Fall 2015

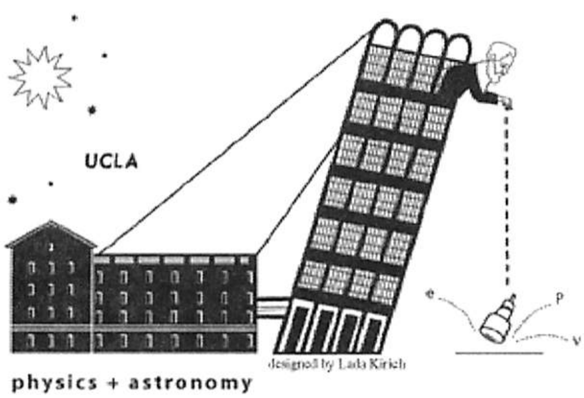
Physics Comprehensive Exam

September 14, 2015 (Part 1) 9:00 – 1:00pm

Part 1: Classical Mechanics and Quantum Mechanics

7 Total Problems/20 Points Each/Total 140 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the problem page.
- Use paper provided for additional pages. **Use one side only.**
- Use Name Labels on EACH of your response pages, including the question page. If you run out of labels, be sure to write your name on each page.
- Return the question page as the first page of your answers.
- When submitting, please clip all pages together in question # order.
- If a part of any question seems ambiguous to you, state clearly your interpretations and answer the question accordingly.



Name:

UCLA Physics Comprehensive Exam – Fall 2015 – Part 1

1. Classical Mechanics

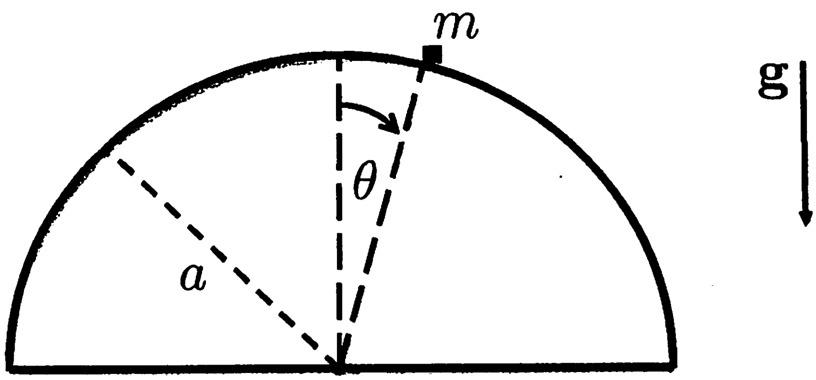
Consider the relativistic motion of a rocket on which no external forces are acting. The rocket is propelled by expelling matter at a constant velocity u with *respect to the rocket*. The velocity u is less than the speed of light c, but u is not necessarily small compared to c. As a result of this propulsion the mass m, and the velocity v of the rocket with respect to a fixed observer, will change over time.

(a) Obtain the differential equation for the change in the rocket velocity v as a function of the change in its mass m, as a function of m, v, and u.

(b) Solve this equation for v as a function of m.

Name:

2. Classical Mechanics



[20 points] A particle of mass m slides down a frictionless hemispherical cap (of radius a) in a uniform gravitational field with acceleration due to gravity $\mathbf{g}$. See the figure.

(a) [10 points] How far does the particle travel before leaving the hemispherical surface if the particle starts at the top with only an infinitesimal initial speed.

(b) [10 points] Now, you push the particle when it is at the top of the hemispherical cap so that its initial speed is v_0 . How does the distance to the lift-off point (where the particle leaves the surface) vary with initial speed v_0? Find the result and sketch a graph.

Name:

3. Quantum Mechanics

Entanglement. Consider two, non-interacting, distinguishable spin-1/2 particles (call them particle 1 and particle 2) prepared in the spin state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(\left|\uparrow_{(1)}\uparrow_{(2)}\right\rangle + \left|\downarrow_{(1)}\downarrow_{(2)}\right\rangle + \left|\uparrow_{(1)}\downarrow_{(2)}\right\rangle\right)$$

where the arrows represent spin up and spin down in the z-basis and the numeric subscripts are just labels referring to particle 1 and particle 2.

(a) If the projection the spin *of particle 2 only* is measured <u>in the x-basis</u>, what is the probability that the measurement outcome is "spin up" (the spin projection along $+x$ is positive)? If you are stuck, it may help to recall that σ_x can be written in terms of z-basis states as $\sigma_x = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|$.

(b) Assume that a measurement of the spin projection *of particle 2 only* in some basis (it doesn't matter what) has been made but that you do not have knowledge of what the outcome of that measurement was. Write the density matrix that represents your best description of the state of the spin of particle 1. Please write your answer in terms of coefficients $a, b, c,$ and d such that

$$\rho_{(1)} = a\left|\uparrow_{(1)}\right\rangle\left\langle\uparrow_{(1)}\right| + b\left|\uparrow_{(1)}\right\rangle\left\langle\downarrow_{(1)}\right| + c\left|\downarrow_{(1)}\right\rangle\left\langle\uparrow_{(1)}\right| + d\left|\downarrow_{(1)}\right\rangle\left\langle\downarrow_{(1)}\right|.$$

(c) How does your answer to part (b) change if you learn that the earlier measurement of particle 2 was made along x and yielded "spin up" (along $+x$) as its outcome?

Name:

4. Quantum Mechanics

4-dimensional Representation of SU(2). Consider a spin-3/2 particle with z-projection eigenstates $|m\rangle$, where m takes values from $\left\{\pm\frac{1}{2}, \pm\frac{3}{2}\right\}$. Work in units of $\hbar \equiv 1$.

(a) Find explicit 4×4 matrix forms for the spin operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ that satisfy the angular momentum commutation relation $\boldsymbol{J} \times \boldsymbol{J} = i\boldsymbol{J}$. Use the standard phase convention implied by $\langle jm'|\hat{J}_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\delta_{m',m\pm1}$ for $\hat{J}_\pm \equiv \hat{J}_x \pm i\,\hat{J}_y$ (this particle of course has $j = \frac{3}{2}$). Express these in the z-basis, *i.e.* such that

$$\hat{J}_z|m\rangle = \hat{J}_z \begin{pmatrix} \delta_{m,\frac{3}{2}} \\ \delta_{m,\frac{1}{2}} \\ \delta_{m,-\frac{1}{2}} \\ \delta_{m,-\frac{3}{2}} \end{pmatrix} = m|m\rangle$$

(b) Find the column vector $|+_x\rangle$ that is the maximally stretched eigenstate of $\hat{J}_x$. That is, find $|+_x\rangle$ that satisfies $\hat{J}_x|+_x\rangle = +\frac{3}{2}|+_x\rangle$.

(c) Assume that the total Hamiltonian for the spin is given by $H = \alpha\hat{J}_x + \beta\,\hat{J}_y + \gamma\hat{J}_z$ for some real constants α, β, and γ. What are the energies of the stationary states of this Hamiltonian?

Name:

5. Quantum Mechanics

Consider a particle with spin s=1/2 and magnetic moment μ in a time-dependent magnetic field $\boldsymbol{H}$(t),

$$\boldsymbol{H_x} = H_1 \cos(\omega_0 t),\ \boldsymbol{H_y} = H_1 \sin(\omega_0 t),\ \boldsymbol{H_z} = H_0,$$

where H_0, H_1 and ω_0 are constants. The Hamiltonian of the spin and the magnetic field interaction: $\hat{H}(t) = -\mu\vec{\boldsymbol{H}}(t)\cdot\vec{\boldsymbol{\sigma}}$, where $\vec{\boldsymbol{\sigma}} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

At the moment t=0, the particle is in a state with $s_z = 1/2$. Determine the probability for s_z=-1/2 at time t>0.

Name:

6. Quantum Mechanics

Consider N particles with spin 1/2 (and no other degrees of freedom), interacting with some isotropic Hamiltonian. All energy eigenstates can be grouped into $(2S+1)$-degenerate levels of total spin S (in units of $\hbar$). Each of the associated sublevels can be uniquely identified according to its total spin projection $M \in [-S, S]$ onto the z axis.

(a) (5 points) Find the total number of distinct N-particle states, $f(M)$, having z-axis spin projection M.

(b) (15 points) How many distinct levels $n(S)$ are there (not counting the $2S+1$ degeneracy) for an arbitrary allowed total spin S? For example, for $N = 1$: $n(1/2) = 1$, while for $N = 2$: $n(0) = 1$ and $n(1) = 1$ (corresponding to the spin singlet and triplet, respectively). *[Hint: you may find it useful to utilize the result from part (a).]*

Name:

7. Quantum Mechanics

An unperturbed (spinless) two-particle state is given by the wave function

$$\psi_\pm(\mathbf{r}_1, \mathbf{r}_2) = \frac{\psi_A(\mathbf{r}_1)\psi_B(\mathbf{r}_2) \pm \psi_A(\mathbf{r}_2)\psi_B(\mathbf{r}_1)}{\sqrt{2}},$$

where ψ_A and ψ_B are two orthonormal single-particle orbitals. The $\pm$ case corresponds to the identical bosons/fermions, respectively. If the particles are subjected to a weak short-ranged interaction potential $V(\mathbf{r}_1 - \mathbf{r}_2) \equiv \lambda\delta(\mathbf{r}_1 - \mathbf{r}_2)$, where $\lambda > 0$ and δ is the Dirac delta function, will they have a larger energy in the case of the bosonic or fermionic statistics? Can you physically interpret your conclusion?

Fall 2015

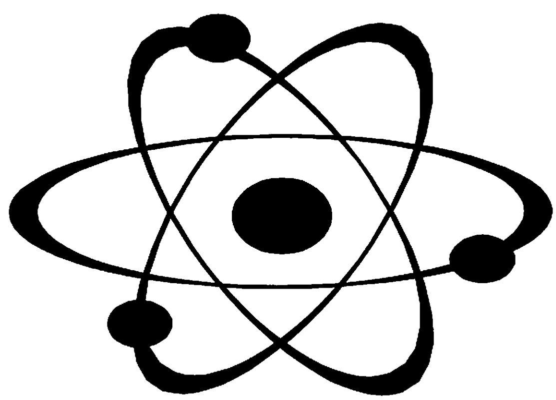
Physics Comprehensive Exam

September 15, 2015 (Part 2) 9:00 - 1:00pm

Part 2: Statistical Mechanics and Electromagnetism

7 Total Problems/20 Points Each/Total 140 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the question page.
- Use paper provided for additional pages. **Use one side only.**
- Use Name Labels on EACH of your response pages, including the question page. If you run out of labels, be sure to write your name on each page.
- Return the question page as the first page of your answers.
- When submitting, please clip all pages together in question # order.
- If a part of any question seems ambiguous to you, state clearly your interpretations and answer the question accordingly.



Name:

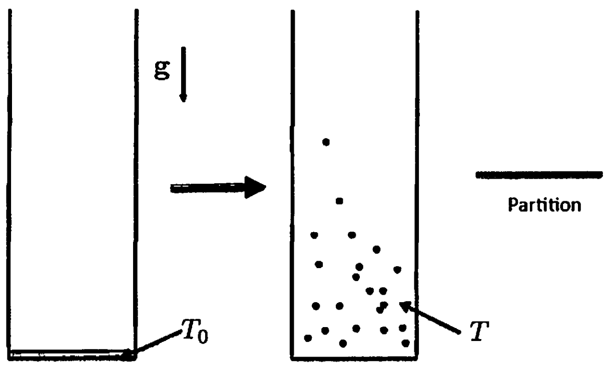
UCLA Physics Comprehensive Exam - Fall 2015 - Part 2

1. Statistical Mechanics

Let a neutron star in equilibrium have mass M_0 and radius R_0. For example, when M_0 equals the solar mass, we would have $R_0 = 12.4$ km. Find the radius R for a neutron star in equilibrium whose mass M is given by $M = 2M_0$. Derive the radius R when M is twice the solar mass. Show all steps needed to derive the answer. Assume no interactions between the neutrons, other than the gravitational force, treat the problem non-relativistically, and effectively at zero temperature.

Name:

2. Statistical Mechanics



A perfectly rigid and perfectly insulated column of cross sectional area A and infinite height contains N gas atoms (which you may treat as an ideal gas) of mass m confined by a partition to the bottom. The temperature of the gas is T_0. Then the partition is removed and the gas expands against a constant gravitational field. What is the final temperature T of the gas?

Name:

3. Statistical Mechanics

Consider two non-interacting identical Fermions, in a system (closed with respect to the number of particles, but in contact with a heat bath) where they can access three different single-particle states, of energy 0, ε, 2ε.

Calculate, at $T = 0$ and at finite T, the energy of the system and the average occupation numbers of the single-particle states.
(Hint: use the partition sum).

Now consider that the system can exchange particles with a "reservoir" of the same particles characterized by a chemical potential $\mu = 2\,\varepsilon + \alpha\,T$. What are now the average occupation numbers of the single-particle states of the original system, and the average number of particles in the original system, at $T = 0$?
(Hint: use occupation numbers).

Name:

4. Electromagnetism

Plane EM waves of frequency ω are incident on a dielectric sphere of radius a with susceptibility χ_e. You may assume that the radius of the sphere is very small compared to the wavelength of the EM wave.

(a) Compute the magnitude of the electric field oscillation inside the dielectric sphere given that the magnitude of the electric field of the incident wave is E_o (assume the wave is linearly polarized in the $\hat{z}$ direction).

(b) Calculate the differential cross-section for scattering from the sphere, assuming that the EM waves are unpolarized (better stated: with time, the polarization of the waves varies randomly in the plane transverse to the incident wavenumber, $\mathbf{k}_{\text{inc}}$).

Name:

5. Electromagnetism

A cylindrical capacitor of length L is composed of an inner cylindrical conductor of radius a and a concentric outer conducting cylindrical shell of radius b. A cylindrical shell of dielectric material of length L and which just fits in between the conductors (inner radius $\sim a$ and outer radius $\sim b$) is inserted so that half is inside of the capacitor (i.e. $L/2$ of the length of the capacitor is now filled with dielectric).

(a) What is the capacitance of this arrangement (you may ignore fringing fields at the ends)?

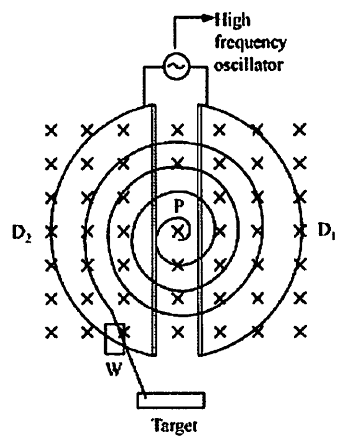
(b) The two conductors are held at a constant potential difference, V, using a battery. What is the force on the dielectric (magnitude and direction)?

(c) Explain the physical origin of this force.

Name:

6. Electromagnetism

A cyclotron consists of two D-shaped regions known as dees. In each dee there is a magnetic field perpendicular to the plane of the page. In the gap separating the dees there is a uniform electric field pointing from one dee to the other. When a charge q is released from rest in the gap it is accelerated by a potential difference V_0 and carried into one of the dees. The magnetic field B in the dee causes the charge to follow a half-circle that carries it back to the gap. While the charge is in the dee the electric field in the gap is reversed, so the charge is once again accelerated across the gap. The cycle continues with the magnetic field in the dees continually bringing the charge back to the gap. Every time the charge crosses the gap it picks up speed. This causes the half-circles in the dees to increase in radius, and eventually the charge emerges from the cyclotron at high speed.



a) Calculate the time it takes for the particle to travel through one of the dees and the frequency of the high frequency oscillator to guarantee continuous acceleration
b) Calculate the maximum output energy for a particle for a cyclotron of radius R
c) When the energy of the particle becomes a significant fraction of the particle rest mass energy, relativistic effects play a role. Calculate the time it takes for the particle to travel through one of the dees in this case. How does the cyclotron have to be modified to be able to reach relativistic energies?

Name:

7. Electromagnetism

A single-frequency laser beam of wavelength λ = 600 nm (in air) shines directly into a power meter that reads 1.00 mW. For this problem you should assume the laser beam to be described as a plane wave and ignore all transverse mode effects.

a) A flat, lossless mirror of power reflectivity R = 99% is placed between the laser and the meter with its normal parallel to the laser beam. What does the power meter read?

b) A second identical mirror is similarly placed at a distance d behind the first mirror (between the first mirror and the detector), to form a Fabry-Perot interferometer. Now the power meter reads 1.00 mW again. Explain why. Calculate the minimum distance d for which this is possible.

c) Now the second mirror is moved outward slowly, and the meter reading varies with the position. Write an expression for the transmission from the double mirror geometry as a function of d. Make a sketch of the power on the meter vs. the displacement d of the second mirror over the range of 2 μm. Label both axes quantitatively.

d) The spectral features that appear in the graph of part (c) have a certain width, even if the laser is truly "single frequency". Estimate this width.