Name: 1C Discussion- Week 5

1. **Radiation in conductors** *(YF 12th ed. 32.58)*. — In a conductor, electromagnetic waves excite currents that are much greater than the displacement current:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + c^2 \partial_t \vec{E} \approx \mu_0 \vec{J} = \mu_0 \sigma \vec{E} \qquad (1)$$

(a) Take an electric field $\vec{E}(x,t) = E_y(x,t)\hat{y}$ propagating in the $\hat{x}$-direction in a conductor. Using equation (1) and Faraday's equation $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$, show that the electric field $E_y(x,t)\hat{y}$ behaves as:

$$\partial_x^2 E_y(x,t) = \mu\sigma\partial_t E_y(x,t) \qquad (2)$$

(b) A solution to equation (2) is

$$E_y(x,t) = E_0 e^{-k_C x} \sin(k_C x - \omega t) \qquad (3)$$

, where $k_C = \sqrt{\omega\mu\sigma/2}$. Show that the electric field amplitude decreases by a factor of $1/e$ in a distance $\delta = 1/k_C$, known as "skin depth". Find δ for copper illuminated by a radio wave with $f = 1.0$MHz ($\sigma = 6.0 * 10^7 \Omega^{-1} m^{-1}$ and $\mu = \mu_0 = 4\pi * 10^{-7}$).
(c) What physical conclusions can you draw from your results?

a)
$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_z(x) \end{vmatrix} = -\partial_x B_z(x)\,\hat{y} = \mu_0 \sigma E_y(x)\,\hat{y} \qquad (\mathrm{I})$$

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y(x) & 0 \end{vmatrix} = \partial_x E_y(x)\,\hat{z} = -\partial_t B_z(x)\,\hat{z} \qquad (\mathrm{II})$$

$\Rightarrow$ Operate with ∂_t on (I) and plug (II) into (I):

$$\partial_x^2 E_y = \mu_0 \sigma \partial_t E_y$$

$\blacksquare$

b) Rewrite $E_y(x,t) = E_0 e^{-k_c x} e^{i(k_c x - \omega t)}$; $\Rightarrow \left| |E_y| \right|_{x=\frac{1}{k_c}} = E_0 e^{-k_c x}\Big|_{x=\frac{1}{k_c}} = \frac{E_0}{e}$

$x = \frac{1}{k_c} = \sqrt{\frac{2}{\underbrace{12}_{4\pi} \cdot 6 \cdot 10^6 \cdot 2\pi}} = \frac{1}{6\sqrt{6}} \times 10^{-3} = 0.03\text{ mm} \sim 30\mu\text{m}$

c)
EM waves do not penetrate conductors, and must be largely reflected.
$\Rightarrow$ metals are shiny ; phone reception in elevators is poor ; EM interaction between Copper and Oxygen results in only very thin (δ) oxidation layer, protecting Statue of Liberty.

2. **Radiation** (*Graduate Qualifying Exam*). — A low-energy electron has a velocity $v_0 << c$ at infinity. The velocity $\vec{v_0}$ is directed towards a fixed, repulsive Coulomb field, the potential energy for which is given by $U(r) = \frac{Ze^2}{r}$. The electron is decelerated until it comes to rest and then is accelerated again in a direction opposite the original direction of motion. Show that when the electron has again reached an infinite distance from the Coulomb scattering center, the kinetic energy of the electron is about $\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2\left(1 - \frac{16v_0^3}{45Zc^3}\right)$ where m is the electron mass and the term depending on v_0^5 representes the energy E radiated away during the deceleration and acceleration. The rate at which energy is emitted from an accelerating charge q with acceleration $a = \frac{dv}{dt}$ is given by

$$P = \frac{\partial E}{\partial t} = \frac{2q^2}{3c^3}\left(\frac{dv}{dt}\right)^2 \tag{4}$$

Hints :
Rewrite $E = \int Pdt = \frac{2q^2}{3c^3}\int \frac{dv}{dt}\cdot\frac{dv}{dt}dt = \frac{2q^2}{3c^3}\int \frac{dv}{dt}dv = \frac{2q^2}{3c^3}\int adv$.
Rewrite $a = \frac{1}{m}F = \frac{1}{m}\left(-\frac{dU(r)}{dr}\right) = \frac{1}{m}\frac{U^2}{Ze^2}$.
Find $U(v)$ with energy conservation.

$U(v) = \frac{1}{2}m(v_0^2 - v^2)$ (ignoring radiative losses over dv)

$a = \frac{1}{m}\frac{m^2}{4}(v_0^2 - v^2)^2 \cdot \frac{1}{Ze^2}$

$E_{Loss} = \frac{2\cancel{e^2}}{3c^3}\frac{m}{4}\cdot\frac{1}{Z\cancel{e^2}}\cdot\int (v_0^2 - v^2)^2\, dv$

Since $\int dv \propto \int_{v_0}^{v(r=\infty)=?} a\,dv$, and acceleration is the same for approach and escape, we can modify $\int_{v_0}^{v(r=\infty)} a\,dv = 2\int_{v_0}^{0} a\,dv = -2\int_0^{v_0} a\,dv$.

$\Rightarrow E_{Loss} = \frac{\cancel{2}m\cancel{(-2)}}{3c^3\cancel{4}Z} \underbrace{\int_0^{v_0}(v_0^4 - 2v_0^2v^2 + v^4)\,dv}_{v_0^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{v_0^5}{15}(15 - 10 + 3) = \frac{8v_0^5}{15}}$

$E_{Loss} = -\frac{8mv_0^5}{45Zc^3} = -\frac{1}{2}mv_0^2\left(\frac{16mv_0^3}{45Zc^3}\right)$