1. (Classical Mechanics)

A planet of mass m is moving in a gravitational central potential around a Sun of mass M. Assume $M \gg m$.

(a) Write down the Lagrangian and the Euler-Lagrange equations for the polar variables r, θ in the plane of motion.

(b) Use the substitution $u = \frac{1}{r}$ to write down a differential equation for the trajectory $u(\theta)$.

(c) What is the equilibrium solution of this equation? What does it represent?

(d) If the planet is not initially on the equilibrium orbit, there will be small oscillations around the equilibrium point. What is the period of these oscillations?

(e) Assume there is a perturbing potential $V = -B/r^2$, calculate the effect of this perturbation on the orbit.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This is the Kepler problem. Since $M \gg m$, we can assume that the sun is fixed in place (if this were not true, we would just replace the mass of the planet m by the reduced mass $\mu \equiv Mm/(M+m)$). We will also assume that the planet is nonrelativistic.

(a) The kinetic energy of the planet is

$$T = \frac{1}{2} m \left|\dot{\mathbf{r}}\right|^2 \tag{1}$$

In polar coordinates (r, θ), the position vector of the planet is given by

$$\mathbf{r} = r\,\hat{\mathbf{r}} = r\cos\theta\,\hat{\mathbf{x}} + r\sin\theta\,\hat{\mathbf{y}} \tag{2}$$

Using the chain rule, we can find the time derivative of the position vector

$$\begin{aligned}
\dot{\mathbf{r}} &= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\mathbf{x}} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{\mathbf{y}} \\
&= \dot{r}\left(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}\right) + r\dot{\theta}\left(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}\right) \\
\dot{\mathbf{r}} &= \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta}
\end{aligned} \tag{3}$$

Plugging into (1), we get

$$\begin{aligned}
T &= \frac{1}{2} m \left|\dot{\mathbf{r}}\right|^2 \\
&= \frac{1}{2} m \left(\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\theta}\right)^2 \\
&= \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\
T &= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2\dot{\theta}^2
\end{aligned} \tag{4}$$

The potential energy of the planet is the gravitational potential energy

$$V = -\frac{GMm}{r} \tag{5}$$

where G is the gravitational constant. For a setup with a nonrelativistic particle and a scalar potential, the Lagrangian is given by

$$L = T - V \tag{6}$$

In this case, the Lagrangian in terms of the polar variables (r, θ) is

$$\boxed{L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2\dot{\theta}^2 + \frac{GMm}{r}} \tag{7}$$

The formula for the Euler-Lagrange equations for a system with generalized coordinates (r, θ) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{8}$$

For the coordinate r, we have

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{GMm}{r^2} \tag{9}$$

Taking the total time derivative of $\frac{\partial L}{\partial \dot{r}}$, we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \tag{10}$$

Therefore, the Euler-Lagrange equation for r is

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} \\
0 &= m\ddot r - \left(mr\dot\theta^2 - \frac{GMm}{r^2}\right) \\
0 &= \ddot r - r\dot\theta^2 + \frac{GM}{r^2} \quad \text{dividing through by } m \qquad (11)
\end{aligned}$$

For the coordinate θ, we have

$$\frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \quad \text{and} \quad \frac{\partial L}{\partial \theta} = 0 \qquad (12)$$

Since $\frac{\partial L}{\partial \theta} = 0$, the Euler-Lagrange equation for θ boils down to

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) \\
0 &= \frac{d}{dt}\left(mr^2\dot\theta\right) \qquad (13)
\end{aligned}$$

Taking the total time derivative using the product and chain rules, we have

$$\begin{aligned}
0 &= 2mr\dot r\dot\theta + mr^2\ddot\theta \\
0 &= 2\dot r\dot\theta + r\ddot\theta \quad \text{dividing through by } m,\ r \qquad (14)
\end{aligned}$$

Gathering our results, we have that the Euler-Lagrange equations for the variables (r, θ) are

$$\boxed{0 = \ddot r - r\dot\theta^2 + \frac{GM}{r^2} \quad \text{and} \quad 0 = 2\dot r\dot\theta + r\ddot\theta} \qquad (15)$$

(b) The problem asks for a single differential equation for $u(\theta)$, where $u \equiv 1/r$. Here, we have two coupled differential equations for r and θ. To reduce these two equations to one, we will make use of a conservation law for the system:

If the Lagrangian does not depend on a generalized coordinate q, then the canonical momentum associated with that coordinate $p_q \equiv \frac{\partial L}{\partial \dot q}$ is conserved.

In this case, the Lagrangian (7) does not depend on θ, only its time derivative $\dot\theta$. Therefore, the canonical momentum associated with the coordinate θ is conserved. Since this canonical momentum is the angular momentum of the system, we will call it L. We already showed in (12) and (13) that

$$L \equiv \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta \quad \text{and} \quad \frac{dL}{dt} = 0 \qquad (16)$$

Note that the second equation from our part (a) answer (15) is just a restatement of angular momentum conservation. We can use this definition of angular momentum to replace the variable $\dot\theta$ in the first equation of (15) with the constant L:

$$\begin{aligned}
0 &= \ddot r - r\dot\theta^2 + \frac{GM}{r^2} \\
&= \ddot r - r\left(\frac{L}{mr^2}\right)^2 + \frac{GM}{r^2} \quad \text{since } L \equiv mr^2\dot\theta \\
0 &= \ddot r - \frac{L^2}{m^2r^3} + \frac{GM}{r^2} \qquad (17)
\end{aligned}$$

Now we have a single ordinary differential equation for the variable r. The next step is to make the substitution $u \equiv 1/r$. To do this, note that by the product rule and the chain rule, we can write the second time derivative of r in terms of u and its derivatives:

$$\begin{aligned}
r &= \frac{1}{u} \\
\frac{dr}{dt} &= -\frac{1}{u^2}\frac{du}{dt} \\
\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) &= \frac{d}{dt}\left(-\frac{1}{u^2}\frac{du}{dt}\right) \\
&= \left(\frac{2}{u^3}\frac{du}{dt}\right)\frac{du}{dt} - \frac{1}{u^2}\frac{d^2u}{dt^2} \\
\frac{d^2r}{dt^2} &= \frac{2}{u^3}\left(\frac{du}{dt}\right)^2 - \frac{1}{u^2}\frac{d^2u}{dt^2} \qquad (18)
\end{aligned}$$

Plugging in these values for $\ddot{r}$ and r in terms of u in (17), we get

$$0 = \frac{2}{u^3}\left(\frac{du}{dt}\right)^2 - \frac{1}{u^2}\frac{d^2u}{dt^2} - \frac{L^2}{m^2}u^3 + GMu^2 \qquad (19)$$

This is a differential equation for u in terms of t, not u in terms of θ. To solve this problem, we can use the chain rule again to swap out time derivatives for derivatives in terms of θ:

$$\begin{aligned}
\frac{du}{dt} &= \frac{du}{d\theta}\frac{d\theta}{dt} \\
&= \frac{du}{d\theta}\cdot\left(\frac{L}{mr^2}\right) \quad \text{since } L = mr^2\dot{\theta} \\
\frac{du}{dt} &= \frac{L}{m}u^2\frac{du}{d\theta} \quad \text{since } u = \frac{1}{r} \qquad (20)
\end{aligned}$$

Taking another time derivative and swapping it for a derivative in terms of θ, we get

$$\begin{aligned}
\frac{d^2u}{dt^2} &= \frac{d}{dt}\left(\frac{du}{dt}\right) \\
&= \frac{d}{d\theta}\left(\frac{du}{dt}\right)\cdot\frac{d\theta}{dt} \\
&= \frac{d}{d\theta}\left(\frac{L}{m}u^2\frac{du}{d\theta}\right)\cdot\frac{d\theta}{dt} \quad \text{by (20)} \\
&= \frac{L}{m}\left[\left(2u\frac{du}{d\theta}\right)\frac{du}{d\theta} + u^2\frac{d^2u}{d\theta^2}\right]\cdot\frac{d\theta}{dt} \quad \text{using the product rule and chain rule} \\
&= \frac{L}{m}\left[\left(2u\frac{du}{d\theta}\right)\frac{du}{d\theta} + u^2\frac{d^2u}{d\theta^2}\right]\cdot\left(\frac{L}{mr^2}\right) \quad \text{since } L = mr^2\dot{\theta} \\
&= \frac{L}{m}\left[\left(2u\frac{du}{d\theta}\right)\frac{du}{d\theta} + u^2\frac{d^2u}{d\theta^2}\right]\cdot\left(\frac{L}{m}u^2\right) \quad \text{since } u = \frac{1}{r} \\
\frac{d^2u}{dt^2} &= \frac{L^2}{m^2}\left[2u^3\left(\frac{du}{d\theta}\right)^2 + u^4\frac{d^2u}{d\theta^2}\right] \qquad (21)
\end{aligned}$$

Plugging in (20) and (21) into (19) and simplifying, we get

$$0 = \frac{2}{u^3}\left(\frac{L}{m}u^2\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\left(\frac{L^2}{m^2}\left[2u^3\left(\frac{du}{d\theta}\right)^2 + u^4\frac{d^2u}{d\theta^2}\right]\right) - \frac{L^2}{m^2}u^3 + GMu^2$$

$$0 = \frac{2L^2}{m^2}u\left(\frac{du}{d\theta}\right)^2 - \frac{2L^2}{m^2}u\left(\frac{du}{d\theta}\right)^2 - \frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2} - \frac{L^2}{m^2}u^3 + GMu^2$$

$$0 = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2} - \frac{L^2}{m^2}u^3 + GMu^2 \quad \text{canceling the first two terms}$$

$$0 = \frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2} \quad \text{multiplying through by } -\frac{m^2}{L^2}\frac{1}{u^2}$$

(In the last step, we are assuming that the planet has nonzero angular momentum, which is true for any orbit.) Therefore, the differential equation for the trajectory $u(\theta)$ is

$$\boxed{0 = \frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2}} \tag{22}$$

(c) There is a general procedure for finding equilibrium points in classical mechanics:

To find the equilibrium points for a system that depends on a generalized coordinate q, set $\dot{q} = \ddot{q} = 0$ and solve for q. In other words, look for points q for which, if $\dot{q} = 0$, then $\ddot{q} = 0$. These correspond to points where, if an object is held in place and then released, the object's q-coordinate is constant over time.

In this case, our differential equation (22) expresses the generalized coordinate u in terms of the polar angle θ instead of the time t, but the logic is the same. Setting $\frac{du}{d\theta} = \frac{d^2u}{d\theta^2} = 0$ and solving for the equilibrium u-value, which we will call u_0, we get

$$0 = u_0 - \frac{GMm^2}{L^2}$$

Therefore, the equilibrium solution of this differential equation is

$$\boxed{u_0 = \frac{GMm^2}{L^2}} \tag{23}$$

The equilibrium value of u is the value of u that remains constant across different polar angles θ in the orbit. Since $u \equiv 1/r$, a constant value of u corresponds to a constant value of r across different polar angles θ. Therefore, the equilibrium solution of equation (22) corresponds to a circular orbit. It is in fact the circular orbit you would get if you set the mass times the centripetal acceleration of the planet equal to the gravitational force on the planet.

(d) There is a general procedure for finding small oscillations about equilibrium in classical mechanics:

Suppose you have found equilibrium points q_0 for a system that depends on a generalized coordinate q. To determine whether each equilibrium is stable, and (if applicable) the details of small oscillations about each equilibrium, set $q = q_0 + \epsilon$, where ϵ is a small parameter. Then expand the equations of motion to lowest nontrivial order in ϵ.
If you have calculated the equilibrium correctly, the first-order contribution in ϵ should vanish, and you will be left with a second-order differential equation for ϵ. If this differential equation is the equation for simple harmonic motion, then the equilibrium is stable, and you can extract the angular frequency from the general form $\ddot{q} + \omega^2 q = 0$.

In this case, we can write

$$u = u_0 + \epsilon = \frac{GMm^2}{L^2} + \epsilon \tag{24}$$

for small ϵ. Expanding the differential equation (22) in ϵ, we get

$$\begin{aligned} 0 &= \frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2} \\ 0 &= \frac{d^2\epsilon}{d\theta^2} + \left(\frac{GMm^2}{L^2} + \epsilon\right) - \frac{GMm^2}{L^2} \\ 0 &= \frac{d^2\epsilon}{d\theta^2} + \epsilon \end{aligned} \tag{25}$$

This is a differential equation for simple harmonic motion (where the time t is swapped out for the polar angle θ). The angular frequency of the simple harmonic motion is given by the square root of the coefficient of ϵ, namely

$$\omega = 1 \tag{26}$$

The period of the oscillations is defined by

$$T = \frac{2\pi}{\omega}$$

so the period of these oscillations is

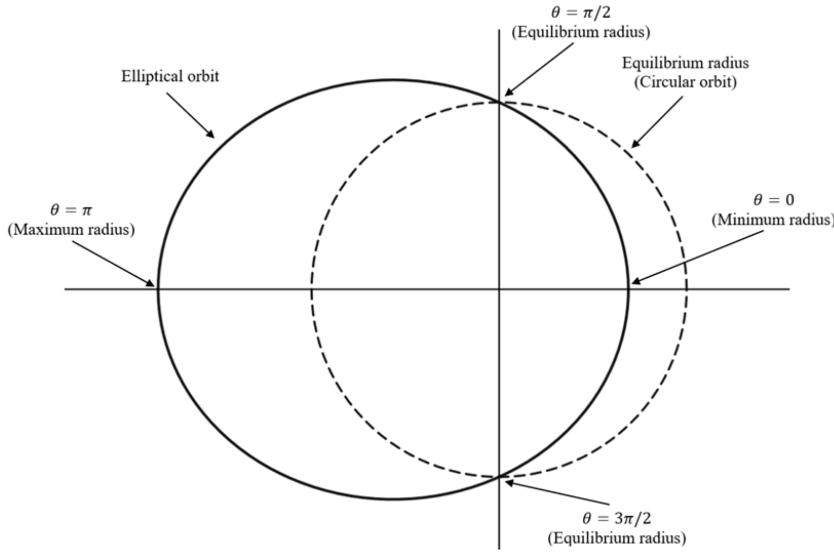
$$\boxed{T = 2\pi \text{ radians}} \tag{27}$$

A word on the units is in order. Ordinarily, ω would be in radians per second, but here we are measuring ϵ as a function of the polar angle θ, not the time t. Therefore, T is measured in radians, not seconds, and ω is in the slightly weird unit of "radians per radian." Here, we measure two quantities in radians:

- The planet's progress through its small oscillations. For example, we could set 0 radians as the point where the planet's radius is at a minimum. Then, at π radians, the planet's radius is at a maximum. At 2π radians, the planet's radius at a minimum, and so on.
- The planet's polar angle in space. After 2π radians of polar angle, the planet is on the same radial line from the sun as it was at 0 radians of polar angle.

ω measures the first kind of radians divided by the second kind. In other words, ω measures how fast the planet progresses through its small oscillations, as compared to how fast the planet progresses through its orbit. In this case, the ratio of the two quantities is one, meaning that the planet's oscillation frequency is the same as its orbital frequency.

The fact that the period of the orbit is 2π radians means that the planet completes one oscillation in the same time as it takes to complete one orbit. This corresponds to an elliptical orbit, as shown on the next page:



(e) To understand the effect of the perturbation $V = -B/r^2$ on the orbit, we need to run through the previous calculations again with the new perturbation added. The calculations that we have already done will be referenced by equation number throughout:

Part (a): Lagrangian

$$L = T - V$$
$$L = \underbrace{\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}}_{\text{from (7)}} + \frac{B}{r^2} \tag{28}$$

Part (a): Euler-Lagrange equation for r

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$$
$$0 = \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 - \frac{GMm}{r^2} - \frac{2B}{r^3}\right)$$
$$0 = m\ddot{r} - mr\dot{\theta}^2 + \frac{GMm}{r^2} + \frac{2B}{r^3}$$
$$0 = \underbrace{\ddot{r} - r\dot{\theta}^2 + \frac{GM}{r^2}}_{\text{from (15)}} + \frac{2B}{mr^3} \quad \text{dividing through by } m \tag{29}$$

Part (b): Replacing $\dot{\theta}$ with L

$$0 = \underbrace{\ddot{r} - \frac{L^2}{m^2r^3} + \frac{GM}{r^2}}_{\text{from (17)}} + \frac{2B}{mr^3} \tag{30}$$

Part (b): Substituting $u \equiv 1/r$

$$0 = \underbrace{\frac{2}{u^3}\left(\frac{du}{dt}\right)^2 - \frac{1}{u^2}\frac{d^2u}{dt^2} - \frac{L^2}{m^2}u^3 + GMu^2}_{\text{from (19)}} + \frac{2B}{m}u^3 \tag{31}$$

Part (b): Substituting time derivatives with derivatives in θ

$$0 = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\theta^2} - \frac{L^2}{m^2}u^3 + GMu^2 + \frac{2B}{m}u^3$$
$$0 = \underbrace{\frac{d^2u}{d\theta^2} + u - \frac{GMm^2}{L^2}}_{\text{from (22)}} - \frac{2Bm}{L^2}u \quad \text{multiplying through by } -\frac{m^2}{L^2}\frac{1}{u^2} \tag{32}$$

Part (c): Solving for the equilibrium value of u

$$0 = u_0 - \frac{GMm^2}{L^2} - \frac{2Bm}{L^2}u_0$$
$$u_0 = \frac{\frac{GMm^2}{L^2}}{1-\frac{2Bm}{L^2}} \quad \text{(reduces to (23) when } B \to 0) \tag{33}$$

Part (d): Finding the period of small oscillations

$$u = u_0 + \epsilon = \frac{\frac{GMm^2}{L^2}}{1-\frac{2Bm}{L^2}} + \epsilon \tag{34}$$

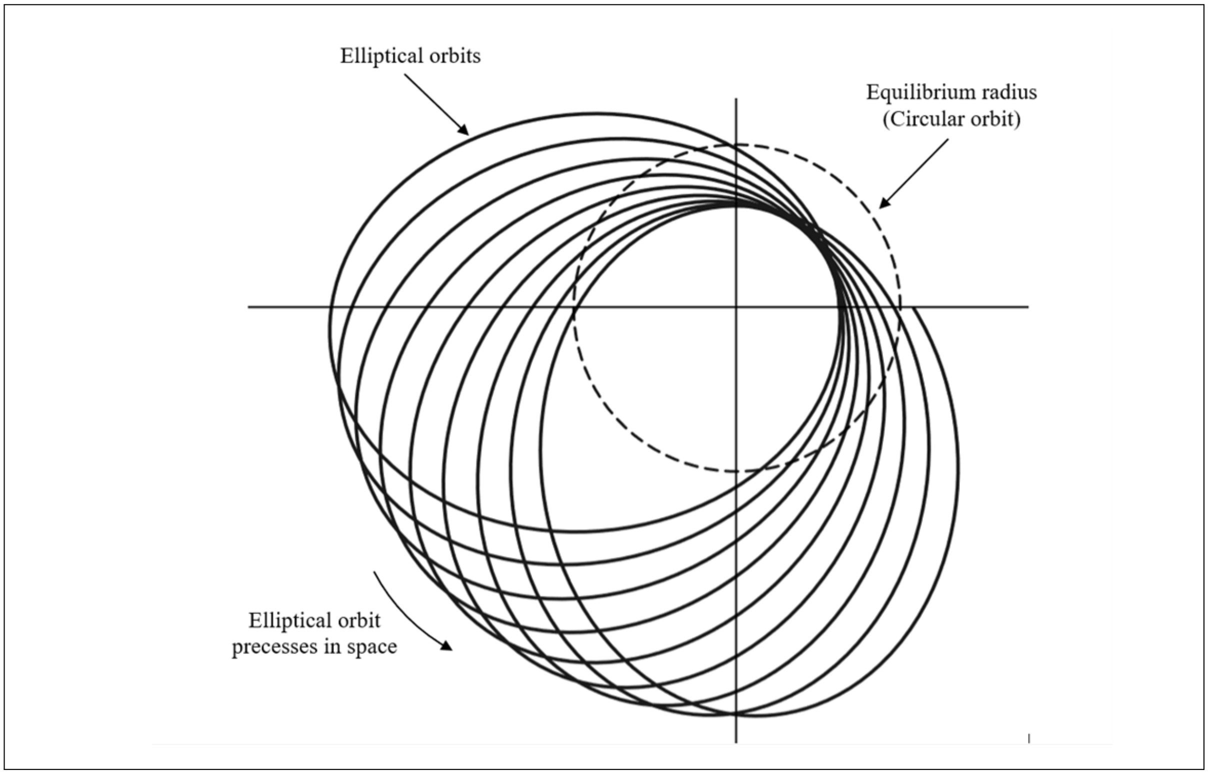
$$0 = \frac{d^2\epsilon}{d\theta^2} + \left(\frac{\frac{GMm^2}{L^2}}{1-\frac{2Bm}{L^2}} + \epsilon\right) - \frac{GMm^2}{L^2} - \frac{2Bm}{L^2}\left(\frac{\frac{GMm^2}{L^2}}{1-\frac{2Bm}{L^2}} + \epsilon\right) \quad \text{plugging into (32)}$$
$$0 = \frac{d^2\epsilon}{d\theta^2} + \left(1-\frac{2Bm}{L^2}\right)\left(\frac{\frac{GMm^2}{L^2}}{1-\frac{2Bm}{L^2}} + \epsilon\right) - \frac{GMm^2}{L^2}$$
$$0 = \frac{d^2\epsilon}{d\theta^2} + \left(1-\frac{2Bm}{L^2}\right)\epsilon \tag{35}$$

$$\text{Simple harmonic oscillation with} \quad \omega = \left(1-\frac{2Bm}{L^2}\right)^{1/2} \tag{36}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(1-\frac{2Bm}{L^2}\right)^{1/2}} \tag{37}$$

From this information, we can spot two differences to the orbit: First, the equilibrium value of u (33) is greater than it was in part (c). Since $r = 1/u$, this implies that the radius of circular orbits is less than it was in part (c).

Second, the period of the oscillations (37) is now greater than 2π. This means that a full oscillation of the orbital radius about its equilibrium (i.e. the full tracing out of each ellipse) takes longer than a full orbit. This causes a precession of the elliptical orbit in space, as shown in the diagram on the next page:


Elliptical orbits
Equilibrium radius
(Circular orbit)
Elliptical orbit
precesses in space