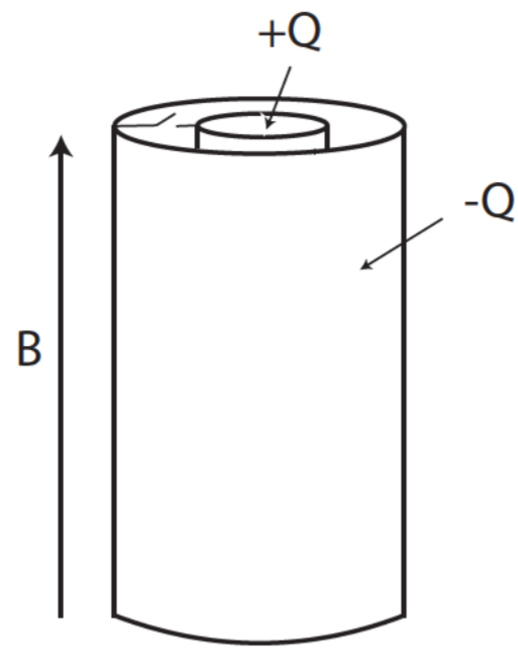
**2. (Electromagnetism)**

Consider a cylindrical capacitor of length $L$ and moment of inertia $I$ with charge $+Q$ on the inner cylindrical metallic shell of radius $a$ and charge $-Q$ on the outer cylindrical metallic shell of radius $b$. The capacitor is immersed in a uniform magnetic field $\mathbf{B}$, which points along the symmetry axis of the capacitor. There is a vacuum between the plates, which are held fixed to one another (e.g. by small insulating posts so that they will rotate together as a rigid body). A thin stainless steel wire with a switch is attached across the two plates of the capacitor. At $t = 0$, the switch is closed, and current flows from one plate to the other through a resistive wire (you can assume resistance $R$). This current creates a torque on the capacitor and it rotates.

Once the capacitor is fully discharged (the charge on each plate is zero), what is the magnitude and direction of the angular velocity, $\boldsymbol{\omega}$, of the capacitor? (For simplicity, ignore fringing fields and assume that the thin stainless wire does not break the symmetry of the system when computing fields.)



**Solution:** *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

In order to find the final angular velocity $\boldsymbol{\omega}$ of the cylindrical capacitor, we need to first find the final mechanical angular momentum of the capacitor. There are two equivalent ways to do this, and we will present both.

Throughout, we will work in cylindrical coordinates $(s, \varphi, z)$, where $s$ points radially outward from the axis of the cylinders and $z$ points in the direction of the magnetic field. In this problem, we will use $L$ to mean the length of the capacitor, while we will use $\mathbf{L}$ to mean angular momentum. We will use $I$ to mean the moment of inertia of the capacitor, while we will use $\mathbf{I}$ to mean the current in the wire connecting the two cylinders.

***Conservation of angular momentum:***
At the beginning of the problem, all angular momentum is in the electromagnetic fields. As the capacitor discharges, the angular momentum in the fields is converted into mechanical angular momentum, and the capacitor begins to rotate.

The linear momentum density stored in electromagnetic fields is given by

$$\mathbf{g} \equiv \epsilon_0 \, \mathbf{E} \times \mathbf{B} \tag{1}$$

and the angular momentum density stored in electromagnetic fields is given by

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{g} = \epsilon_0 \left[ \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \right] \tag{2}$$

To find the initial angular momentum stored in the electromagnetic fields, we need to find the initial electric field everywhere. Since we assume that the wire does not break the symmetry, we are allowed to assume that the setup has cylindrical symmetry. Since we ignore fringing fields, we can treat the cylindrical capacitor as being of infinite length. With that in mind, the electric field points radially and depends only on the distance from the axis: $\mathbf{E} = E(s)\,\hat{\mathbf{s}}$.

We can apply Gauss' law to a Gaussian cylinder of radius $s$ of length $d$, centered on-axis. By the integral form of Gauss' law,

$$\begin{aligned} 2\pi s d \, E(s) &= \int_{\text{cylinder}} \mathbf{E} \cdot d\mathbf{a} \\ &= \frac{Q_{\text{encl}}}{\epsilon_0} \end{aligned} \tag{3}$$

Suppose the Gaussian cylinder lies between the two cylinders in the capacitor. In other words, suppose that $a < s < b$. In that case, since the linear charge density of the inner cylinder is $Q/L$, the charge enclosed by the Gaussian cylinder is equal to

$$Q_{\text{encl}} = Q \frac{d}{L} \quad \text{if } a < s < b \tag{4}$$

Therefore, for $a < s < b$, we have

$$\begin{aligned} 2\pi s d \, E(s) &= \frac{Qd/L}{\epsilon_0} \\ E(s) &= \frac{Q}{2\pi \epsilon_0 L s} \quad \text{for} \quad a < s < b \end{aligned}$$

For all other values of $s$, there is no charge enclosed in the Gaussian cylinder, so the electric field is equal to zero. We are ignoring fringing fields, so the electric field above and below the capacitor is equal to zero. Therefore, the electric field everywhere is

$$\mathbf{E}(s) = \begin{cases} \frac{Q}{2\pi\epsilon_0 L s} \hat{\mathbf{s}} & \text{for} \quad a < s < b, \text{ inside the capacitor} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The electromagnetic angular momentum density (2) is nonzero only in regions where $\mathbf{E}$ and $\mathbf{B}$ are both nonzero. In this problem, the only such region is the region between the cylinders ($a < s < b$). In this region, the initial linear momentum density (1) is equal to

$$\begin{aligned}
\mathbf{g} &= \epsilon_0\,\mathbf{E}\times\mathbf{B} \\
&= \epsilon_0 \frac{Q}{2\pi\epsilon_0 L s}\,\hat{\mathbf{s}}\times B\,\hat{\mathbf{z}} \\
&= \frac{QB}{2\pi L s}\,(\hat{\mathbf{s}}\times\hat{\mathbf{z}}) \\
\mathbf{g} &= -\frac{QB}{2\pi L s}\hat{\varphi} \quad \text{since } \hat{\mathbf{s}}\times\hat{\mathbf{z}} = -\hat{\mathbf{z}}\times\hat{\mathbf{s}} = -\hat{\varphi}
\end{aligned} \tag{6}$$

and the initial angular momentum density (2) is equal to

$$\begin{aligned}
\boldsymbol{\ell} &= \mathbf{r}\times\mathbf{g} \\
&= (s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}})\times\left(-\frac{QB}{2\pi L s}\hat{\varphi}\right) \\
&= -\frac{QB}{2\pi L}\,(\hat{\mathbf{s}}\times\hat{\varphi}) - \frac{QBz}{2\pi L s}\,(\hat{\mathbf{z}}\times\hat{\varphi}) \\
&= -\frac{QB}{2\pi L}\hat{\mathbf{z}} + \frac{QBz}{2\pi L s}\hat{\mathbf{s}} \quad \text{since } \hat{\mathbf{s}}\times\hat{\varphi} = \hat{\mathbf{z}} \text{ and } \hat{\mathbf{z}}\times\hat{\varphi} = -\hat{\varphi}\times\hat{\mathbf{z}} = -\hat{\mathbf{s}} \\
\boldsymbol{\ell} &= -\frac{QB}{2\pi L}\hat{\mathbf{z}} + [\ldots]\,\hat{\mathbf{s}}
\end{aligned} \tag{7}$$

Here, the ellipses $\ldots$ represent the radial component of the angular momentum density. To get the initial angular momentum, we integrate the angular momentum density $\boldsymbol{\ell}$ over the region between the cylinders.

We can ignore the radial component of the angular momentum density $[\ldots]\,\hat{\mathbf{s}}$ because over any circle of constant radius $s$, this component is equal to a constant times $\hat{\mathbf{s}}$. The unit vector $\hat{\mathbf{s}}$, which points radially outward, integrates to zero over any full circle. Therefore, the $[\ldots]\,\hat{\mathbf{s}}$ component of the angular momentum density does not contribute to the total angular momentum.

To get the initial angular momentum, we need only to integrate the $z$-component of $\boldsymbol{\ell}$ over the region between the two cylinders:

$$\begin{aligned}
\mathbf{L} &= \int \boldsymbol{\ell}\,dV \\
&= \int\left(-\frac{QB}{2\pi L}\hat{\mathbf{z}}\right)dV \\
\mathbf{L} &= -\frac{QB}{2\pi L}V\,\hat{\mathbf{z}} \quad \text{where } V \text{ is the volume of the region}
\end{aligned} \tag{8}$$

The region we are interested in is the region between the two cylinders. This region has length $L$, and its cross-sectional area is equal to the difference between the area of a circle of radius $b$ and the area of a circle of radius $a$. In other words,

$$\begin{aligned}
V &= A\,L \quad \text{where } A \text{ is the cross-sectional area} \\
&= \left(\pi b^2 - \pi a^2\right)L \\
V &= \pi(b^2 - a^2)L
\end{aligned} \tag{9}$$

Plugging this into (8), we get that the initial angular momentum stored in the electromagnetic fields is

$$\begin{aligned}
\mathbf{L} &= -\frac{QB}{2\pi L}\pi(b^2 - a^2)L\,\hat{\mathbf{z}} \\
\mathbf{L} &= -\frac{QB}{2}(b^2 - a^2)\hat{\mathbf{z}}
\end{aligned} \tag{10}$$

Once the capacitor has fully discharged, the electric field is equal to zero, so there is no more angular momentum in the electromagnetic fields. There is no radiation in the problem, so none of the electromagnetic angular momentum has radiated away to infinity. Therefore, all the angular momentum must have been converted into mechanical angular momentum. The mechanical momentum of an object can be written in terms of the moment of inertia of the object and the angular velocity vector of the object:

$$\mathbf{L} = I\boldsymbol{\omega} \tag{11}$$

Setting (10) equal to (11) and solving for the angular velocity vector, we get

$$-\frac{QB}{2}(b^2 - a^2)\,\hat{\mathbf{z}} = I\boldsymbol{\omega}$$

$$\boxed{\boldsymbol{\omega} = -\frac{QB}{2I}(b^2 - a^2)\,\hat{\mathbf{z}}} \tag{12}$$

***Direct calculation of torque:***
Another way to solve this problem is to directly calculate the torque on the stainless steel wire through which the current $\mathbf{I}$ travels between the two cylinders. To directly calculate the torque on the wire due to the magnetic field, we start by calculating the force on an section of the wire with oriented line element $d\boldsymbol{\ell}$. From the Lorentz force law, we get that the force on a section of the wire with oriented line element $d\ell$ is

$$\begin{aligned}
d\mathbf{F} &= dq\,\mathbf{v} \times \mathbf{B} \\
&= (\lambda d\ell)\;\mathbf{v} \times \mathbf{B} \quad \text{where } \lambda \text{ is the linear charge density in the wire} \\
&= \mathbf{I}\,d\ell\, \times \mathbf{B} \quad \text{since } \mathbf{I} = \lambda \mathbf{v}
\end{aligned} \tag{13}$$

In this case, $Q$ is the charge on the inner cylinder. Since the capacitor is discharging, we have $\frac{dQ}{dt} < 0$, and the current flows radially outward (from the positively charged cylinder to the negatively charged cylinder). Putting both facts together and using the conservation of charge, we get that the current is given by

$$\mathbf{I} = -\frac{dQ}{dt}\,\hat{\mathbf{s}} \tag{14}$$

Here, the differential line element $d\ell$ is equal to $ds$. Therefore, the differential force element on a section of the wire is

$$\begin{aligned}
d\mathbf{F} &= \mathbf{I}\,d\ell\, \times \mathbf{B} \\
&= -\frac{dQ}{dt}\,\hat{\mathbf{s}}\,ds \times B\hat{\mathbf{z}} \\
&= -B\frac{dQ}{dt}\,ds\,(\hat{\mathbf{s}} \times \hat{\mathbf{z}}) \\
d\mathbf{F} &= \frac{dQ}{dt}\,B\,ds\,\hat{\varphi} \quad \text{since } \hat{\mathbf{s}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{s}} = -\hat{\varphi}
\end{aligned} \tag{15}$$

The differential torque can be written in terms of the differential force as

$$\begin{aligned}
d\boldsymbol{\tau} &= \mathbf{r} \times d\mathbf{F} \\
&= (s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}}) \times \frac{dQ}{dt}\,B\,ds\,\hat{\varphi} \\
&= \frac{dQ}{dt}\,B\,s\,ds\,(\hat{\mathbf{s}} \times \hat{\varphi}) + \frac{dQ}{dt}\,B\,s\,ds\,(\hat{\mathbf{z}} \times \hat{\varphi}) \\
&= \frac{dQ}{dt}\,B\,s\,ds\,\hat{\mathbf{z}} - \frac{dQ}{dt}\,B\,s\,ds\,\hat{\mathbf{s}} \quad \text{since } \hat{\mathbf{s}} \times \hat{\varphi} = \hat{\mathbf{z}} \text{ and } \hat{\mathbf{z}} \times \hat{\varphi} = -\hat{\varphi} \times \hat{\mathbf{z}} = -\hat{\mathbf{s}} \\
d\boldsymbol{\tau} &= \frac{dQ}{dt}\,B\,s\,ds\,\hat{\mathbf{z}} + [\ldots]\,\hat{\mathbf{s}} \qquad (16)
\end{aligned}$$

Here, the ellipses ... represent the radial component of the differential torque. Since we are only interested in rotation about the $z$-axis, we will ignore this radial component.

The net torque about the $z$-axis is therefore

$$\begin{aligned}
\boldsymbol{\tau} &= \int d\boldsymbol{\tau} \\
&= \int_{s=a}^{s=b} \frac{dQ}{dt}\,B\,s\,ds\,\hat{\mathbf{z}} \\
\boldsymbol{\tau} &= \frac{dQ}{dt} B \left(\frac{b^2 - a^2}{2}\right) \qquad (17)
\end{aligned}$$

Torque is the derivative of mechanical angular momentum ($\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$), so

$$\begin{aligned}
\mathbf{L}_{\text{mech},f} - \mathbf{L}_{\text{mech},i} &= \int_{t_i}^{t_f} dt\,\boldsymbol{\tau} \quad \text{where } t_i \text{ and } t_f \text{ are the initial and final times} \\
&= \int_{t_i}^{t_f} dt\,\frac{dQ}{dt} B \left(\frac{b^2 - a^2}{2}\right) \\
&= \int_{Q_i}^{Q_f} dQ\,B \left(\frac{b^2 - a^2}{2}\right) \\
\mathbf{L}_{\text{mech},f} - \mathbf{L}_{\text{mech},i} &= (Q_f - Q_i)\ B \left(\frac{b^2 - a^2}{2}\right) \qquad (18)
\end{aligned}$$

Here, the initial mechanical angular momentum $\mathbf{L}_{\text{mech},i}$ is equal to zero (since the cylinders start at rest). $Q$ refers to the charge on the inner cylinder. The initial charge on the inner cylinder is $Q$ ($Q_i = Q$), and the final charge on the inner cylinder is zero ($Q_f = 0$). Plugging this in, we get

$$\begin{aligned}
\mathbf{L}_{\text{mech},f} &= -QB \left(\frac{b^2 - a^2}{2}\right) \\
\mathbf{L}_{\text{mech},f} &= -\frac{QB}{2}(b^2 - a^2) \qquad (19)
\end{aligned}$$

The mechanical momentum of an object can be written in terms of the moment of inertia of the object and the angular velocity vector of the object:

$$\mathbf{L} = I\boldsymbol{\omega} \qquad (20)$$

Setting (19) equal to (20) and solving for the angular velocity vector, we get

$$-\frac{QB}{2}(b^2 - a^2)\,\hat{\mathbf{z}} = I\boldsymbol{\omega}$$

$$\boxed{\boldsymbol{\omega} = -\frac{QB}{2I}(b^2 - a^2)\,\hat{\mathbf{z}}} \qquad (21)$$