2. *Quantum Mechanics* (Spring 2005)

Show that in one space dimension any attractive potential, no matter how weak, always has at least one bound state. *Hint:* Use the variational principle with some appropriate trial wave function such as the normalized Gaussian

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where $b$ is a parameter.

- $\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$
- Using variational principle $\bar{H} = \dfrac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = E(b)$
- attractive potential properties:
  - $\{0 > E(b) \geq E_0\}$
  - $\lim_{x\to\pm\infty} |V(x)| = 0$
  - peicewise continuous
  - Since $V(x) < 0 \Rightarrow V(x) = -|V(x)|$

$$H = \frac{\hat{p}^2}{2m} - V(x) = -\frac{\hbar^2}{2m}\vec{\nabla}_x^2 - |V(x)|$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx' \left[\left(\frac{2b}{\pi}\right)^{1/4} e^{-bx'^2}\right]^* \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx'^2}$$

$$= \left(\frac{2b}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} dx'\, e^{-2bx'^2} = \left(\frac{2b}{\pi}\right)^{1/2} \sqrt{\frac{\pi}{2b}} = 1$$

$$\bar{H} = E(b) = \int_{-\infty}^{\infty} dx'\, \psi_b^*(x')\left(-\frac{\hbar^2}{2m}\nabla_{x'}^2 - |V(x')|\right)\psi_b(x')$$

$$\langle T\rangle_b = -\int_{-\infty}^{\infty} dx'\, \psi_b^*(x') \frac{\hbar^2}{2m} \nabla_{x'}^2 \psi_b(x')$$

$$\vec{\nabla}_{x'}\left[\psi_b^*(\vec{\nabla}_{x'}\psi_b)\right] = \psi_b^* \vec{\nabla}_{x'}^2 \psi_b + (\vec{\nabla}_{x'}\psi_b)^2$$

$$= -\frac{\hbar^2}{2m}\left\{ \cancel{\psi_b^*(\vec{\nabla}_{x'}\psi_b)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\vec{\nabla}_{x'}\psi_b)^2 dx' \right\}$$

$\uparrow$ $\psi_b$ must vanish @ $\pm\infty$

$$= \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left(\frac{d}{dx'}\psi_b\right)^2 dx' = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left[(-2bx')\psi_b\right]^2 dx' = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} A^2 4b^2 x'^2 e^{-2bx'^2} dx'$$

$$\frac{d}{db} e^{-2bx'^2} = -2x'^2 e^{-2bx'^2}$$

$$= -\frac{\hbar^2 b^2 A^2}{m}\frac{d}{db}\int_{-\infty}^{\infty} dx'\, e^{-2bx'^2} = -\frac{\hbar^2 b^2}{m}A^2\frac{d}{db}\sqrt{\frac{\pi}{2b}} = -A^2\sqrt{\frac{\pi}{2}}\frac{\hbar^2 b^2}{m}\left(-\frac{1}{2}\right)b^{-3/2}$$

$$= \sqrt{\frac{\pi}{2}}\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2} = \frac{\hbar^2 b}{2m}$$

$$\langle V\rangle_b = -\langle|V(x)|\rangle$$

We want $\langle|V|\rangle_b = \int_{-\infty}^{\infty} dx'\, \psi_b^* |V(x')| \psi_b > \langle T\rangle$

such that $E(b) < 0$

## Method 1:

Defining $I(b) = \int_{-\infty}^{\infty} dx'\, |V(x')| e^{-2bx'^2}$

$$E(b) = \langle T\rangle_b + \langle V\rangle_b = \frac{\hbar^2 b}{2m} - \sqrt{\frac{2b}{\pi}}\, I(b)$$

as $b \to 0$, $\dfrac{E(b)}{\sqrt{b}} \to -I(0) = -\int_{-\infty}^{\infty} dx'\, |V(x')|$

Since $I(0) = \int_{-\infty}^{\infty} dx'\, |V(x')| > 0$ (strictly positive)

$\boxed{\text{then } \lim_{b\to 0} E(b) < 0 \text{ s.t. for sufficently small } b \text{ param there must be @ least one bound state}}$

## Method 2

$$\frac{dE(b)}{db} = 0$$

$$\frac{d\langle T\rangle_b}{db} = -\frac{d\langle V\rangle_b}{db}$$

$$\frac{\hbar^2}{2m} = -\left\{\sqrt{\frac{2b}{\pi}}\frac{dI(b)}{db} + \sqrt{\frac{2}{\pi}}\frac{1}{2}b^{-1/2} I(b)\right\}$$

$$\frac{\hbar^2}{2m} = -\left\{\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} dx'\,(-2x'^2)|V| e^{-2bx'^2} + \sqrt{\frac{2}{\pi}}\frac{1}{2}\frac{1}{\sqrt{b}}\int_{-\infty}^{\infty} dx'\, |V| e^{-2bx'^2}\right\}$$

$$\frac{\hbar^2}{2m} + \sqrt{\frac{1}{2\pi b}}\, I(b) = 2\sqrt{\frac{2b}{\pi}}\langle x^2|V|\rangle \text{ @ minimum } b$$

$\uparrow$ $I(b) > 0$

$\therefore \langle x^2|V|\rangle > 0$ as well for finite $b$

Minimized:

$$E(b) = \frac{\hbar^2 b}{2m} - \sqrt{\frac{2b}{\pi}}\, I(b) = \frac{\hbar^2 b}{2m} - \sqrt{\frac{2b}{\pi}}\left\{4b\langle x^2|V|\rangle - \sqrt{2\pi b}\,\frac{\hbar^2}{2m}\right\}$$

$\lim_{b\to 0} E(b) \sim -b^{3/2}\langle x^2|V|\rangle \Rightarrow$ for small $b$, $E(b) < 0$

@ least one bound state