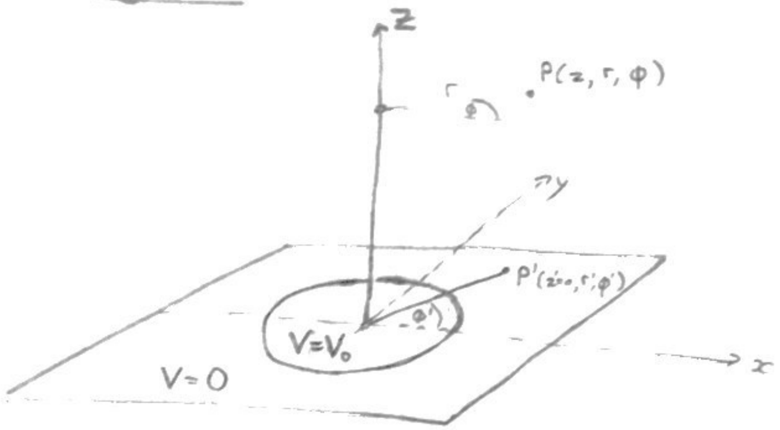
F'07 Q9

a) $\Phi(z \geq 0, r, \phi) = ?$; Hint: Find appropriate Green's function.

b) $\Phi(z \geq 0, r=0, \phi) = ?$

Diagram:



$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} G(\vec{x},\vec{x}')\,\rho(\vec{x}')\,dV' - \frac{1}{4\pi}\int_{A'}\left(\Phi(\vec{x})\frac{\partial G(\vec{x},\vec{x}')}{\partial n'} - G_D(x,x')\frac{\partial \Phi}{\partial n'}\right)dA'$$

(from Green's second identity)

not recursive if $\Phi(\vec{x})\Big|_{surface}$ is known, as in this Q:

$$\Phi(\vec{x})\Big|_{z=0} = \begin{cases} V_0 & \text{if } r \leq a \\ 0 & \text{otherwise} \end{cases}$$

$\rho(\vec{x}') = 0$, no sources.

Dirichlet B.C.'s: $G_D(x,x')\Big|_{surface} = 0$.

$$\Rightarrow \Phi(\vec{x}) = \frac{-V_0}{4\pi}\int_0^a r'dr'\int_0^{2\pi} d\phi' \left.\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right|_{z'=0}$$

$$\rightarrow G(\vec{x},\vec{x}') = \frac{1}{\sqrt{\underbrace{(x-x')^2}_{x_{12}} + \underbrace{(y-y')^2}_{y_{12}} + (z-z')^2}} - \frac{1}{\sqrt{x_{12}^2 + y_{12}^2 + (z+z')^2}}$$

(from $\nabla^2 F = 0$, $G_D|_s = 0$)

$\rightarrow \hat{n}' = -\hat{z}' \Rightarrow$ (out of Volume of interest ($z \geq 0$))

$$\left.\frac{\partial G(\vec{x}',\vec{x})}{\partial n'}\right|_{z'=0} = \left.-\frac{\partial G}{\partial z'}\right|_{z'=0} = \left.\frac{-(z-z')}{(x_{12}^2 + y_{12}^2 + (z-z')^2)^{3/2}}\right|_{z'=0} + \left.\frac{-(z+z')}{(x_{12}^2 + y_{12}^2 + (z+z')^2)^{3/2}}\right|_{z'=0} =$$

$$= \frac{-2z}{(x_{12}^2 + y_{12}^2 + z^2)^{3/2}}$$

$\Rightarrow$ convert $\left[(x'-x)^2+(y-y')^2+z^2\right]^{1/2}$ to cylindrical to match coordinates of surface integral = (polar, $z'=0$):

$$\Rightarrow \underset{z\geq 0}{\phi(z,r,\phi)} = V_0 z \int_0^a r'dr' \frac{1}{\big(\underbrace{r^2+r'^2-2rr'\cos(\phi-\phi')}+z^2\big)^{3/2}}$$

could use cos-rule : orthogonal

b) $$\phi(z\geq 0, r=0, \phi) = V_0 z \int_0^a dr' \frac{r'}{\left(r'^2+z^2\right)^{3/2}} =$$

$$= V_0 z \left[\frac{-1}{\left(r'^2+z^2\right)^{1/2}}\right]_0^a =$$

$$= V_0 z \left[\frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}}\right] =$$

$$\Rightarrow \boxed{\phi(z\geq 0, r=0, \phi) = V_0\left[1 - \frac{z}{(a^2+z^2)^{1/2}}\right]}$$

Check: $\phi(z=0,0,\phi) = V_0$