0 and } \pi \end{aligned} \tag{86}$$

Therefore, if $p_0 \equiv \mathbf{p}_1 = \mathbf{p}_2$ and $\theta = \theta_1 = \theta_2$, we can write the physical momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ as

$$\mathbf{p}_1 = p_0\,(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \quad \text{and} \quad \mathbf{p}_2 = p_0\,(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \tag{87}$$

Therefore, the initial and final momenta are mirrored over the x-axis, as mentioned by the problem statement.

(d) We use the conserved quantities we found in part (c). Let x_d be the penetration depth, i.e., the maximum x-coordinate in the particle's trajectory. At the penetration depth, the particle turns around in the x-direction, so the x-component of the momentum is zero:

$$p_{d,x} = 0 \tag{88}$$

Since the particle's trajectory is assumed to be in the xy-plane, by part (b), the physical momentum at the penetration depth is in the y-direction. Now, we apply both conservation laws we found in part (c) to derive an equation for the y-component of the physical momentum at the penetration depth:

***Conservation of energy*:**
Since the energy $E = \mathbf{p}^2/(2m)$ is conserved, the magnitude of the momentum, which we call p_0,

is conserved throughout the trajectory. Therefore, the physical momentum at the penetration depth is

$$\mathbf{p}_d = \pm p_0\,\hat{\mathbf{y}} \tag{89}$$

Note the $\pm$ sign, which reflects the fact that the electron could be moving either up or down at the penetration depth. The $+$ sign corresponds to the rightmost trajectory in the figure above (electron moving in the $+y$ direction at the penetration depth), while the $-$ sign corresponds to the leftmost trajectory in the figure above (electron moving in the $-y$ direction at the penetration depth).

Conservation of the y-component of the canonical momentum:
Before the electron enters the magnetic field, the canonical momentum is equal to the physical momentum. By (82) we get that the y-component of the canonical momentum just before the electron enters the magnetic field is

$$p_{C,1,y} = p_{1,y} = p_0\cos\theta \quad \text{by (87)} \tag{90}$$

At the penetration depth, the y-component of the canonical momentum contains a contribution from the vector potential. By (82), we get that the y-component of the canonical momentum at the penetration depth is

$$p_{C,d,y} = p_{d,y} + eA_y(x_d, 0) \tag{91}$$

We have set $z = 0$ since the particle is in the xy-plane.

Since $p_{C,y}$ is conserved, we can set these two quantities equal to one another to get

$$p_0\cos\theta = p_{d,y} + eA_y(x_d, 0) \tag{92}$$

We can now combine (89) and (92) to get a single equation that can be solved for x_d in terms of p_0:

$$\boxed{p_0\cos\theta = \pm p_0 + eA_y(x_d, 0)} \tag{93}$$

Recall from the discussion above that the upper sign $(+)$ refers to the rightmost trajectory in the figure, and the lower sign $(-)$ refers to the leftmost trajectory in the figure.

We now need to find the vector potential for the given magnetic field $\mathbf{B} = G\,(\hat{\mathbf{x}}\,z - \hat{\mathbf{z}}\,x)$. Recall that $\mathbf{B} = \nabla\times\mathbf{A}$, so for $\mathbf{A} = \hat{\mathbf{y}}\,A_y(x,z)$, we have

$$\begin{aligned}\mathbf{B} = \nabla\times\mathbf{A} &= \left(\hat{\mathbf{x}}\,\frac{\partial}{\partial x} + \hat{\mathbf{y}}\,\frac{\partial}{\partial y} + \hat{\mathbf{z}}\,\frac{\partial}{\partial z}\right)\times\hat{\mathbf{y}}\,A_y(x,z) \\ &= -\hat{\mathbf{x}}\frac{\partial A_y}{\partial z} + \hat{\mathbf{z}}\frac{\partial A_y}{\partial x} \quad \text{since } \hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}} \text{ and } \hat{\mathbf{z}}\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}\end{aligned} \tag{94}$$

Matching this to the given magnetic field, we get

$$Gz = -\frac{\partial A_y}{\partial z} \quad \text{and} \quad -Gx = \frac{\partial A_y}{\partial x} \tag{95}$$

This means A_y is given by

$$A_y(x,z) = -\frac{G}{2}\left(x^2 + z^2\right) + \text{ constant} \tag{96}$$

Since A_y is zero for $x < 0$ and since the problem tells us the magnetic field has no z-component at $z = 0$, continuity of A_y requires that the constant of integration be equal to zero. (If the constant were not equal to zero, the circulation of $\mathbf{A}$ over a rectangular Amperian loop in the

xy-plane centered at $x = 0$ would be nonzero. This would imply by Ampere's law that the magnetic flux through that loop would be nonzero, implying that the z-component of $\mathbf{B}$ is nonzero at $z = 0$—a contradiction with the problem statement.) This means that we have

$$A_y(x, z) = -\frac{G}{2}(x^2 + z^2) \tag{97}$$

We can now plug this into equation (93):

$$p_0 \cos\theta = \pm p_0 - \frac{eG}{2} x_d^2 \tag{98}$$

Solving for x_d^2 and using the fact that $e = -|e|$, we get

$$x_d^2 = \frac{2}{eG}\left(\pm p_0 - p_0 \cos\theta\right) = -\frac{2p_0}{|e|G}\left(\pm 1 - \cos\theta\right) \tag{99}$$

The last step to solve for x_d is to take the square root. In order to get a reasonable answer, we need to make sure we are taking the square root of a positive number. p_0 and $|e|$ are both defined to be positive, so it all depends on the sign of G.

If $G > 0$, then x_d^2 is only positive if we take the lower sign ($-$) of the $\pm$ symbol. Recall from above that this corresponds to the leftmost trajectory in the figure above. Once we do this, we can solve for x_d:

$$x_d^2 = -\frac{2p_0}{|e|G}\left(-1 - \cos\theta\right) = \frac{2p_0}{|e|G}\left(1 + \cos\theta\right)$$

$$\boxed{G > 0: \qquad x_d = \sqrt{\frac{2p_0}{|e|G}\left(1 + \cos\theta\right)} \quad \text{(leftmost diagram)}} \tag{100}$$

If $G > 0$, then x_d^2 is only positive if we take the upper sign ($+$) of the $\pm$ symbol (note that $1 - \cos\theta \geq 0$). Recall from above that this corresponds to the rightmost trajectory in the figure above. Once we do this, we can solve for x_d:

$$x_d^2 = -\frac{2p_0}{|e|G}\left(1 - \cos\theta\right) = \frac{2p_0}{|e|\,|G|}\left(1 - \cos\theta\right)$$

$$\boxed{G < 0: \qquad x_d = \sqrt{\frac{2p_0}{|e|\,|G|}\left(1 - \cos\theta\right)} \quad \text{(rightmost diagram)}} \tag{101}$$

If you wanted to solve this problem assuming the electron was relativistic, in part (a) you would use the contribution to the Lagrangian of a relativistic free particle

$$L_{\text{rel}} = -mc^2\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} \qquad \Longrightarrow \qquad L = -mc^2\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} + e\,\dot{\mathbf{r}}\cdot\mathbf{A} \tag{102}$$

The canonical momentum is then (using the relativistic physical momentum $\mathbf{p} = \gamma m\mathbf{v}$)

$$\mathbf{p}_{C,\text{rel}} = \frac{1}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} m\dot{\mathbf{r}} + e\mathbf{A} \equiv \gamma m\dot{\mathbf{r}} + e\mathbf{A} = \mathbf{p} + e\mathbf{A} \tag{103}$$

meaning that (71) still holds. Applying the same formulas as before, you would get the relativistic Hamiltonian

$$H_{\text{rel}} = \gamma mc^2 = \sqrt{(\mathbf{p}c)^2 + (mc^2)^2} \tag{104}$$

This is also equal to the relativistic energy of the electron. Other than that, parts (b), (c), and (d) are essentially the same as in the nonrelativistic version.