Name:

Comprehensive exam, Fall 2010

1. *Quantum Mechanics*

A particle of mass m moves in a one dimensional potential

$$V(x) = \begin{cases} -\alpha\delta(x) & -\infty < x < x_0 \\ +\infty & x > x_0 \end{cases}$$

where $\alpha > 0$ and $x_0 > 0$.

a) For $E < 0$ find the solutions of the time independent Schroedinger equation in the regions $I : -\infty < x < 0$ and $II : 0 < x < x_0$. Do not worry about normalizing the wavefunction.

b) What is the boundary condition at $x = x_0$ and what is the matching condition at $x = 0$?

c) Using these results, find an equation to determine the energy(s) of possible bound state(s). How many bound states do exist ? Explain your reasoning.

d) What happens in the limit $x_0 \to 0$?

Name:

2. *Quantum Mechanics*

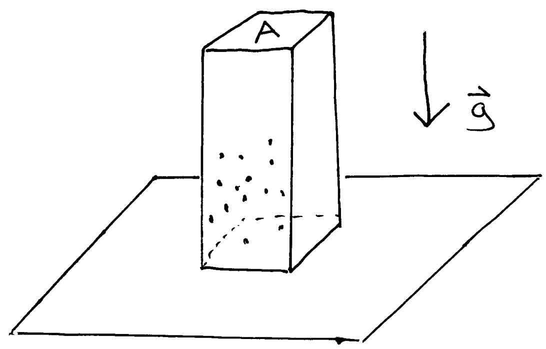
Three Hermitian 256×256 matrices, M_i with $i = 1, 2, 3$, are known to obey the commutation relations $[M_1, M_2] = iM_3$ (and its two cyclic permutations). The eigenvalues of M_1 are as follows: ± 2, each with multiplicity 1; $\pm 3/2$, each with multiplicity 8; ± 1, each with multiplicity 28; $\pm 1/2$, each with multiplicity 56; and 0, with multiplicity 70. Derive the 256 eigenvalues of the matrix $M^2 \equiv M_1^2 + M_2^2 + M_3^2$, (i.e. obtain its different eigenvalues, and their multiplicities).

Name:

3. *Quantum Mechanics*

A collection of $N \gg 1$ electrons are confined to the interior of a semi-infinite square tube of cross sectional area A, as shown. The tube is in a gravitational field, characterized by the gravitational acceleration g, as shown in the figure. Ignoring interactions between the electrons, compute the approximate volume of the tube occupied by the electrons.

Due to quantum mechanical uncertainty, the "occupied volume" is not uniquely defined, so as part of your answer you should state how you are defining it. Use any simplifications appropriate to the $N \gg 1$ limit.



Name:

4. *Quantum Mechanics*

A particle moves on a 1-dimensional periodic lattice with $2N$ sites labeled by an integer $n = 1, \ldots, 2N$. We write $|n\rangle$ for the state in which the particle occupies the nth site. These states form an orthonormal set, and satisfy $\langle n'|n\rangle = \delta_{n,n'}$ for $1 \leq n, n' \leq 2N$. The Hamiltonian is defined by

$$\begin{aligned} \langle n|H|n\rangle &= A_0(-1)^n \\ \langle n+1|H|n\rangle = \langle n|H|n+1\rangle &= A_1 \end{aligned}$$

all other matrix elements of H being zero. Here, A_0 and A_1 are real constants, and lattice periodicity is realized via the relation $|2N+1\rangle = |1\rangle$. We define operators K and T by

$$\begin{aligned} K|n\rangle &= (-1)^n|n\rangle \\ T|n\rangle &= |n+1\rangle \end{aligned}$$

(a) Prove that the operators K and T satisfy $KT + TK = 0$ and $[K, T^2] = 0$;

(b) Express H in terms of K and T, and show that H commutes with T^2;

(c) Compute, in terms of the states $|n\rangle$, the simultaneous eigenstates $|\pm, \ell\rangle$ of K and T^2, such that $K|\pm, \ell\rangle = \pm|\pm, \ell\rangle$, and the integer ℓ labeling the eigenvalues of T^2;

(d) Show that the states $|\pm, \ell\rangle$, found in (c), may be normalized so that

$$T|\pm, \ell\rangle = e^{i\pi\ell/N}|\mp, \ell\rangle \qquad\qquad \ell = 0, 1, \cdots, N-1$$

(e) Compute the eigenvalues of H and their multiplicities.

Name:

5. *Quantum Mechanics*

Although one encounters local potentials most frequently, some non-local potentials are useful and lead to simple and interesting dynamics. This problem is about such a non-local potential for which an *exact solution* may be obtained for all values of the coupling λ. Consider the 3-dimensional Hamiltonian H with *non-local potential* V, given by

$$H = \frac{\mathbf{p}^2}{2M} + V \qquad\qquad \langle \mathbf{r}'|V|\mathbf{r}\rangle = -\frac{\lambda\hbar^2}{2M}u(r')u(r)$$

where $r = |\mathbf{r}|$, $r' = |\mathbf{r}'|$, and $u(r)$ is a real-valued function. Throughout this problem, derive your results for all (real) values of the coupling constant parameter λ.

(a) Write down the integro-differential equation obeyed by a wave function $\psi_E(\mathbf{r})$ at energy E, in terms of the function $u(r)$.

(b) Show that only the s-wave is affected by this interaction.

(c) Establish the Lippmann-Schwinger equation, in integral form, for the scattering of an incoming plane wave of wave vector $\mathbf{k}$.

(d) Show that the scattering amplitude $f(\mathbf{k}', \mathbf{k})$ is given by

$$f(\mathbf{k}', \mathbf{k}) = 4\pi\lambda|v(k)|^2 \left[1 + \frac{2\lambda}{\pi}\int d^3q \, \frac{|v(q)|^2}{k^2 - q^2 + i\epsilon}\right]^{-1}$$

where $v(k)$ is the Fourier transform of $u(r)$, given by, $v(k) = \frac{1}{k}\int_0^\infty dr\, r\sin(kr)u(r)$, and where $k = |\mathbf{k}|$ and $q = |\mathbf{q}|$.

Formulas and notations:

The Lippmann-Schwinger equation for a *local potential* is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int d^3\mathbf{r}' \, G(\mathbf{r}, \mathbf{r}', \mathbf{k}^2) U(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}') \qquad\qquad V(\mathbf{r}) = \frac{\hbar^2}{2M}U(\mathbf{r})$$

where $\phi_{\mathbf{k}}$ is the normalized incoming free wave, given by

$$\phi_{\mathbf{k}}(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}}$$

The Green function is defined by $(\Delta + \mathbf{k}^2)G(\mathbf{r}, \mathbf{r}'; \mathbf{k}^2) = \delta^{(3)}(\mathbf{r}, \mathbf{r}')$, and is given by

$$G(\mathbf{r}, \mathbf{r}', \mathbf{k}^2) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{k}^2 - \mathbf{q}^2 + i\varepsilon} = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

The scattering amplitude f is defined by $(r = |\mathbf{r}|,\ k = |\mathbf{k}|)$,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{r}\right]$$

Name:

6. *Statistical Mechanics*

When electrons are confined to move in a two-dimensional plane (e.g. in a semiconductor quantum well or heterostructure), they will form a triangular lattice if their density is too low. Such a state is called a *Wigner crystal*. The transverse vibrations of such a crystal are similar to those of the ionic lattice, but the longitudinal vibrations satisfy $\omega_{\vec{k}} = a\sqrt{k}$ as a result of the unscreened long-ranged Coulomb interaction between the electrons. What is the contribution to the low-temperature specific heat of a Wigner crystal coming from longitudinal vibrations?

Name:

7. *Statistical Mechanics*

Calculate the speed of sound c in an ideal gas for two cases: **(a)** isothermal compression and **(b)** adiabatic compression. Recall the general hydrodynamic relation

$$c = \sqrt{\frac{dP}{d\rho}}\,,$$

where P is the pressure and ρ the mass density of the substance. Express your answer in terms of Boltzmann's constant k_B, temperature T, mass of the constituent particles m, and, for part (b), the adiabatic exponent γ defined through the relation $PV^\gamma = \text{const.}$

(c) Would the sound propagate faster in a gas of H (hydrogen) atoms or rigid H_2 molecules, at the same temperature? By how much?

Name:

8. *Statistical Mechanics*

Consider a system composed of $N \gg 1$ identical but distinguishable noninteracting atoms, each of which has only two nondegenerate energy levels: 0 and $\omega > 0$. Let $\epsilon = E/N$ be the energy per atom.

(a) What is the maximum possible value of ϵ if the system is not necessarily in equilibrium? What is the maximum value for ϵ attainable for the system equilibrated at positive temperatures? Please answer the same question for negative temperatures.

(b) In the case of thermodynamic equilibrium, compute the entropy per atom, $s = S/N$, as a function of ϵ.

(c) Over what range of ϵ is the temperature positive, and over what range is it negative?

Name:

9. *Statistical Mechanics*

A parallel plate capacitor consists of two square plates, each of area a^2, separated by the distance d. The first plate, which is located at $z = 0$, is made of conducting metal and is grounded. The second plate, which is located at $z = d$, is made of dielectric and is maintained at a positive potential V with respect to the metal plate. The whole system is at some very high temperature T so that the electrons emitted from the hot metal of the first plate form a dilute gas which is in equilibrium and which fills this capacitor. There is no conductivity of electrons between the gas and the dielectric plate. Assume that the capacitor is so large $(a >> d)$ that the edge effects can be disregarded.

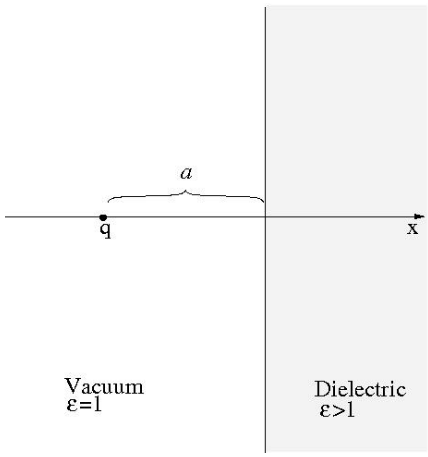
(a) Write out the system of equations and boundary conditions that determines the potential $\varphi(z)$, and the density of electrons $n(z)$, inside the capacitor as functions of z $(d > z > 0)$. Note: since in equilibrium there is no net flux of electrons across $z = 0$, you may assume that $n(z)$ has vanishing gradient there.

(b) Assuming a weak potential, $\frac{|eV|}{kT} << 1$, find $\varphi(z)$ and $n(z)$ inside the capacitor to first order in $\frac{eV}{kT}$.

Name:

10. *Electromagnetism*

Consider a point charge q located at $x = -a$ facing a dielectric of dielectric constant $\epsilon > 1$ which fills the half infinite space defined by $x \geq 0$.



a) Using the method of image charges, find the electric field $\vec{E}(\vec{r})$.

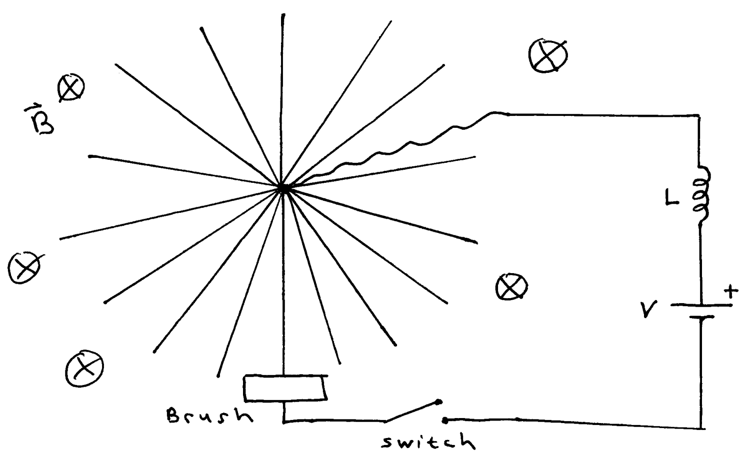
b) Imposing continuity of components of $\vec{D}$ and $\vec{E}$, express the image charge(s) in terms of ϵ and q.

c) Discuss the limiting cases $\epsilon \to 1$ and $\epsilon \to \infty$ of this problem. What is/are the images charge(s) in these two cases ?

Name:

11. *Electromagnetism*

A wheel consisting of a large number of thin conducting spokes is free to pivot about an axle. A brush makes electrical contact with one spoke at a time at the bottom of the wheel. A battery of voltage V feeds current through an inductor into the axle, through a spoke, to the brush. A permanent magnet provides a uniform magnetic field B into the plane of the paper. At time $t = 0$ a switch is closed, allowing current to flow. The radius and moment of inertia of the wheel are R and J respectively. The inductance of the current path is L, and the wheel in initially at rest. Neglecting friction and resistivity, calculate the current and angular velocity of the wheel as functions of time.



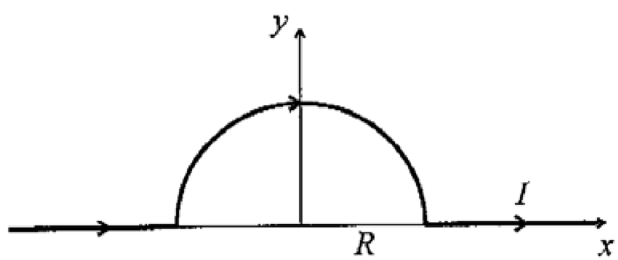
Name:

12. *Electromagnetism*

A semicircular wire of radius R is centered at the origin, while straight segments extend to infinity along the X-axis, as shown below. A uniform current I is suddenly turned on at $t = 0$, remaining constant thereafter.

(a) Calculate the vector ($\vec{A}$) and scalar potential (V) as a function of time at the origin.

(b) Calculate $\vec{E}$ and $\vec{B}$ as a function of time at the origin.



Name:

13. *Electromagnetism*

An isolated conducting sphere of radius a is placed inside a thin conducting spherical shell of radius b. The centers of the two spheres are not coincident, but are instead displaced from each other by a small distance δ, with $\delta \ll a, b$. The total charge of the inner sphere is q, and the outer sphere is grounded. Find the distribution of surface charge σ on the inner sphere and the force F acting on it, to first order in δ.

Name:

14. *Electromagnetism*

A plane electromagnetic wave with wavelength λ is incident on a rectangular aperture of width $2w_x \times 2w_y$. A 2D detector is placed in the far field to measure the diffraction intensity (i.e. the Fraunhofer approximation holds).

(a) Calculate the diffraction intensity on the detector.

(b) Plot the diffraction intensity distribution along the horizontal axis and verify the Heisenberg uncertainty principle.

Name:

Comprehensive exam, Fall 2010

1. *Quantum Mechanics*

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$$V(x) = \begin{cases} -\alpha\delta(x) & -\infty < x < x_0 \\ +\infty & x > x_0 \end{cases}$$

where $\alpha > 0$ and $x_0 > 0$.

a) For $E < 0$ find the solutions of the time independent Schroedinger equation in the regions $I : -\infty < x < 0$ and $II : 0 < x < x_0$. Do not worry about normalizing the wavefunction.

b) What is the boundary condition at $x = x_0$ and what is the matching condition at $x = 0$?

c) Using these results, find an equation to determine the energy(s) of possible bound state(s). How many bound states do exist ? Explain your reasoning.

d) What happens in the limit $x_0 \to 0$?

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$$\frac{\partial \psi_+}{\partial x} - \frac{\partial \psi_-}{\partial x} = \int_{-\varepsilon}^{+\varepsilon} \frac{\partial^2 \psi}{\partial x^2}\, dx$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$

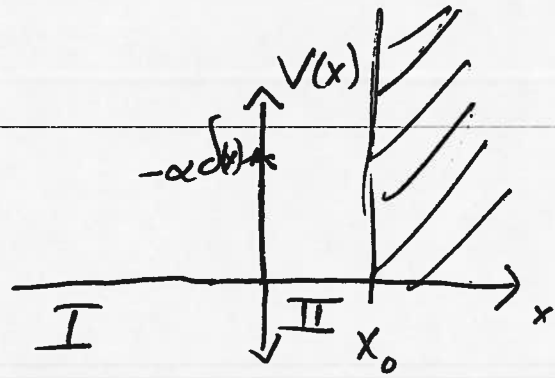
$$= \cancel{\int_{-\varepsilon}^{\varepsilon} -\frac{2mE}{\hbar^2}\psi\, dx} + \int_{-\varepsilon}^{\varepsilon} -\frac{2m\alpha\,\delta(x)}{\hbar^2}\psi\, dx$$

0

$$= -\frac{2m\alpha}{\hbar^2}\psi(0)$$

Question # 1

Page # 1



a) in I + II the time indep S.E. is

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi = k^2\psi$$

so $\psi_{I} = Ae^{kx} + \cancel{B}e^{-kx}$ (→ 0)

$\psi_{II} = Ce^{kx} + De^{-kx}$

for bound states

$k \to ik$ for scattering

$B=0$ since ψ is normalized

b) at $x = x_0$ $\psi(x) \to 0$

at $x=0$ $\psi_I(x=0) = \psi_{II}(x=0)$

$$\frac{\partial\psi_+}{\partial x} - \frac{\partial\psi_-}{\partial x} = \int_{-\varepsilon}^{\varepsilon}\frac{\partial^2\psi}{\partial x^2}\,dx$$

$$= \int^{\varepsilon} -\frac{2m(E-V)}{\hbar^2}\psi\,dx$$

Question # 1

Page # 2

ψ smooth

$$= \int_{-\varepsilon}^{\varepsilon} \cancel{\frac{-2mE}{\hbar^2}\psi\,dx} + \int_{-\varepsilon}^{\varepsilon} \frac{2mV}{\hbar^2}\psi\,dx$$

$\rightarrow 0$

✓

$$= \frac{-2m\alpha}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi\,dx = \frac{-2m\alpha}{\hbar^2}\psi(0) = \frac{d\psi_{II}}{dx}\Big|_{x=0} - \frac{d\psi_{I}}{dx}$$

c) $Ce^{kx_0} + De^{-kx_0} = 0$

$A = C + D$

$Ck - Dk - Ak = \frac{-2m\alpha}{\hbar^2}A$

$C = -De^{-2kx_0}$ $\quad \cancel{A = D - De^{-2kx_0}}$ $\quad A = D(1 - e^{-2kx_0})$

$C - D - A = -\frac{2m\alpha}{\hbar^2 k}A$

$C = A - D$

$\cancel{A} - D - D - \cancel{A} = -\frac{2m\alpha}{\hbar^2 k}A$

$\cancel{-}\cancel{2}D = \cancel{+}\frac{\cancel{2}m\alpha}{\hbar^2 k}A$

$D = \frac{m\alpha}{\hbar^2 k}\cancel{A} = \frac{\cancel{A}}{1 - e^{-2kx_0}}$

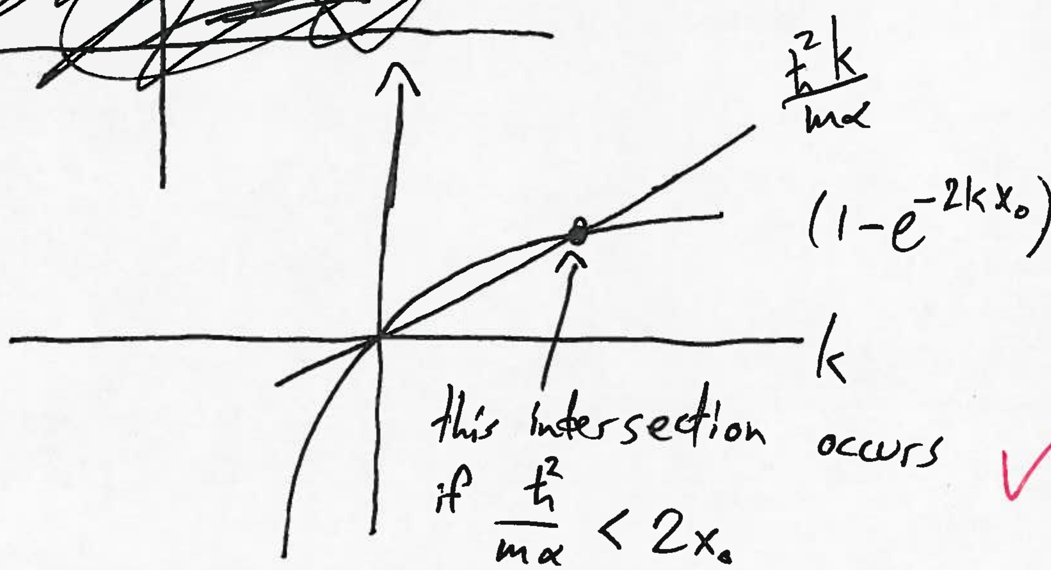
Question # 1

Page # 3

$$\cancel{\text{max}} \quad \frac{m\alpha}{\hbar^2}\left(1-e^{-2kx_0}\right) = k$$

$$\left(1-e^{-2kx_0}\right) = \frac{\hbar^2}{m\alpha}k \quad \checkmark$$

Sol'ns k to this eqn will produce bound states



this intersection occurs if $\frac{\hbar^2}{m\alpha} < 2x_0$ ✓

we can produce at most one nontrivial $(k \neq 0)$ bound state since $\frac{\hbar^2 k}{m\alpha}$ has constant derivative and $(1-e^{-2kx_0})$ has derivative always > 0 ✓

Question # 1

Page # 4

d) as $x_0 \to 0$ eqn becomes $k=0$ is only solution so no bound state exists which is expected since the potential is now $V(x) = \begin{cases} 0 & x<0 \\ \infty & x \geq 0 \end{cases}$

✓

20

2. *Quantum Mechanics*

Three Hermitian 256×256 matrices, M_i with $i = 1, 2, 3$, are known to obey the commutation relations $[M_1, M_2] = iM_3$ (and its two cyclic permutations). The eigenvalues of M_1 are as follows: ± 2, each with multiplicity 1; $\pm 3/2$, each with multiplicity 8; ± 1, each with multiplicity 28; $\pm 1/2$, each with multiplicity 56; and 0, with multiplicity 70. Derive the 256 eigenvalues of the matrix $M^2 \equiv M_1^2 + M_2^2 + M_3^2$, (i.e. obtain its different eigenvalues, and their multiplicities).

98

99

102

56

118

$5 + 32 + 81 + 96 + 42$

$37 + 81$

$118 + 96$

208

214

42

256

$1 \times 5 + 8 \times 4 + 27 \times 3 + 48 \times 2 + 42 \times 1 =$

Question # 2

Page # 1

The M_i matrices have the same algebra as angular momentum operators

Consider $M_1 \sim J_z$

~~M_1, ℓ, m~~ eigenvalues:

± 2	1
$\pm 3/2$	8
± 1	28
$\pm 1/2$	56
~~2~~ 0	70

The matrix M_1 is in a representation that is the sum of $\ell=0, \ell=\frac{1}{2}, \ell=1, \ell=\frac{3}{2}, \ell=2$ angular momenta

Calculate the multiplicity of each:

We have one $\ell=2$

so one ~~/~~ ± 1 eigenvalue belongs to $\ell=2$ the other 27 are from $\ell=1$ reps

for 0 eigenvalues 1 belongs to $\ell=2$, 27 belong to $\ell=1$

so $70-28=42$ are from $\ell=0$ reps.

Question # 2

Page # 2

We have 8 spin $3/2$ ~~pt~~ reps.

and applying the same logic as before $56-8=48$ spin-$\frac{1}{2}$

reps

This gives our representations as

$l=2 \times 1$

$l=\frac{3}{2} \times 8$

$l=1 \times 27$

$l=\frac{1}{2} \times 48$

$l=0 \times 42$

Check dimensions:

$$1 \times 5 + 8 \times 4 + 27 \times 3 + 48 \times 2 + 42 \times 1 = 256 \checkmark$$

Now M^2 is equivalent to J^2 so the corresponding eigenvalues and multiplicities are:

rep	eigenvalue	multiplicity
$l=2$	$2 \cdot 3 = 6$	5
$l=3/2$	$\frac{3}{2}\left(\frac{5}{2}\right) = \frac{15}{4}$	$8 \times 4 = 32$

Question # 2

Page # 3

rep	eigenvalue	mult
$l=1$	$1(2)=2$	$27\times 3=81$
$l=\frac{1}{2}$	$\frac{1}{2}\left(\frac{3}{2}\right)=3/4$	$48\times 2=96$
$l=0$	0	$42\times 1=42$

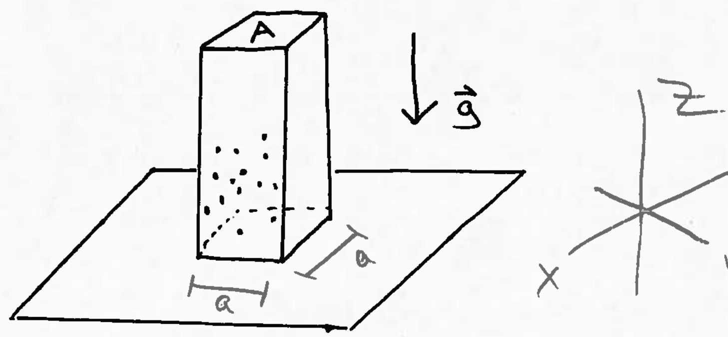
sum = 256

10

3. *Quantum Mechanics*

A collection of $N \gg 1$ electrons are confined to the interior of a semi-infinite square tube of cross sectional area A, as shown. The tube is in a gravitational field, characterized by the gravitational acceleration g, as shown in the figure. Ignoring interactions between the electrons, compute the approximate volume of the tube occupied by the electrons.

Due to quantum mechanical uncertainty, the "occupied volume" is not uniquely defined, so as part of your answer you should state how you are defining it. Use any simplifications appropriate to the $N \gg 1$ limit.



Since the particles are confined within this tube, their wave functions must vanish at the boundaries,

$$\Psi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

Assume a seperable product solution

$$X(0) = X(a) = 0$$
$$Y(0) = Y(a) = 0$$
$$Z(0) = Z(\infty) = 0$$

$$V(x,y,z) = \begin{cases} \infty & \text{when } |x| > a \text{ and/or } |y| > a \text{ and/or } z < 0 \\ mgz & \text{when } |x| < a \text{ and } |y| < a \text{ and } z > 0 \end{cases}$$

gravitational potential energy

Question # 3

Page # 2

We can solve for the components of the total wave function

$$-\frac{\hbar^2}{2m}\frac{\partial^2 X(x)}{\partial x^2} + \underbrace{V(x)}_{=0}X(x) = E\,X(x)$$

The potential is independent of x

The solutions will just be,

$$X(x) = A\cdot\sin(kx) + B\cdot\cos(kx)$$ where $k = \frac{n\pi}{a}$, $n = 1,2,3\ldots$

In order to satisfy the B.C.,

$$B = 0,$$

So,

$$X(x) = \sum_{n=1}^{\infty} A_n \cdot \sin\left(\frac{n\pi}{a}x\right)$$

Sum of solutions is also a solution

where A_n is some constant

Similarly, for $Y(y) \Rightarrow Y(y) = \sum_{m=1}^{\infty} B_m \cdot \sin\left(\frac{m\pi}{a}y\right)$

for $Z(z)$, we have,

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}Z(z) + \underbrace{(mgz)}_{V(z)}Z(z) = E\,Z(z)$$

So, $$\frac{\partial^2}{\partial z^2}Z(z) - \frac{2m}{\hbar^2}(mgz - E)\,Z(z) = 0$$

The solutions to this are the Airy functions $(Ai(z))$

So, $$\psi(x,y,z) = Ai(z)\cdot\sum_{n,m=1} A_n B_n \cdot \sin\left(\frac{n\pi x}{a}\right)\cdot\sin\left(\frac{m\pi y}{a}\right)$$

next →

Question # 3

Page # 3

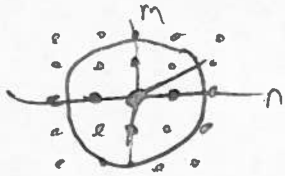
If we are ignoring interparticle interactions, then this wave function would be valid for all the electrons in the box

But electrons are fermions, there can only be 2 electrons per state $(= 2(\frac{1}{2})+1)$ due to spin w/down degeneracy

So even if we are ignoring their mutual Coulomb repulsion, we cannot ignore the fact that they cannot all be in whatever state they want, independently of the states of the other electrons

It would be nice to know what the density of states is.

In the 2d case (ignoring the z-potential right now) we have allowed energies of



circle of radius k

$$E_k = \frac{\hbar^2 k^2}{4ma^2} \qquad k^2 = n^2 + m^2$$

# of states is $\frac{1}{4}$ the area of this circle in k-space

only $\frac{1}{4}$ of the area since we only care about the region where $n, m > 0$

Question # 3

Page # 4

So, for large N, N>>1, we can count the number of states to be approximately

$$\frac{1}{4}\pi k^2 = N = \text{\# of states}$$

Remember that $E = \frac{\hbar^2 k^2}{4ma^2} \Rightarrow k^2 = \frac{4ma^2 E}{\hbar^2}$

So,

$$N = \frac{1}{4}\pi \frac{4ma^2 E}{\hbar^2} = \frac{\pi m a^2}{\hbar^2} E$$

So, $\frac{dN}{dE} = \mathcal{D}(E) =$ density of states

$$= \frac{\pi m a^2}{\hbar^2} = \frac{\pi m A}{\hbar^2}$$

If we now take into account the spin degeneracy we need to divide this out by 2

(Since 2 electrons can go in each state)

So, $\mathcal{D}(E) = \frac{\pi m A}{2\hbar^2}$

$$dN = \frac{2\pi A\, p\, dp}{h^2}$$

$E = \frac{p^2}{2m} \Rightarrow p^2 = 2mE$

$p = \sqrt{2mE}$ $\quad dp = \sqrt{2m}\,\frac{1}{2}E^{-1/2}\,dE$

$$= \frac{2\pi A}{h^2}\sqrt{2m}\,E^{1/2}\cdot\sqrt{2m}\,\frac{1}{2}E^{-1/2}\,dE = \frac{2\pi A}{h^2}\frac{(2m)}{2}\cdot 2 \leftarrow \text{spin degeneracy}$$

dE

$$= \frac{4m\pi A}{h^2} = \frac{4m\pi A}{h^2}dE$$

doing it another way

$$\mathcal{D}(E) = \frac{4\pi m A}{h^2}$$

Question # 3

Page # 5

We could define our volume taken up as The Sum of The uncertainties in the position of each particle

$$V_{\text{per particle}} = \sigma_x \sigma_y \sigma_y$$

$$V_{\text{total}} = N \cdot V_{\text{per particle}}$$

(Could add in a factor of $\frac{1}{2}$ to take into account 2 particles per state

We could assume that they all pack together tightly near the bottom of the box,

20/20

4. *Quantum Mechanics*

A particle moves on a 1-dimensional periodic lattice with $2N$ sites labeled by an integer $n = 1, \ldots, 2N$. We write $|n\rangle$ for the state in which the particle occupies the nth site. These states form an orthonormal set, and satisfy $\langle n'|n\rangle = \delta_{n,n'}$ for $1 \le n, n' \le 2N$. The Hamiltonian is defined by

$$\begin{aligned} \langle n|H|n\rangle &= A_0(-1)^n \\ \langle n+1|H|n\rangle = \langle n|H|n+1\rangle &= A_1 \end{aligned}$$

all other matrix elements of H being zero. Here, A_0 and A_1 are real constants, and lattice periodicity is realized via the relation $|2N+1\rangle = |1\rangle$. We define operators K and T by

$$\begin{aligned} K|n\rangle &= (-1)^n|n\rangle \\ T|n\rangle &= |n+1\rangle \end{aligned}$$

(a) Prove that the operators K and T satisfy $KT + TK = 0$ and $[K, T^2] = 0$;

(b) Express H in terms of K and T, and show that H commutes with T^2;

(c) Compute, in terms of the states $|n\rangle$, the simultaneous eigenstates $|\pm, \ell\rangle$ of K and T^2, such that $K|\pm, \ell\rangle = \pm|\pm, \ell\rangle$, and the integer ℓ labeling the eigenvalues of T^2;

(d) Show that the states $|\pm, \ell\rangle$, found in (c), may be normalized so that

$$T|\pm, \ell\rangle = e^{i\pi\ell/N}|\mp, \ell\rangle \qquad \ell = 0, 1, \cdots, N-1$$

(e) Compute the eigenvalues of H and their multiplicities.

a) $KT|n\rangle = K|n+1\rangle = (-1)^{n+1}|n+1\rangle$

$TK|n\rangle = T(-1)^n|n\rangle = (-1)^n|n+1\rangle$

4

So $(KT+TK)|n\rangle = (-1)^{n+1}|n+1\rangle + (-1)^n|n+1\rangle = 0.$ ✓

$[K,T^2]|n\rangle = (KT^2 - T^2K)|n\rangle = K|n+2\rangle - T^2(-1)^n|n\rangle = (-1)^{n+2}|n+2\rangle - (-1)^n|n+2\rangle = 0.$ ✓

b) $\boxed{H = A_0 K + A_1 T + A_1 T^+.}$

$[H,T^2] = A_0\overset{0}{[K,T^2]} + A_1\overset{0}{[T,T^2]} + A_1[T^+,T^2].$

4

$[T^+,T^2]|n\rangle = (T^+T^2 - T^2T^+)|n\rangle = T^+|n+2\rangle - T^2|n-1\rangle = |n+1\rangle - |n+1\rangle = 0.$

So $\boxed{[H,T^2] = 0.}$

Question # 4

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c) $|\pm, l\rangle = \sum_{n=1}^{2N} e^{i\phi n} c_n |n\rangle$

$K|+, l\rangle = (+)|+, l\rangle = \sum_{n=1}^{2N} e^{i\phi n}(-1)^n c_n |n\rangle = +\sum_{n=1}^{2N} e^{i\phi n} c_n |n\rangle$

So $C_n = 0$ when is odd. ✓

$\Rightarrow |+, l\rangle = \sum_{n\ \text{even}} e^{i\phi n}|n\rangle$ ✓

and similarly

4 $|-, l\rangle = \sum_{n\ \text{odd}} e^{i\phi n}|n\rangle$ ✓

$T^2|+, l\rangle = \sum_{n\ \text{even}} e^{i\phi n}|n+2\rangle = e^{-2i\phi}\sum_{n=\text{even}} e^{i\phi(n+2)}|n+2\rangle = e^{-2i\phi}|+, l\rangle$

$T^2|-, l\rangle = \sum_{n\ \text{odd}} e^{i\phi n}|n+2\rangle = e^{-2i\phi}\sum_{n\ \text{odd}} e^{i\phi(n+2)}|n+2\rangle = e^{-2i\phi}|-, l\rangle$

determine ϕ.

d) $T|+, l\rangle = \sum_{n\ \text{even}} e^{i\phi n}|n+1\rangle = e^{-i\phi}\sum_{n\ \text{even}} e^{i\phi(n+1)}|n+1\rangle = e^{-i\phi}|-, l\rangle$

4 $T|-, l\rangle = \sum_{n\ \text{odd}} e^{i\phi n}|n+1\rangle = e^{-i\phi}\sum_{n\ \text{odd}} e^{i\phi(n+1)}|n+1\rangle = e^{-i\phi}|+, l\rangle$

Let $\phi = -\frac{\pi l}{N}$ ✓

$l = 0, 1, \ldots N-1$

$\rightarrow N$ distinct eigenvalues of T.

Then

$T^2|\pm, l\rangle = \sum_{n\ \text{even}} e^{\frac{-i\pi l}{N}}|n+2\rangle = e^{i\frac{2\pi l}{N}}|\pm, l\rangle.$

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e) $H|\pm,\ell\rangle = A_0(\pm)|\pm,\ell\rangle + A_1 e^{\frac{i\pi\ell}{N}}|\mp,\ell\rangle + A_1 e^{-\frac{i\pi\ell}{N}}|\mp,\ell\rangle$

$= A_0(\pm)|\pm,\ell\rangle + 2A_1|\mp,\ell\rangle \cos\frac{\pi\ell}{N}$

So $\langle +,\ell|H|+,\ell\rangle = A_0$ $\quad\langle -,\ell|H|+,\ell\rangle = 2A_1\cos\frac{\pi\ell}{N}$

4 $\quad\langle -,\ell|H|-,\ell\rangle = -A_0$ $\quad\langle +,\ell|H|-,\ell\rangle = 2A_1\cos\frac{\pi\ell}{N}$

$\det\begin{pmatrix} A_0 - E & 2A_1\cos\frac{\pi\ell}{N} \\ 2A_1\cos\frac{\pi\ell}{N} & -A_0 - E \end{pmatrix} = 0 \rightarrow -(A_0-E)(A_0+E) = 4A_1^2\cos^2\frac{\pi\ell}{N}$

$\Rightarrow A_0^2 - E^2 = -4A_1^2\cos^2\frac{\pi\ell}{N} \Rightarrow \boxed{E = \pm\left(A_0^2 + 4A_1^2\cos^2\frac{\pi\ell}{N}\right)^{\frac{1}{2}}}$

So

$E_\pm = \pm\sqrt{A_0^2 + 4A_1^2\cos^2\frac{\pi\ell}{N}}$ each with multiplicity N.

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5. *Quantum Mechanics*

Although one encounters local potentials most frequently, some non-local potentials are useful and lead to simple and interesting dynamics. This problem is about such a non-local potential for which an *exact solution* may be obtained for all values of the coupling λ. Consider the 3-dimensional Hamiltonian H with *non-local potential* V, given by

$$H = \frac{\mathbf{p}^2}{2M} + V \qquad \langle \mathbf{r}'|V|\mathbf{r}\rangle = -\frac{\lambda\hbar^2}{2M}u(r')u(r)$$

where $r = |\mathbf{r}|$, $r' = |\mathbf{r}'|$, and $u(r)$ is a real-valued function. Throughout this problem, derive your results for all (real) values of the coupling constant parameter λ.

(a) Write down the integro-differential equation obeyed by a wave function $\psi_E(\mathbf{r})$ at energy E, in terms of the function $u(r)$.

(b) Show that only the s-wave is affected by this interaction.

(c) Establish the Lippmann-Schwinger equation, in integral form, for the scattering of an incoming plane wave of wave vector $\mathbf{k}$.

(d) Show that the scattering amplitude $f(\mathbf{k}', \mathbf{k})$ is given by

$$f(\mathbf{k}', \mathbf{k}) = 4\pi\lambda|v(k)|^2\left[1 + \frac{2\lambda}{\pi}\int d^3q \frac{|v(q)|^2}{k^2 - q^2 + i\epsilon}\right]^{-1}$$

where $v(k)$ is the Fourier transform of $u(r)$, given by, $v(k) = \frac{1}{k}\int_0^\infty dr\, r\sin(kr)u(r)$, and where $k = |\mathbf{k}|$ and $q = |\mathbf{q}|$.

<u>Formulas and notations</u>:

The Lippmann-Schwinger equation for a *local potential* is

$$\psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int d^3\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}', \mathbf{k}^2)U(\mathbf{r}')\psi_{\mathbf{k}}(\mathbf{r}') \qquad V(\mathbf{r}) = \frac{\hbar^2}{2M}U(\mathbf{r})$$

where $\phi_{\mathbf{k}}$ is the normalized incoming free wave, given by

$$\phi_{\mathbf{k}}(\mathbf{r}) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^{3/2}}$$

The Green function is defined by $(\Delta + \mathbf{k}^2)G(\mathbf{r}, \mathbf{r}'; \mathbf{k}^2) = \delta^{(3)}(\mathbf{r}, \mathbf{r}')$, and is given by

$$G(\mathbf{r}, \mathbf{r}', \mathbf{k}^2) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}}{\mathbf{k}^2 - \mathbf{q}^2 + i\varepsilon} = -\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

The scattering amplitude f is defined by ($r = |\mathbf{r}|$, $k = |\mathbf{k}|$),

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}', \mathbf{k})\frac{e^{ikr}}{r}\right]$$

Question # 5

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$$H|\Psi\rangle = E|\Psi\rangle \qquad \langle\vec{r}|H|\vec{r}'\rangle\langle\vec{r}'|\Psi\rangle = E\langle\vec{r}|\Psi\rangle$$

$$-\frac{\hbar^2}{2m}\Delta_{\vec{r}}\Psi_E(\vec{r}) - \frac{\lambda\hbar^2}{2m}u(r)\int d^3\vec{r}'\, u(r')\Psi_E(\vec{r}') = E\Psi_E(\vec{r}) \checkmark$$

5

$$\Psi_E(\vec{r}) = \sum_l \sum_m R_{l,m,E}(r)\, Y_l^m(\theta,\varphi)$$

5 $$-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R_{l,m,E}\right) - \frac{l(l+1)}{r^2}R_{l,m,E}\right) = E\,R_{l,m,E} \quad \text{for } l \geq 1$$

$\Rightarrow$ only s-wave is affected

$$\Psi_{\vec{k}}(\vec{r}) = \phi_k(\vec{r}) + \chi_{\vec{k}}(\vec{r}) \qquad -\frac{\hbar^2}{2m}\Delta\phi_k = E\phi_k$$

5 $$\chi_{\vec{k}}(\vec{r}) = \frac{2m}{\hbar^2}\int d^3\vec{y}\; G(\vec{r},\vec{y};\vec{k}^2)\int d^3\vec{z}\left(-\frac{\lambda\hbar^2}{2m}\right)u(|\vec{y}|)\,u(|\vec{z}|)\Psi_k(\vec{z}) \checkmark$$

$$\Psi_{\vec{k}}(\vec{r}) = \phi_{\vec{k}}(\vec{r}) + \frac{2m}{\hbar^2}\left(-\frac{\lambda\hbar^2}{2m}\right)\int d^3\vec{y}\; G(\vec{r},\vec{y};\vec{k}^2)\,u(|\vec{y}|)\int d^3\vec{z}\; u(|\vec{z}|)\Psi_{\vec{k}}(\vec{z})$$

$$\int d^3\vec{z}\; u(|\vec{z}|)\Psi_{\vec{k}}(\vec{z}) = \varphi(\vec{k})$$

$$\varphi(\vec{k}) = \int d^3\vec{z}\; u(|\vec{z}|)\phi_k(\vec{z}) - \lambda\int d^3\vec{r}\int d^3\vec{y}\; u(|\vec{r}|)\,u(|\vec{y}|)\,G(r,y;k^2)\varphi(k)$$

$$G(r,y,k^2) = \int\frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{i\vec{q}(\vec{r}-\vec{y})}}{\vec{q}^{\,2}-\vec{k}^2+i\varepsilon}$$

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$$\int d^3\vec{x}\; u(|\vec{x}|)\, \phi_{\vec{k}}(\vec{x}) = \int d^3x \frac{1}{(2\pi)^{3/2}} u(|\vec{x}|)\, e^{i\vec{k}\cdot\vec{x}} =$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} r^2 dr \frac{1}{(2\pi)^{3/2}} u(r)\, e^{ikr\cos\theta}$$

$$= \frac{2\pi}{(2\pi)^{3/2}} \int_0^{\infty} r^2 dr\, u(r) \int_0^{\pi} -d\cos\theta\; e^{ikr\cos\theta} = \sqrt{\frac{2}{\pi}}\, v(k)$$

$$\varphi(\vec{k}) = \sqrt{\frac{2}{\pi}}\, v(k) - \frac{2\lambda}{\pi} \varphi(\vec{k}) \int d^3q \frac{1}{q^2-k^2+i\varepsilon} |v(q)|^2$$

5

$$\Rightarrow \varphi(\vec{k}) \left(1 + \frac{2\lambda}{\pi} \int d^3q \frac{|v(q)|^2}{q^2-k^2+i\varepsilon}\right) = \sqrt{\frac{2}{\pi}}\, v(k)$$

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) = \phi_{\vec{k}}(\vec{r}) - \lambda \int d^3\vec{y} \left(-\frac{1}{4\pi r} e^{ikr - i\vec{k}'\cdot\vec{y}}\right) u(|\vec{y}|) \frac{\sqrt{\frac{2}{\pi}}\, v(k)}{\left(1 + \frac{2\lambda}{\pi}\int d^3q \frac{|v(q)|^2}{q^2-k^2+i\varepsilon}\right)}$$

$$\Rightarrow \psi_{\vec{k}}(\vec{r}) = \phi_{\vec{k}}(\vec{r}) + \frac{\lambda}{4\pi r} e^{ikr} \frac{2}{\pi} (2\pi)^{3/2} \frac{|v(k)|^2}{\left(1 + \frac{2\lambda}{\pi}\int d^3q \frac{|v(q)|^2}{q^2-k^2+i\varepsilon}\right)}$$

using the def. of $f(k', k)$ we get;

$$f(\vec{k}', \vec{k}) = \frac{4\pi\lambda}{\hbar^2} \frac{|v(k)|^2}{\left(1 + \frac{2\lambda}{\pi}\int d^3q \frac{|v(q)|^2}{q^2-k^2+i\varepsilon}\right)}$$

6. *Statistical Mechanics*

When electrons are confined to move in a two-dimensional plane (e.g. in a semiconductor quantum well or heterostructure), they will form a triangular lattice if their density is too low. Such a state is called a *Wigner crystal.* The transverse vibrations of such a crystal are similar to those of the ionic lattice, but the longitudinal vibrations satisfy $\omega_{\vec{k}} = a\sqrt{k}$ as a result of the unscreened long-ranged Coulomb interaction between the electrons. What is the contribution to the low-temperature specific heat of a Wigner crystal coming from longitudinal vibrations?

20

$$\omega_{\vec{k}} = a\sqrt{k}$$

~~the~~ energy of a wave $= \hbar\omega_{\vec{k}}$

Since for matter waves, the number of phonons is not conserved, the chemical potential is zero (they are bosons), then the distribution function,

$$n_k = \frac{1}{e^{\hbar\omega/T} - 1}$$

total energy of longitudinal waves $E = \int \frac{\hbar\omega}{e^{\hbar\omega/T} - 1} g(\omega)\, d\omega$

~~where~~ $g(\omega) d\omega$ gives the ~~de~~ density of states.

at low temperature we could approximate $g(\omega) d\omega$ with ~~$\frac{d^3 x}{h^3}$~~ $\frac{d^2 p\, d^2 x}{h^2}$.

$$E_{\text{longitudinal}} = \int \frac{\hbar\omega_{\vec{k}}}{e^{\hbar\omega_{\vec{k}}/T} - 1} \cdot \frac{d^2 p\, d^2 x}{h^2} = \frac{A}{h^2} \int \frac{\hbar\omega_{\vec{k}}}{e^{\hbar\omega_{\vec{k}}/T} - 1} d^2 p$$

where A is the area of the crystal plane.

Question # 6

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$\underline{P = \hbar\, \underline{k}}$

$$E_{longit} = \frac{A}{h^2} \int \cancel{\frac{\hbar a\sqrt{k}}{e^{\hbar a\sqrt{k}/T}}} \frac{\hbar\omega_k}{e^{\hbar\omega_k/T}-1} 2\pi p\, dp$$

$p = \hbar k$

but $\omega = a\sqrt{k}$

$\omega^2 = a^2 k$

$\therefore \quad P = \frac{\hbar \omega^2}{a^2}$

$dp = \frac{\hbar}{a^2} 2\omega d\omega$

$$E_{longit} = \frac{A}{h^2} \int \frac{\hbar\omega}{e^{\hbar\omega/T}-1} 2\pi \frac{\hbar}{a^2}\omega^2 \cdot \frac{\hbar}{a^2} 2\omega d\omega$$

$$= \frac{A}{h^2} \int_0^\infty \frac{\hbar \times 4\pi \left(\frac{\hbar}{a^2}\right)^2 \omega^4 d\omega}{e^{\hbar\omega/T}-1} = \frac{A}{h^2} \frac{4\pi\hbar^3}{a^4} \int_0^\infty \frac{\omega^4 d\omega}{e^{\hbar\omega/T}-1}$$

$$= \frac{A}{h^2} \frac{4\pi\hbar^3}{a^4} \int_0^\infty \frac{x^4 T^4/\hbar^4 \times T dx/\hbar}{(e^x - 1)}$$

let $x = \frac{\hbar\omega}{T}$ $\quad \omega = \frac{x}{\hbar}$

$dx = \frac{\hbar}{T} d\omega$

$$= \frac{A}{h^2} \frac{4\pi\hbar^3}{a^4} \frac{T^5}{\hbar^5} \int_0^\infty \frac{x^4 dx}{(e^x-1)}$$

$$E_{long} = \frac{A}{h^2} \frac{4\pi}{a^4\hbar^2} T^5 \times C$$

where $C = \int_0^\infty \frac{x^4}{(e^x-1)} dx$ = a numerical value

then Specific heat of longitudinal vibrations $= \frac{\partial E_{long}}{\partial T} = C = \frac{A}{h^2} \frac{4\pi}{a^4\hbar^2} C\, 5T^4$

$T = \frac{PV}{N}$

$\cancel{ST} = PV^{\gamma}$

7. *Statistical Mechanics*

Calculate the speed of sound c in an ideal gas for two cases: **(a)** isothermal compression and **(b)** adiabatic compression. Recall the general hydrodynamic relation

$$c = \sqrt{\frac{dP}{d\rho}},$$

$PV = NT$

where P is the pressure and ρ the mass density of the substance. Express your answer in terms of Boltzmann's constant k_B, temperature T, mass of the constituent particles m, and, for part (b), the adiabatic exponent γ defined through the relation $PV^{\gamma} = \text{const}$.

(c) Would the sound propagate faster in a gas of H (hydrogen) atoms or rigid H_2 molecules, at the same temperature? By how much?

$dU = TdS - PdV$ $\quad \cancel{dV}$ $\quad PV = NT$ $\quad PV^{\gamma}$ $\quad \frac{3}{2} C_V$

$dU = 0$ $\quad TdS = PdV$

$c^2 \quad \frac{dP}{d\rho} = \frac{d}{d\rho}(P)$

so $\quad TdS = pdV$ $\quad \rho = \frac{m}{V}$ $\quad \frac{d}{d\rho} = \frac{dV}{m}$

$dU = -PdV$ $\quad Q = 0$ $\quad P = \frac{NT}{V}$

$TdS = PdV$

$\frac{dP}{d\rho}$

$Q = pdV$ $\quad \frac{d}{d\rho}\left(\frac{NT}{V}\right)$

$\frac{d}{d\rho}\left(\frac{NT}{V}\right) = d$ $\quad \frac{1}{m}\frac{d}{d(\frac{1}{V})}\left(NT\left(\frac{1}{V}\right)\right)$ $\quad \gamma/\gamma$

$\frac{d}{d\rho}$ $\quad \frac{1}{m}\frac{d}{d\rho}(\rho T)$ $\quad \frac{NT}{m}$ $\quad \gamma = C_p/C_V \quad 5/2 \,/\, 3/2$

$= \frac{1}{m} T = \sqrt{\frac{T}{m}}$

Question # 7

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$k_B = 1$

(a) isothermal compression $\Rightarrow$ $\Delta T = 0$

$$\therefore \frac{dP}{d\rho} = \frac{d}{d\rho}\left(\frac{NT}{V}\right) = \frac{1}{m}\frac{d}{d\rho}\left(\frac{mNT}{V}\right) = \frac{1}{m}\frac{d}{d\rho}\left(\rho T\right) = \frac{T}{m}$$

$$\Rightarrow \boxed{C = \sqrt{\frac{T}{m}}} \qquad \text{or} \qquad C = \sqrt{\frac{Tk}{m}}$$

3/3

(b) adiabatic $\Rightarrow$ $PV^{\gamma} = \text{const}$

$$\therefore \frac{dP}{d\rho} = \frac{1}{m}\frac{d}{d\rho}(\rho T) = \frac{1}{m}\frac{d}{d\rho}\left(\rho\frac{PV}{N}\right) = \frac{1}{m}\frac{\partial}{\partial\rho}\left(\rho\frac{PV^{\gamma}}{N}\frac{1}{V^{\gamma-1}}\right)$$

$$= \frac{1}{mN}PV^{\gamma}\frac{\partial}{\partial\rho}\left(\rho\frac{1}{V^{\gamma-1}}\right) = \frac{1}{mN}PV^{\gamma}\frac{\partial}{\partial\rho}\left(\rho\left(\frac{mN}{V}\right)^{\gamma-1}\frac{1}{(mN)^{\gamma-1}}\right)$$

$$= \frac{V^{\gamma}}{(mN)^{\gamma}}P\frac{\partial}{\partial\rho}\left(\rho\rho^{\gamma-1}\right)$$

$$= \frac{P}{\rho^{\gamma}}\frac{\partial}{\partial\rho}\left(\rho^{\gamma}\right) = \gamma\frac{P}{\rho} \qquad \therefore \quad C = \sqrt{\gamma\frac{P}{\rho}}$$

3/3

$$P = \frac{NT}{V} \qquad \rho = \frac{mN}{V} \qquad \boxed{C = \sqrt{\gamma\frac{T}{m}}} \qquad \text{or} \quad c = \sqrt{\frac{Tk\gamma}{m}}$$

(c) I assume ~~that~~ ~~the~~ b/c H_2 is "rigid", the type of compression it would undergo would be a slightly different than the compression H atoms would undergo. In both cases sound wave would not introduce heat into the volume and thus $Q=0$ for both cases. $\Rightarrow$ we must use adiabatic compression for both

$$\frac{M_H}{M_{H_2}} = 2$$

$$\boxed{\gamma_{H_2} = \frac{C_P}{C_V} = \frac{7}{5} \quad \gamma_H = \frac{5}{3}}$$

$$\frac{C_{H_2}}{C_H} = \sqrt{\frac{\gamma_{H_2}}{m_{H_2}}\frac{m_H}{\gamma_H}} = \sqrt{\frac{21}{2.25}} \simeq \frac{2}{3}$$

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The ratio $\frac{C_{H_2}}{C_H} = \frac{1}{\sqrt{2}}$ ✓ if the compression was isothermal but the propagation of sound => vibrations => $dU \neq 0$.

H_2 diatomic so $C_V = \frac{5}{2}N$ $C_p = C_V + N$ for ideal gas

H monatomic so $C_V = \frac{3}{2}N$

4/4

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8. *Statistical Mechanics*

Consider a system composed of $N \gg 1$ identical but distinguishable noninteracting atoms, each of which has only two nondegenerate energy levels: 0 and $\omega > 0$. Let $\epsilon = E/N$ be the energy per atom.

(a) What is the maximum possible value of ϵ if the system is not necessarily in equilibrium? What is the maximum value for ϵ attainable for the system equilibrated at positive temperatures? Please answer the same question for negative temperatures.

(b) In the case of thermodynamic equilibrium, compute the entropy per atom, $s = S/N$, as a function of ϵ.

(c) Over what range of ϵ is the temperature positive, and over what range is it negative?

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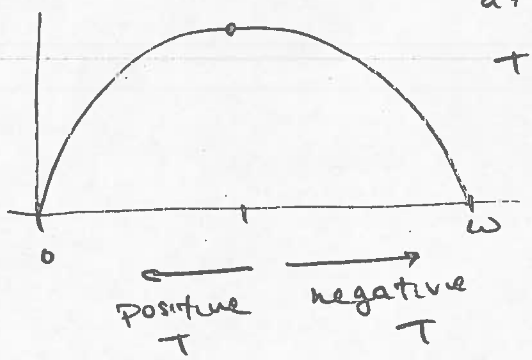
(a) The maximum possible average energy is ω. ✓

For positive T, $\frac{\partial S}{\partial u} \geq 0$ $\therefore$ max $\varepsilon = \frac{\omega}{2}$ ✓ ($T = \infty$ at this point. For negative T, max ε is once again ω. ✓



4/4

(b)

$$\Omega = \binom{N}{N_\omega} = \frac{N!}{N_\omega!(N-N_\omega)!} \Rightarrow S = \log(N!) - \log(N_\omega!) - \log(N-$$

$$-N\log(N-N_\omega) + N_\omega \log(N-N_\omega)$$

for $N \gg 1$ $S \simeq N\log N - N_\omega \log N_\omega$

$$= N\log\left[\frac{N}{N-N_\omega}\right] + N_\omega \log\left[\frac{N-N_\omega}{N}\right]$$

$$\frac{S}{N} = \log\left[\frac{N}{N-N_\omega}\right] + \frac{N_\omega}{N}\log\left[\frac{N-N_\omega}{N}\right]$$

Now $\varepsilon = \frac{\omega N_\omega}{N} \Rightarrow \frac{N_\omega}{N} = \frac{\varepsilon}{\omega}$ $\quad N_\omega = \frac{\varepsilon N}{\omega}$ $\quad N - N_\omega = N\left(1-\frac{\varepsilon}{\omega}\right)$

$$\Rightarrow \frac{S}{N} = -\log\left[1-\frac{\varepsilon}{\omega}\right] + \frac{\varepsilon}{\omega}\log\left[1-\frac{\varepsilon}{\omega}\right]$$

$$= \boxed{\left(\frac{\varepsilon}{\omega}-1\right)\log\left[1-\frac{\varepsilon}{\omega}\right]}$$

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$$S = N\log N - N_w \log N_w - N\log(N-N_w) + N_w\log(N-N_w)$$

$$= N\log\left(\frac{N}{N-N_w}\right) + N_w\log\left(\frac{N-N_w}{N_w}\right)$$

$$\varepsilon = \frac{E}{N} = \frac{N_w w}{N} \Rightarrow N_w = \frac{N\varepsilon}{w} \qquad N - N_w = N - \frac{N\varepsilon}{w} = N\left(1-\frac{\varepsilon}{w}\right)$$

$$S = N\log\left(\frac{N}{N(1-\frac{\varepsilon}{w})}\right) + \frac{N\varepsilon}{w}\log\left(\frac{\cancel{N}(1-\frac{\varepsilon}{w})}{\cancel{N}\varepsilon/w}\right)$$

$$\boxed{\frac{S}{N} = -\log\left(1-\frac{\varepsilon}{w}\right) + \frac{\varepsilon}{w}\log\left(\frac{w}{\varepsilon}-1\right)}$$ ✓ 3/3

$$\frac{\partial S}{\partial E} = \frac{\partial S/N}{\partial \varepsilon} = \frac{1}{1-\frac{\varepsilon}{w}}\left(\frac{1}{w}\right) + \frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right) + \frac{1}{\frac{w}{\varepsilon}-1}\left(-\frac{w}{\varepsilon^2}\right)\left(\frac{\varepsilon}{w}\right)$$

$$= \frac{1}{w-\varepsilon} + \frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right) - \frac{1}{\frac{w}{\varepsilon}\frac{\varepsilon^2}{w} - \frac{\varepsilon^2}{w}}\left(\frac{\varepsilon}{w}\right)$$

$$= \frac{1}{w-\varepsilon} + \frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right) - \frac{1}{\varepsilon - \frac{\varepsilon^2}{w}}\left(\frac{\varepsilon}{w}\right)$$

$$= \frac{1}{w-\varepsilon} - \frac{w/\varepsilon}{w-\varepsilon}\left(\frac{\varepsilon}{w}\right) \quad \frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right)$$

$$= \cancel{\frac{1-w/\varepsilon}{w-\varepsilon}} + \frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right)$$

3/3 ✓

$$T = \frac{1}{\frac{1}{w}\log\left(\frac{w}{\varepsilon}-1\right) + \cancel{\frac{1-w/\varepsilon}{w-\varepsilon}}} = \boxed{\frac{w}{\log\left(\frac{w}{\varepsilon}-1\right)}}$$

$$\boxed{\begin{array}{ll} T > 0 & 0 < \varepsilon < \frac{w}{2} \\ T < 0 & \frac{w}{2} < \varepsilon < w \end{array}}$$

10/10

Page 1.

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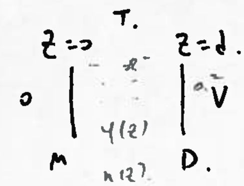
9. *Statistical Mechanics*

A parallel plate capacitor consists of two square plates, each of area a^2, separated by the distance d. The first plate, which is located at $z = 0$, is made of conducting metal and is grounded. The second plate, which is located at $z = d$, is made of dielectric and is maintained at a ~~positive~~ potential V with respect to the metal plate. The whole system is at some very high temperature T so that the electrons emitted from the hot metal of the first plate form a dilute gas which is in equilibrium and which fills this capacitor. There is no conductivity of electrons between the gas and the dielectric plate. Assume that the capacitor is so large ($a >> d$) that the edge effects can be disregarded.

(a) Write out the system of equations and boundary conditions that determines the potential $\varphi(z)$, and the density of electrons $n(z)$, inside the capacitor as functions of z ($d > z > 0$). Note: since in equilibrium there is no net flux of electrons across $z = 0$, you may assume that $n(z)$ has vanishing gradient there.

(b) Assuming a weak potential, $\frac{|eV|}{kT} << 1$, find $\varphi(z)$ and $n(z)$ inside the capacitor to first order in $\frac{eV}{kT}$.



$\varphi(z) = V(z)$.

(a). $\nabla^2\varphi(z) = -\frac{\rho(z)}{\varepsilon_0}$ $\rho(z) = -e\,n(z)$.

$\Rightarrow \nabla^2\varphi(z) = \frac{e\,n(z)}{\varepsilon_0}$ $\varphi|_{z=0} = 0$. $\varphi|_{z=d} = V$. $\frac{\partial n(z)}{\partial z}\Big|_{z=0} = 0$.

+3 +4 +1

$e > 0$.

(b). $n(z)$:

$$n(z)\,a^2 dz = \int e^{-\beta(\frac{p^2}{2m} - eV(z))}\frac{d^2x\,dz\,d^3p}{h^3} = \frac{N}{Z}\int_s e^{-\beta(\frac{p^2}{2m} - eV(z))}d^2x\,d^3p$$

$$Z = \int_v e^{-\beta(\frac{p^2}{2m} - eV(z))}d^3\vec{x}\,d^3p \qquad \Rightarrow n(z)\,a^2 dz = N\cdot\frac{a^2\cdot e^{\frac{eV(z)}{kT}}dz}{\int_0^d e^{\frac{eV(z)}{kT}}dz}.$$

$$\Rightarrow n(z) = \frac{N\exp[\frac{eV(z)}{kT}]}{\int_0^d \exp[\frac{eV(z)}{kT}]dz}. \qquad \frac{|eV|}{kT} << 1.$$

$$= CN\exp[\frac{eV(z)}{kT}] \quad C \text{ is a const.} \quad \exp[\frac{eV(z)}{kT}] \approx 1 + \frac{eV(z)}{kT}.$$

+5

$\varphi(z) = V(z)$.

$$\Rightarrow \boxed{\frac{d^2V(z)}{dz^2} = \frac{e}{\varepsilon_0}(1 + \frac{eV(z)}{kT})\cdot CN = B(1 + \frac{eV(z)}{kT})} \qquad \boxed{B = \frac{e}{\varepsilon_0}CN} \text{ is a constant}$$

$$\Rightarrow V(z) = A e^{-\sqrt{\frac{Be}{kT}}z} + D e^{\sqrt{\frac{Be}{kT}}z} - \frac{kT}{e} \quad \text{+ 2}$$

$e > 0$.

$V(z=0) = 0 \Rightarrow A + D = \frac{kT}{e}$. $V(z=d) = V \Rightarrow A e^{-\sqrt{\frac{Be}{kT}}d} + D e^{\sqrt{\frac{Be}{kT}}d} - \frac{kT}{e} = V$.

$\frac{\partial n(z)}{\partial z}\Big|_{z=0} = 0 \Rightarrow \frac{dV(z)}{dz}\Big|_{z=0} = 0 \Rightarrow D - A = 0$.

+2

P2. $\Rightarrow A = D = \frac{kT}{2e}$ and $\boxed{B = \frac{kT}{ed^2}\cosh^{-1}(\frac{eV}{kT} + 1)^2 = \frac{eCN}{\varepsilon_0}}$

Question # 9

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$$\Rightarrow C = \frac{\varepsilon_0}{eN} \cdot \frac{kT}{ed^2} \cosh^{-1}\left[\frac{eV}{kT}+1\right].$$

$$\Rightarrow \frac{kT}{e} \cosh\left(\sqrt{\frac{eB}{kT}}z\right) - \frac{(kT)^2}{2e^2}$$

$$\Rightarrow V(z) = \frac{kT}{2e}\left[\frac{e^{-\sqrt{\frac{eB}{kT}}z} + e^{\sqrt{\frac{eB}{kT}}z}}{2} - \frac{kT}{e}\right].$$ B is determined from ②.

$$n(z) = \frac{\varepsilon_0}{e} \frac{kT}{ed^2} \cosh^{-1}\left[\frac{eV}{kT}+1\right] \exp\left[\frac{eV(z)}{kT}\right].$$

+2

10. *Electromagnetism*

Consider a point charge q located at $x = -a$ facing a dielectric of dielectric constant $\epsilon > 1$ which fills the half infinite space defined by $x \geq 0$.

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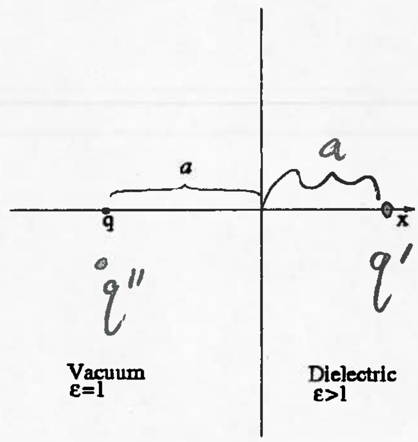

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a) Using the method of image charges, find the electric field $\vec{E}(\vec{r})$.

b) Imposing continuity of components of $\vec{D}$ and $\vec{E}$, express the image charge(s) in terms of ϵ and q.

c) Discuss the limiting cases $\epsilon \to 1$ and $\epsilon \to \infty$ of this problem. What is/are the images charge(s) in these two cases ?

$$\Phi_{vac} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{x}+a\hat{x}|} + \frac{q'}{|\vec{x}-a\hat{x}|}\right]$$ ✓

$$\Phi_{del.} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q''}{|\vec{x}+a\hat{x}|}\right]$$ ✓

should be ε, but I'll just absorb this in q''

2

- For continuity at $x=0$, $q+q' = q''$ ✓
- For continuity of $D^{\perp}$ at $x=0$,

$$\varepsilon_0 \left.\frac{\partial \Phi_v}{\partial x}\right|_{x=0} = \varepsilon \left.\frac{\partial \Phi_d}{\partial x}\right|_{x=0}$$ ✓

$D = \varepsilon E$

Question # 10

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- $\frac{\partial}{\partial x} \frac{1}{|\vec{x} \pm a\hat{x}|} = \partial_x \left(\frac{1}{\sqrt{(x \pm a)^2 + y^2 + z^2}} \right) = \frac{-\frac{1}{2} \partial (x \pm a)}{(\quad)^{3/2}}$

$\xrightarrow{x=0} \frac{\mp a}{\sqrt{a^2+y^2+z^2}}$ ✓

$\Rightarrow \quad \varepsilon_0(-aq) + \varepsilon_0(+aq') = \varepsilon(-aq'')$

- Thus, $\begin{cases} q + q' = q'' \\ q - q' = \varepsilon/\varepsilon_0 \, q'' \end{cases}$

o.k.

$\Rightarrow \begin{cases} 2q = (1 + \varepsilon/\varepsilon_0) q'' \\ 2q' = (1 - \varepsilon/\varepsilon_0) q'' \end{cases} \Rightarrow \begin{cases} \boxed{q'' = \frac{2}{1+\varepsilon/\varepsilon_0} q} \\ \boxed{q' = \frac{1-\varepsilon/\varepsilon_0}{1+\varepsilon/\varepsilon_0} q} \end{cases}$

- $\boxed{\text{As } \varepsilon \to \varepsilon_0, \quad q'' \to q \ \& \ q' \to 0}$ ✓ no dielectric ✓

$\boxed{\text{As } \varepsilon \to \infty, \quad q'' \to 0 \ \& \ q' \to -q}$ ✓ conductor ✓

no field in conductor

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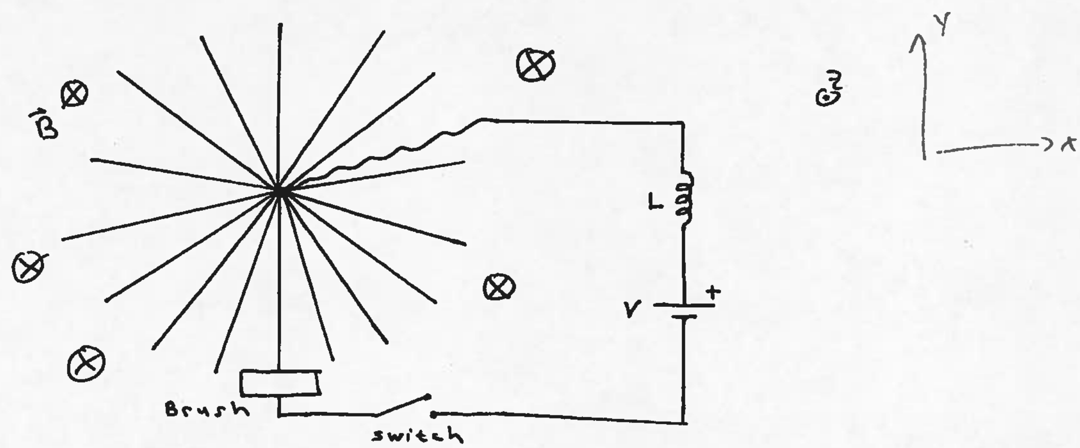
11. *Electromagnetism*

A wheel consisting of a large number of thin conducting spokes is free to pivot about an axle. A brush makes electrical contact with one spoke at a time at the bottom of the wheel. A battery of voltage V feeds current through an inductor into the axle, through a spoke, to the brush. A permanent magnet provides a uniform magnetic field B into the plane of the paper. At time $t = 0$ a switch is closed, allowing current to flow. The radius and moment of inertia of the wheel are R and J respectively. The inductance of the current path is L, and the wheel in initially at rest. Neglecting friction and resistivity, calculate the current and angular velocity of the wheel as functions of time.



Since there are many spokes, im going to assume the spoke in contact with the brush is always vertical, so $I \sim \hat{y}$

$$V_L = L \cdot \frac{dI}{dt}$$

$$F_{spoke} = \frac{\mu_0}{4\pi} \cdot \bar{I} \wedge \bar{B} = \frac{\mu_0}{4\pi} \cdot I \cdot B \cdot \hat{z}$$

$$\tau = \frac{\mu_0 RIB}{4\pi} \cdot \hat{z}$$

~~Power dissipated is transfered to the wheel so~~

$$\cancel{P = I V_{wheel}}$$

~~then~~

$$\cancel{V_L = V - \mathcal{E} = V - \frac{d\Phi}{dt}}$$

No resistivity => no losses

$$P = I \cdot V = \frac{d}{dt}\left(\frac{1}{2} L I^2 + \frac{1}{2} J \omega^2\right)$$

Question # 11

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$$\Rightarrow \cancel{\tfrac{1}{2}} \cdot L \cdot I \cdot \dot{I} + \cancel{\tfrac{1}{2}} J \cdot \omega \dot{\omega} = IV \qquad \dot{\omega} = \frac{\tau}{J}$$

$$\Rightarrow \cancel{\tfrac{1}{2}} \cdot L I \cdot \dot{I} + \cancel{\tfrac{1}{2}} \omega \cdot \tau = I \cdot V$$

$$\Rightarrow \cancel{\tfrac{1}{2}} \cdot L \cdot I \cdot \dot{I} + \cancel{\tfrac{1}{2}} \omega \frac{\mu_0 R B}{4\pi} I = I \cdot V$$

$$\Rightarrow \cancel{\tfrac{1}{2}} \dot{I} \cdot L = V - \cancel{\tfrac{1}{2}} \frac{\omega \mu_0 R \cdot B}{4\pi}$$

$\frac{d}{dt}$:

$$\Rightarrow \frac{L}{\cancel{2}} \cdot \ddot{I} = - \frac{\mu_0 R B}{\cancel{4}8\pi} \dot{\omega} = - \frac{\mu_0 R B}{\cancel{4}8\pi} \frac{\tau}{J} = \cancel{-\tfrac{1}{2}} \left(\frac{\mu_0 R B}{4\pi} \right)^2 \frac{1}{J} \cdot I$$

$$\Rightarrow \ddot{I} + \left(\frac{\mu_0 R B}{4\pi} \right)^2 \frac{1}{JL} \cdot I = 0 \qquad k = \frac{\mu_0 R B}{4\pi} \cdot \frac{1}{\sqrt{JL}}$$

$$\Rightarrow I = A \cdot \cos(kt) + B \cdot \sin(kt) = \cancel{I_0} \cdot \cancel{\cos(kt)} B \sin(kt) + \cancel{A} I_0 \cos(kt)$$

$$\cancel{\tau = \sqrt{JL} \cdot k \cdot I = J \cdot \dot{\omega}}$$

$$\cancel{\Rightarrow \omega = \sqrt{\frac{L}{J}} \cdot k \cdot \int I \cdot dt = \sqrt{\frac{L}{J}} I_0 \sin(kt) + \omega_0}$$

~~If wheel starts at rest $\omega_0 = 0$, so~~

$$\cancel{I = I_0 \cdot \cos(kt)}$$

$$\cancel{\omega = \sqrt{\frac{L}{J}} \cdot I_0 \cdot \sin(kt)}$$

~~at $t=0$, the power equation gives~~

$$\cancel{\tfrac{1}{2} L \dot{I} = V} \quad \Rightarrow$$

At $t = 0$, $\omega = 0$, so

$$L \cancel{\tfrac{1}{2}} \dot{I} = \cancel{I_0} \cdot V$$

$$\Rightarrow \cancel{\tfrac{1}{2}} \cdot k \cdot B = \cancel{I_0} \cdot \frac{V}{L} \Rightarrow B = \frac{V}{kL}$$

Question # 11 Name ______

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$$\Rightarrow I(t) = I_0 \left\{ \cos(kt) + \frac{V}{kL} \cdot \sin(kt) \right\}$$

$$\tau(t) = \sqrt{JL} \cdot k \cdot I(t) = J \cdot \dot{\omega}(t)$$

$$\Rightarrow \dot{\omega}(t) = \sqrt{\frac{L}{J}} \cdot k \cdot I(t)$$

$$\Rightarrow \omega(t) = \sqrt{\frac{L}{J}} \cdot \cancel{I_0} \cdot \left\{ \sin(kt) - \frac{V}{kL} \cos(kt) \right\} + C_1$$

$$\omega(0) = 0, \quad \Rightarrow \quad C_1 = \frac{V}{kL} \cdot \sqrt{\frac{L}{J}} \cdot \cancel{I_0}$$

$$\cancel{\Rightarrow \omega(t) = \sqrt{\frac{L}{J}} I_0 \left\{ \sin(kt - \frac{V}{k} \cdot (\cos(kt) - 1) \right\}}$$

$$\omega(t) = \sqrt{\frac{L}{J}} I_0 \sin(kt) + \frac{V}{k} \frac{1}{\sqrt{JL}} (1 - \cos(kt))$$

If $I_0 = 0$,

$$I(t) = \frac{V}{kL} \sin(kt)$$

$$\omega(t) = \frac{V}{k\sqrt{JL}} \cdot (1 - \cos(kt))$$

$$k = \frac{\mu_0 R B}{4\pi}$$

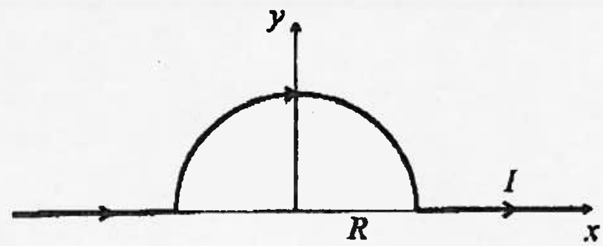
12. *Electromagnetism*

A semicircular wire of radius R is centered at the origin, while straight segments extend to infinity along the X-axis, as shown below. A uniform current I is suddenly turned on at $t = 0$, remaining constant thereafter.

(a) Calculate the vector ($\vec{A}$) and scalar potential (V) as a function of time at the origin.

(b) Calculate $\vec{E}$ and $\vec{B}$ as a function of time at the origin.

~~20~~ 19



(a) $\vec{A}(x,t) = \frac{\mu_0}{4\pi}\int \frac{J(x',t_r)}{|x-x'|}\,dv'$ where t_r = retarded time.

$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x',t_r)}{|x-x'|}\,dv' = 0$ coz there are no free charges.

~~$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int$~~ ~~also note~~ ~~that~~

consider the following situation:

the vector potential at origin,

$A = \frac{\mu_0}{4\pi}\int \frac{J(x,t_r)}{|x-x'|}\,dv'$

$= \frac{\mu_0}{4\pi} I \int \frac{dz'\,\hat{z}}{|x-x'|}$ ~~from~~

$\therefore$ with this the vector potential at the center close to the straight line current would always be zero.

Question # 12

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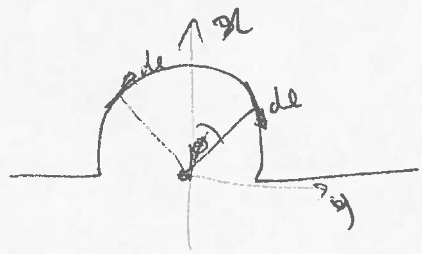
~~$\vec{A}(\vec{r},t) = \frac{\mu_0 I}{4\pi} \oint \frac{dl}{|r-r'|}$~~

~~for time~~ at time $t < R/c$

A is zero. coz the information of the current from the loop takes R/c time to reach the center.

for time $t > R/c$, A = contribution from loop + contribution from strong wire

Contribution from loop $= \frac{\mu_0 I}{4\pi} \int \frac{d\underline{l}}{|x-x'|}$ $\quad |x-x'| = R$



$= \frac{\mu_0 I}{4\pi R} \int d\underline{l} = \frac{\mu_0 I}{4\pi R} \int R\, d\phi\, \hat{\phi}$

$d\underline{l} = R\, d\phi\, \hat{\phi}$

$\hat{\phi} = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y}$

$\therefore A = \frac{\mu_0 I}{4\pi R} \int_{-\pi/2}^{\pi/2} R\, d\phi \cdot (-\sin\phi\, \hat{x} + \cos\phi\, \hat{y})$

$= \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} (-\sin\phi\, \hat{x} + \cos\phi\, \hat{y})\, d\phi$

$= \frac{\mu_0 I}{4\pi} \left[\cos\phi\, \hat{x} + \sin\phi\, \hat{y} \right]_{-\pi/2}^{\pi/2}$

$= \frac{\mu_0 I}{4\pi} \left[\sin\pi/2 - \sin(-\pi/2) \right] \hat{y}$

$= \frac{\mu_0 I}{4\pi} \cdot 2\hat{y} = \frac{\mu_0 I}{2\pi} \hat{y}$ //

Question # ~~1~~ 2

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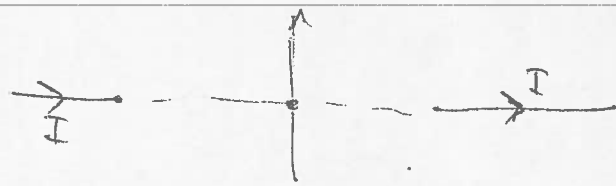
The contribution from straight part:



for the ~~contribution~~ contribution is finite after time $t > R/c$

for time $t > R/c$: $\frac{\mu_0 I}{4\pi}\left[\int_{-l}^{-R}\frac{dz'}{|r-r'|} + \int_{R}^{+l}\frac{dz'}{|r-r'|}\right]$

$|r-r'| = |-r'| = |z'|$

to find l:
$t = \frac{l}{c}$; $l = ct$

$$\therefore A = \frac{\mu_0 I}{4\pi}\left[\int_{-l}^{-R}\frac{dz'}{|z'|} + \int_{R}^{+l}\frac{dz'}{|z'|}\right]$$

For z' -ve $|z'| = -z'$.

$$A = \frac{\mu_0 I}{4\pi}\left[\int_{-l}^{-R}\frac{dz'}{-z'} + \int_{R}^{l}\frac{dz'}{z'}\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[-\left[\ln(z')\right]_{-l}^{-R} + \left[\ln(z')\right]_{R}^{l}\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[-\left(\ln(-R) - \ln(-l)\right) + \ln(l/R)\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[-\ln\left(R/l\right) + \ln\left(l/R\right)\right] = \frac{\mu_0 I}{2\pi}\ln\left(l/R\right)$$

then A_{total} for $t > R/c$ = $\left(\frac{\mu_0 I}{2\pi} + \frac{\mu_0 I}{2\pi}\ln\left(l/R\right)\right)\hat{y}$

$$= \left(\frac{\mu_0 I}{2\pi} + \frac{\mu_0 I}{2\pi}\ln\left(\frac{ct}{R}\right)\right)$$

$V = \frac{d}{t}$

$t = \frac{s}{v}$

Question # 12

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(b). $\underline{E} = 0$ for ~~R~~ $t < R/c$.

for $t > R/c$, $\underline{E} = -\nabla\phi + \frac{\partial A}{\partial t}$.

$= -\frac{\partial A}{\partial t}$

$= \frac{\mu_0 I}{2\pi} \frac{R}{ct} \cdot \frac{c}{R} \hat{y}$

$= \frac{\mu_0 I}{2\pi t} \hat{y}$ ✓

$\therefore$ ~~Current~~ E goes to 0 as $t \to \infty$

as expected.

~~for~~ $\underline{B} = \nabla \times \underline{A} = 0$ for $t < R/c$

the

$\vec{B}$? $t > \frac{R}{c}$

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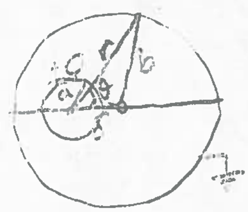
13. *Electromagnetism*

An isolated conducting sphere of radius a is placed inside a thin conducting spherical shell of radius b. The centers of the two spheres are not coincident, but are instead displaced from each other by a small distance δ, with $\delta \ll a, b$. The total charge of the inner sphere is q, and the outer sphere is grounded. Find the distribution of surface charge σ on the inner sphere and the force F acting on it, to first order in δ.



$$b^2 = \delta^2 + r^2 - 2\epsilon r\cos\theta \approx r^2 - 2\delta r\cos\theta$$

$$r \approx \frac{1}{2}\left(2\delta\cos\theta + \sqrt{4\epsilon^2\cos^2\theta + b^2}\right) \approx b + \delta\cos\theta$$

+3

$\nabla^2\Phi = 0$ we have axial symmetry

The potential at a point btw the two spheres is:

$$\Phi = \sum_0^\infty \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \qquad \ell = 0, 1 \text{ for lowest orders.}$$

$$\rightarrow \phi = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

+3

The surface of the inner sphere is an equipotential. Hence it doesn't depend on θ.

$$A_1 a + \frac{B_1}{a^2} = 0$$

The charge density is: $\sigma = -\epsilon_0\left(\frac{\partial\Phi}{\partial r}\right)_{r=a}$ $\qquad \sigma = \frac{Q}{4\pi a^2}$

$$\int_0^\pi \sigma\, 2\pi a^2 \sin\theta\, d\theta = Q \qquad \rightarrow \qquad B_0 = \frac{Q}{4\pi\epsilon_0}$$

The outer sphere is grounded $\rightarrow \Phi = 0$ for $r \approx b + \delta\cos\theta$

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$$A_0 + \frac{B_0}{b+\delta\cos\theta} + \left[A_1(b+\delta\cos\theta) + \frac{B_1}{(b+\delta\cos\theta)^2}\right]\cos\theta = 0$$

To first order in δ we get

$$(b+\delta\cos\theta)^{-1} = b^{-1}\left(1+\frac{\delta}{b}\cos\theta\right)^{-1} \approx \frac{1}{b}\left(1-\frac{\delta}{b}\cos\theta\right)$$

$$(b+\delta\cos\theta)^{-2} = b^{-2}\left(1+\frac{\delta}{b}\cos\theta\right)^{-2} \approx \frac{1}{b^2}\left(1-\frac{2\delta}{b}\cos\theta\right)$$

$$\Rightarrow A_0 + \frac{B_0}{b} + \left(-\frac{B_0\delta}{b^2} + A_1 b + \frac{B_1}{b^2}\right)\cos\theta \approx 0$$

neglecting $\delta\cos^2\theta$ & higher orders, we require

$$A_0 + \frac{B_0}{b} = 0$$

$$-\frac{B_0\delta}{b^2} + A_1 b + \frac{B_1}{b^2} = 0 \quad \rightarrow A_0 = -\frac{Q}{4\pi\epsilon_0 b} \quad \rightarrow A_1 = \frac{Q\delta}{4\pi\epsilon_0(b^3-a^3)}$$

$$\rightarrow B_1 = -\frac{Q\delta a^3}{4\pi\epsilon_0(b^3-a^3)}$$

$$\Rightarrow \Phi = \frac{Q}{4\pi\epsilon_0}\left\{\frac{1}{r} - \frac{1}{b} + \frac{\delta r}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta\right\}$$

+4

The force is $Q\vec{\nabla}\phi \Rightarrow \vec{\nabla}\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\Phi\right)\hat{r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\phi\right)\hat{\theta}$

$$\frac{\partial\phi}{\partial r} = \frac{Q}{4\pi\epsilon_0}\left\{-\frac{1}{r^2} + \frac{\delta}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta + \frac{\delta r}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1} 3\left(\frac{a}{r}\right)^2\left(\frac{-a}{r^2}\right)\cos\theta\right\}$$

$$r^2\frac{\partial\phi}{\partial r} = \frac{Q}{4\pi\epsilon_0}\left\{-1 + \frac{\delta r^2}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta + \frac{3\delta r^3}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1}\left(\frac{a}{r}\right)^2\left(\frac{-a}{r^2}\right)\cos\theta\right\}$$

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$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=\frac{\varphi}{4\pi\epsilon_0}\left\{\frac{2\delta r}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta+\frac{\delta r^2}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1}\left[(-3)\left(\frac{a}{r}\right)^2\left(\frac{-a}{r^2}\right)\right]\cos\theta\right.$$

$$+\frac{9\delta r^2}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1}\left(\frac{a}{r}\right)^2\left(\frac{-a}{r^2}\right)\cos\theta+$$

$$+\frac{3\delta r^3}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-2}(-3)\left(\frac{a}{r}\right)^2\left(-\frac{a}{r^2}\right)\left(\frac{a}{r}\right)^2\left(\frac{-a}{r^2}\right)\cos\theta$$

$$+\frac{3\delta r^3}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1}2\left(\frac{a}{r}\right)\left(\frac{-a}{r^2}\right)\left(-\frac{a}{r^2}\right)\cos\theta$$

$$\left.+\frac{3\delta r^3}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]^{-1}\left(\frac{a}{r}\right)^2\left(\frac{a}{r^3}\right)\cos\theta\right\}\equiv\alpha$$

$$\frac{\partial}{\partial\theta}\phi=\frac{-\varphi}{4\pi\epsilon_0}\frac{\delta r}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]\sin\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{\partial}{\partial\theta}\phi\right)=\frac{-\varphi}{4\pi\epsilon_0}\frac{\delta r}{b^3-a^3}\left[1-\left(\frac{a}{r}\right)^3\right]3\sin^2\theta\cos\theta\equiv\beta$$

$$\Rightarrow\vec{F}=\frac{\alpha}{r^2}\varphi\,\hat{r}+\frac{\beta}{r^2\sin\theta}\varphi\,\hat{\theta}$$

not yet

+4

14. *Electromagnetism*

A plane electromagnetic wave with wavelength λ is incident on a rectangular aperture of width $2w_x \times 2w_y$. A 2D detector is placed in the far field to measure the diffraction intensity (i.e. the Fraunhofer approximation holds).

(a) Calculate the diffraction intensity on the detector.

(b) Plot the diffraction intensity distribution along the horizontal axis and verify the Heisenberg uncertainty principle.

a.) The Fraunhofer approximation is essentially a Fourier transform of the aperture.

$$\text{aperture: } \begin{cases} 0, & |x| > w_x \text{ or } |y| > w_y \\ 1, & |x| < w_x \text{ and } |y| < w_y \end{cases} \qquad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

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$$\Rightarrow F(\vec{k},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{x}'} A(\vec{x}')\, d^3x'$$

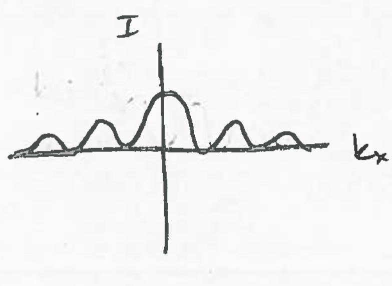
$$= \frac{1}{2\pi}\int_{-w_x}^{w_x} e^{-ik_x x'}\, dx' \int_{-w_y}^{w_y} e^{-ik_y y'}\, dy'$$

$$= \frac{1}{2\pi}\left(-\frac{1}{ik_x}\left(e^{-ik_x w_x} - e^{ik_x w_x}\right)\cdot -\frac{1}{ik_y}\left(e^{-ik_y w_y} - e^{ik_y w_y}\right)\right)$$

$$= \frac{1}{2\pi}\cdot\frac{1}{k_x}\cdot\frac{1}{k_y}\cdot 4\sin(k_x w_x)\sin(k_y w_y)$$

$$\Rightarrow I \propto |F|^2 \propto \frac{\sin^2(k_x w_x)}{k_x^2}\cdot\frac{\sin^2(k_y w_y)}{k_y^2}$$

b.)



$\Delta x \sim 2w_x$

$$p = \hbar k \Rightarrow \Delta p = \hbar\Delta k, \quad k = \frac{2\pi}{\lambda} \sim \frac{2\pi}{x} \Rightarrow \Delta x \sim \frac{2\pi}{\Delta k} \quad ?$$

$$\Rightarrow \Delta p\Delta x \sim \frac{2\pi}{\Delta k}\hbar\Delta k = 2\pi\hbar > \frac{\hbar}{2\pi}$$