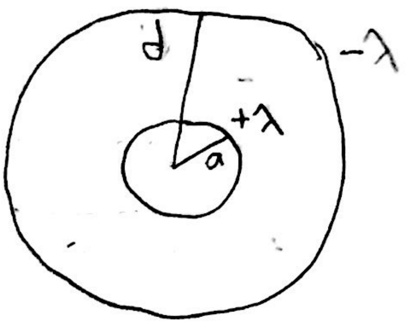
Week 7 Solutions

1.)



a.) $$\vec{E}(a \le r \le d) = \frac{\lambda_{encl}(a \le r \le d)}{2\pi\epsilon_0 r}\hat{r}$$

$$= \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

$$\Delta V_1 = V_d - V_a = -\int_a^d E(a \le r \le d)\,dr$$

$$= -\int_a^d \frac{\lambda\,dr}{2\pi\epsilon_0 r} = \frac{-\lambda}{2\pi\epsilon_0}\ln\left(\frac{d}{a}\right)$$

b.) $$\frac{C_1}{\ell} = \frac{\lambda}{|\Delta V_1|} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d}{a}\right)} = \frac{2\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}$$

c.) $$\vec{E}(a \le r \le b) = \vec{E}(c \le r \le d) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

$\vec{E}(b \le r \le c) = 0$ (inside conductor)

$$\Delta V_2 = -\int_a^b E(a \le r \le b)\,dr - \int_c^d E(c \le r \le d)\,dr$$

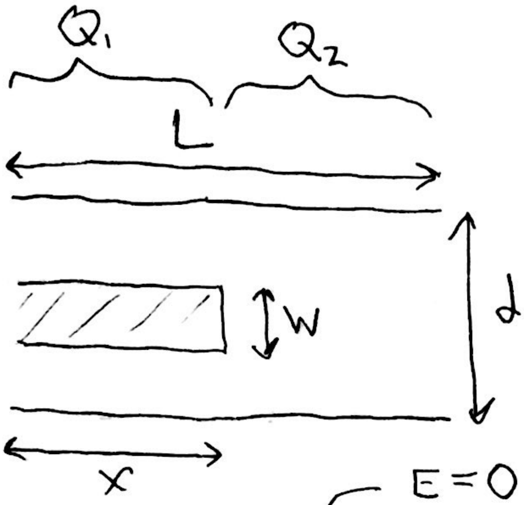
$$= -\int_a^b \frac{\lambda}{2\pi\epsilon_0 r}\,dr - \int_c^d \frac{\lambda}{2\pi\epsilon_0 r}\,dr$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right) - \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{d}{c}\right)$$

$$\frac{C_2}{\ell} = \frac{\lambda}{|\Delta V_2|} = \frac{2\pi\epsilon_0}{\ln(b/a) + \ln(d/c)} = \frac{2\pi\epsilon_0}{\ln\left(\frac{d}{a}\cdot\frac{b}{c}\right)}$$

$C_2 > C_1$, since $|\Delta V_2| < |\Delta V_1|$, since the conductor zeros out the field in part of the region $a \le r \le d$

2.)



E = 0 inside conductor

$$\Delta V_1 = E_1(d-w) = \frac{Q_1(d-w)}{\epsilon_0 x L}$$

$$C_1 = \frac{Q_1}{|\Delta V_1|} = \frac{\epsilon_0 x L}{d-w}$$

$$\Delta V_2 = E_2 d = \frac{Q_2 d}{\epsilon_0 (L-x) L}$$

$$C_2 = \frac{Q_2}{|\Delta V_2|} = \frac{\epsilon_0 (L-x) L}{d}$$

$$C_i = \epsilon_0 L \cdot \frac{L}{d}$$

$$C_f = C_1 + C_2 = \epsilon_0 L \left(\frac{x}{d-w} + \frac{L-x}{d} \right) > C_i$$

Before slab is inserted: $U_i = \frac{Q^2}{2C_i} = \frac{Q^2 d}{2\epsilon_0 L^2}$

After slab is inserted:

$$U_f = \frac{Q^2}{2C_f} = \frac{Q^2}{2\epsilon_0 L} \frac{d(d-w)}{xd + (d-w)(L-x)} = U_i \frac{L(d-w)}{xd + (d-w)(L-x)}$$

Because $C_f > C_i$, and "Q" is conserved (no battery), it follows that $U_f < U_i$. The energy "lost" by the capacitor can be thought of as work done by the electric field of the capacitor in pulling the slab further in between the plates, to its final position "x".

$$V_f = \frac{Q}{C_f} = \frac{Q}{\epsilon_0 L\left(\frac{x}{d-w} + \frac{L-x}{d}\right)} = \frac{Q}{\epsilon_0 L}\left[\frac{d(d-w)}{xd+(d-w)(L-x)}\right]$$

$$Q_1 = C_1 V_f = \frac{\epsilon_0 x L}{d-w} V_f = \frac{Q\ xd}{xd+(d-w)(L-x)}$$

The region with the slab.

$$Q_2 = C_2 V_f = \frac{\epsilon_0 (L-x) L}{d} V_f = \frac{Q\ (d-w)(L-x)}{xd+(d-w)(L-x)}$$

The region without the slab.

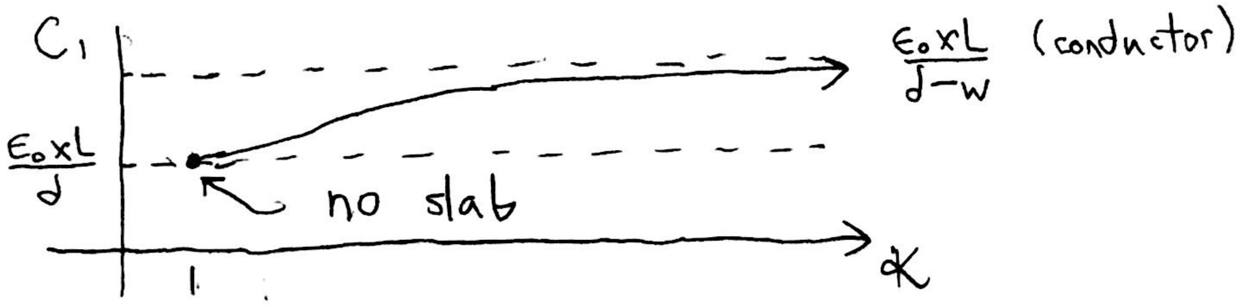
Note that $Q_1 + Q_2 = Q$.

d.) If slab has dielectric constant "κ":

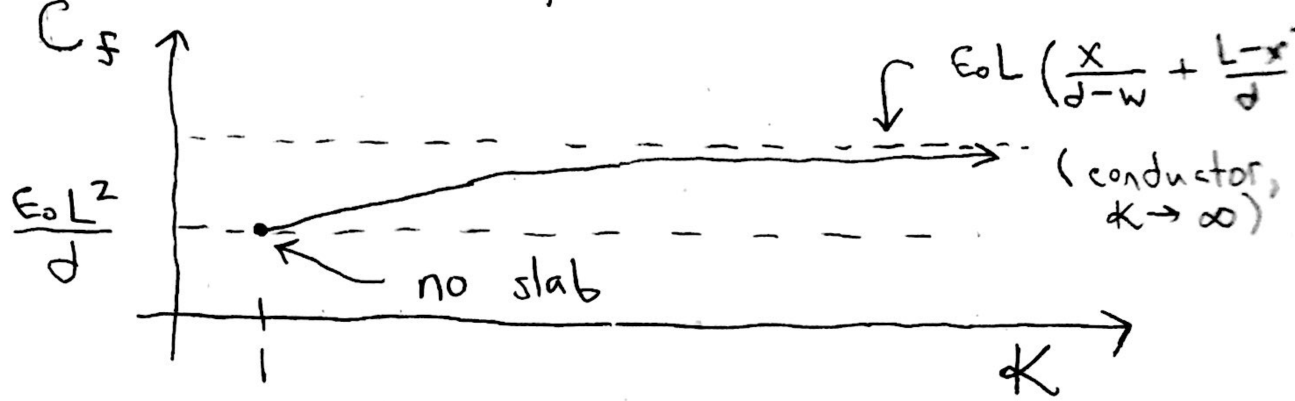
$$C_{1a} = \frac{\kappa \epsilon_0 x L}{w} \qquad C_{1b} = \frac{\epsilon_0 x L}{d-w}$$

$$\frac{1}{C_1} = \frac{1}{C_{1a}} + \frac{1}{C_{1b}} = \frac{1}{\epsilon_0 x L}\left(\frac{w}{\kappa} + d-w\right) = \frac{w+\kappa(d-w)}{\kappa \epsilon_0 x L}$$

$$C_1 = \frac{\kappa \epsilon_0 x L}{w+\kappa(d-w)} \qquad C_f = C_1 + \frac{\epsilon_0 (L-x) L}{d}$$



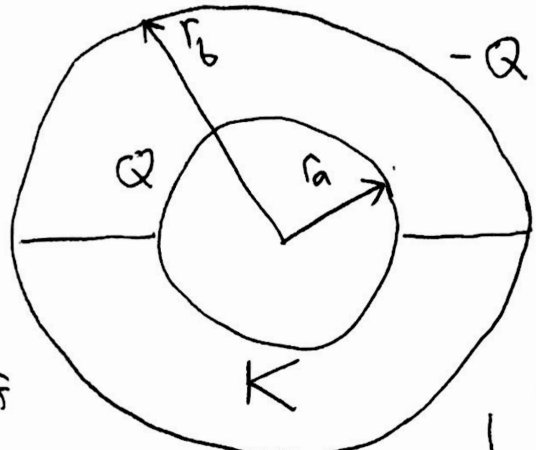
Total capacitance



$$C_f(K=1) = \frac{\epsilon_0 L^2}{d} \quad : \quad \text{(no slab)}$$

$$C_f(K \to \infty) = \epsilon_0 L \left(\frac{x}{d-w} + \frac{L-x}{d} \right) \quad \text{(conductor)}$$

3.)



At a given "r" where "$r_a \leq r \leq r_b$", $E(r)$ is the same in both regions, at all points on a Gaussian sphere of radius "r". We divide the Gaussian sphere into two hemispheres, one in the dielectric and one in the empty space.

General form of Gauss's law

$$\int K \epsilon_0 \vec{E} \cdot d\vec{a} = Q_{encl}$$

$$\epsilon_0 (K \cdot 2\pi r^2 + 1 \cdot 2\pi r^2) E(r) = Q_{encl}(r)$$

$$\vec{E}(r) = \frac{Q_{encl}(r)}{2\pi \epsilon_0 r^2 (K+1)} \hat{r} = \frac{Q}{2\pi \epsilon_0 r^2 (K+1)} \hat{r}$$

$$\Delta V = V_b - V_a = -\int_{r_a}^{r_b} E(r)\, dr$$

$$= -\int_{r_a}^{r_b} \frac{Q\, dr}{2\pi \epsilon_0 r^2 (K+1)}$$

$$= \frac{Q}{2\pi \epsilon_0 (K+1)} \left(\frac{1}{r_b} - \frac{1}{r_a} \right)$$

$$C = \frac{Q}{|\Delta V|} = 2\pi \epsilon_0 (K+1) \left(\frac{1}{r_a} - \frac{1}{r_b} \right)^{-1}$$

$$= 2\pi \epsilon_0 (K+1) \frac{r_a r_b}{r_b - r_a}$$