Name:

Comprehensive exam, Fall 2011

1. *Quantum Mechanics*

Consider a particle in the ground state of an infinitely deep square potential well of width a in one dimension. What is the probability distribution function for the particle to have momentum in the range $(p, p + dp)$?

Name:

2. *Quantum Mechanics*

A collection of hydrogen atoms in the ground state is contained between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor so as to a produce a homogeneous electric field

$$\mathcal{E} = 0 \quad t < 0\ , \qquad \mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \quad t > 0$$

a) After a long time, $t \gg \tau$, what fraction of the atoms will be in the $|n = 2, \ell = 1, m = 0\rangle$ state? Assume that $\mathcal{E}_0$ is small, and work to lowest nontrivial order in $\mathcal{E}_0$.

b) Working to the same order in $\mathcal{E}_0$ as in part (a), what fraction of the atoms are in the $|n = 2, \ell = 0, m = 0\rangle$ state?

Relevant hydrogen atom wavefunctions $\psi_{nlm}(r, \theta, \phi)$ are

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}\ , \quad \psi_{210} = \sqrt{\frac{3}{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}} \cos\theta$$

where a_0 is the Bohr radius. You may express your answer in terms of any convenient variables, e.g. the Bohr radius, fine structure constant, etc.

Name:

3. *Quantum Mechanics*

Consider a three dimensional harmonic oscillator

$$H_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2$$

a) Find the energy and degeneracy of the first excited state of H_0.

b) The perturbation

$$H_1 = \lambda(xy + yz + zx)$$

is added. Find the corrections to the energy and to the states, to first order in λ, for all the states consider in (a).

Name:

4. *Quantum Mechanics*

We consider a quantum system of 3 distinguishable spin 1/2 particles (for example one may be an electron, the second a proton, and the third a neutron), whose spin operators are denoted by $\vec{S}_1, \vec{S}_2, \vec{S}_3$, with components S_1^a, S_2^a, S_3^a for $a = 1, 2, 3$. The Hamiltonian is,

$$H = \alpha \, \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_3)$$

where α is a real constant.

(a) Show that the total angular momentum operator $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ commutes with H.

(b) Calculate the energy and multiplicity of the states with total angular momentum 3/2.

(c) Determine the remaining eigenvalues of H and their respective multiplicities.

Name:

5. *Quantum Mechanics*

A non-relativistic particle of mass m is scattered by the central potential

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(r/a)}$$

where a is a constant.

a) Reduce the Schrödinger equation for the s-wave to the following equation,

$$\frac{d^2\phi(x)}{dx^2} + \frac{2}{\cosh^2 x}\phi(x) + \alpha^2\phi(x) = 0$$

For each value of α^2, this equation has the following two solutions,

$$\phi_\pm(x) = e^{\pm i\alpha x}\big(\tanh x \mp i\alpha\big)$$

b) Calculate the s-wave contribution to the total scattering cross section.

c) Does this system possess any bound states ? Justify your answer.

<u>**Useful Formulas**</u>:

$$\begin{aligned}
\psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikr}}{r}\right] \\
e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \\
f(\mathbf{k}',\mathbf{k}) &= \sum_{l=0}^{\infty}(2l+1)\frac{e^{2i\delta_l}-1}{2ik}P_l(\cos\theta)
\end{aligned}$$

Name:

6. *Statistical Mechanics*

In this problem you are to determine the density of particles $n(z)$ in the atmosphere as a function of height z. Assume that the atmosphere is made up of an ideal gas of particles of mass m. Also, you may assume that the atmosphere is at a uniform temperature T, and that the gravitational field g is constant.

a) Consider a thin layer of atmosphere at height z. Taking into account the gravitational potential energy, calculate the Helmhotz free energy of the layer.

b) Obtain an expression for the chemical potential

c) Use the equilibrium condition on $\mu(z)$ between layers (under exchange of particles) to derive a differential equation and solve for the density $n(z)$. Express your result in terms of $n(0)$.

Name:

7. *Statistical Mechanics*

Consider an idealized crystal that has N lattice points, and the same number of interstitial positions (places between the lattice points where excited atoms can reside). Let E be the energy necessary to displace an atom from a lattice site to an interstitial position and let n be the number of atoms occupying interstitial sites in equilibrium (at a finite temperature).

a) What is the entropy S of this crystal?

b) What is the temperature T of this crystal?

Assume that $n \gg 1$ and $N - n \gg 1$.

Name:

8. *Statistical Mechanics*

Consider two identical particles in a box of volume V, under conditions such that classical (Boltzmann) statistics applies. The particles interact through the potential:

$$U(r) = \begin{cases} -\varepsilon \ , & r \leq b \\ 0 \ , & r > b \end{cases}$$

where r is the distance between the (point-like) particles.

a) Calculate the average pressure at temperature T (ignore the fact that the fluctuations in this quantity are large due to the presence of only two particles).

b) For $\varepsilon > 0$ comment on the physical meaning of your result in the limits $\varepsilon/T \gg 1$ and $\varepsilon/T \ll 1$.

Name:

9. *Statistical Mechanics*

A gas that deviates slightly from ideal behavior exhibits an equation of state given by

$$p\nu = RT - \frac{a}{\nu}$$

where p is the pressure, T is the absolute temperature, R is the gas constant, "a" is a small constant coefficient, and ν is the volume per unit mole, *i.e.* $\nu = V/N_m$. The system consists of N particles corresponding to N_m moles.

a) Deduce the dependence of the partition function Z on volume for this gas.

b) Use your knowledge of the perfect ideal gas to identify the fully normalized partition function for this system. The answer should include the proper quantum normalization for phase-space and account for indistinguishability. The particle mass is m.

c) If the average energy for this system is given by $\overline{E} = \frac{3}{2}N_m RT + a\frac{N_m^3}{V}$, find the specific heat at constant pressure, *i.e.* c_p . Give your answer in terms of T and ν.

Name:

10. *Electromagnetism*

A small sphere of polarizability α and radius a is placed at a great distance from a conducting sphere of radius b, which is maintained at a potential V relative to infinity. Find an approximate expression for the force on the dielectric sphere valid for $r \gg a$.

Name:

11. *Electromagnetism*

A thin spherical shell of radius R has a constant surface charge density σ and is rotating with angular frequency ω. Find the magnetic field inside and outside the shell.

Name:

12. *Electromagnetism*

A plane polarized electromagnetic wave $E = E_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}$ is incident normally on a flat uniform sheet of an excellent conductor ($\sigma \gg \omega\varepsilon_0$) having a thickness D.

a) Assuming that in space and in the conduction sheet $\mu/\mu_0 = \varepsilon/\varepsilon_0 = 1$, calculate the amplitude of the transmitted wave and of the reflected wave

b) Calculate the amount of light absorbed by the metal in the approximation $D \to \infty$

Hint: for the complex index of refraction of a metal you can use $n^2 = 1 + i\sigma/\omega\varepsilon_0$

Name:

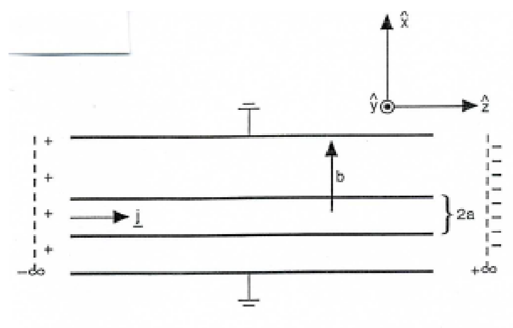
13. *Electromagnetism*

Consider a conducting sheet of thickness $2a$ along the x-direction and having conductivity σ. It is forced to carry a uniform current density j along the z-direction. The current sheet is surrounded by two parallel walls at a distance b from the center of the sheet. The walls can be considered to be perfect conductors at zero electrostatic potential. The sheet and the walls can be approximated as infinite in the other two dimensions (y, z). Here (x, y, z) refer to a Cartesian coordinate system with origin at the center of the sheet. The region between the current sheet and the walls is vacuum.

a) Find the electric field vector everywhere in this system.

b) Find the Poynting vector inside and outside the current sheet for $x > 0$.

c) Use conservation of energy to deduce the value of the perpendicular component of the Poynting vector at the surface of the sheet, *i.e.* at $x = a$.



Name:

14. *Electromagnetism*

a) Consider a point particle with some mass and charge subjected to static gravitational and electric fields $\vec{g}$ and $\vec{E}$, as produced by some arrangement of static external particles. Prove that it is impossible for the point particle to be held in *stable* equilibrium (Earnshaw's theorem).

b) Now consider a diamagnetic (*i.e.* $\mu < \mu_0$) sphere in a constant gravitational field $\vec{g}$ and spatially varying magnetic field $\vec{B}(\vec{x})$. Show that stable equilibrium is possible for such a situation. You may use the following approximate formula (valid for $|\chi| \ll 1$) for the magnetic moment of a sphere of radius a

$$\vec{m} = -\frac{|\chi|\frac{4}{3}\pi a^3}{\mu_0}\vec{B}(\vec{x})$$

Comprehensive exam, Fall 2011

1. *Quantum Mechanics*

Consider a particle in the ground state of an infinitely deep square potential well of width a in one dimension. What is the probability distribution function for the particle to have momentum in the range $(p, p + dp)$?

QM #1

[redacted]

1

Comprehensive Exam [redacted]

1. *QM1*

Consider a particle in an infinitely deep square potential well of width a in one dimension. What is the probability distribution function for the particle to have momentum in the range $(p, p + dp)$?

Solution:

Expansion coefficients of the ground-state wave function $(-a/2 < x < a/2)$

$$\psi(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{\pi}{a}x\right)$$

in the momentum eigenfunction basis are given by

$$a(p) = \int dx e^{-ipx/\hbar}\psi(x) = \sqrt{\frac{2}{a}} \int_{-a/2}^{a/2} dx e^{-ipx/\hbar} \cos\left(\frac{\pi}{a}x\right) = \frac{2^{3/2}\pi\hbar^2\sqrt{a}}{\pi^2\hbar^2 - p^2a^2} \cos\frac{pa}{2\hbar} .$$

The probability distribution function is thus

$$f(p) \equiv \frac{dP}{dp} = \frac{|a(p)|^2}{2\pi\hbar} = \frac{4\pi\hbar^3 a}{(\pi^2\hbar^2 - p^2a^2)^2} \cos^2\left(\frac{\pi}{a}x\right) .$$

This can be checked by considering periodic boundary conditions at $x = \pm L/2$ with $L \to \infty$. In this case, $\tilde{a}(p) = a(p)/\sqrt{L}$ for properly normalized plane waves and $dn = dpL/2\pi\hbar$ (the number of discrete states with momentum in the range of dp). Then, $f(p) = |\tilde{a}(p)|^2 dn/dp$, which agrees with our result above.

2. *Quantum Mechanics*

A collection of hydrogen atoms in the ground state is contained between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor so as to a produce a homogeneous electric field

$$\mathcal{E} = 0 \quad t < 0\,, \qquad \mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \quad t > 0$$

a) After a long time, $t \gg \tau$, what fraction of the atoms will be in the $|n = 2, \ell = 1, m = 0\rangle$ state? Assume that $\mathcal{E}_0$ is small, and work to lowest nontrivial order in $\mathcal{E}_0$.

b) Working to the same order in $\mathcal{E}_0$ as in part (a), what fraction of the atoms are in the $|n = 2, \ell = 0, m = 0\rangle$ state?

Relevant hydrogen atom wavefunctions $\psi_{nlm}(r, \theta, \phi)$ are

$$\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}\,, \quad \psi_{210} = \sqrt{\frac{3}{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}} \cos\theta$$

where a_0 is the Bohr radius. You may express your answer in terms of any convenient variables, e.g. the Bohr radius, fine structure constant, etc.

QM #2

QM 4 Solution

(4a)

a) Amplitude to excite state $|nlm\rangle$ is

$$A_{nlm} = -\frac{i}{\hbar}\int_0^\infty dt\ \langle nlm|\Delta H|100\rangle\, e^{i\omega t}$$

$$\hbar\omega = E_{nlm} - E_{100}$$

$$E_n = -\frac{m\alpha^2}{2\hbar^2}\frac{1}{n^2}$$

$$\Delta H = -\mathcal{E}_0\, z\, e^{-t/\tau}$$

$$= -\mathcal{E}_0\, r\cos\theta\ e^{-t/\tau}$$

$$A_{210} = \frac{i\mathcal{E}_0}{\hbar}\langle 210|r\cos\theta|100\rangle \underbrace{\int_0^\infty dt\, e^{(i\omega-\frac{1}{\tau})t}}_{= -\frac{1}{i\omega-\frac{1}{\tau}} = \frac{\tau}{1-i\omega\tau}}$$

$$= 2\pi\int_0^\infty dr\, r^2\int_{-1}^{1} d(\cos\theta)\ \psi^*_{210}(r,\theta)\, r\cos\theta\, \psi_{100}(r,\theta)$$

$$= \sqrt{\frac{3}{2\pi}}\frac{1}{a_0^2}\int_0^\infty dr\left(r^3 - \frac{r^4}{2a_0}\right)e^{-\frac{3r}{2a_0}}\underbrace{\int_{-1}^{1} d(\cos\theta)\cos^2\theta}$$

$$= \sqrt{\frac{3}{2\pi}}\, a_0\int_0^\infty dx\,\left(x^3 - \tfrac{1}{2}x^4\right)e^{-\frac{3}{2}x}\cdot\frac{2}{3}$$

$$= \left(\frac{2}{3}\right)^{9/2}\frac{1}{\sqrt{\pi}}\, a_0\int_0^\infty dy\,\left(y^3 - \tfrac{1}{3}y^4\right)e^{-y}$$

$$= \left(\frac{2}{3}\right)^{9/2}\frac{1}{\sqrt{\pi}}\, a_0(6-2) = 4\left(\frac{2}{3}\right)^{9/2}\frac{a_0}{\sqrt{\pi}}$$

Kraus 10

4b

$$\Rightarrow \quad A_{210} = 4\left(\frac{2}{3}\right)^{9/2} \frac{a_0}{\sqrt{\pi}} \frac{i\mathcal{E}_0}{\hbar} \frac{\tau}{1 - i\omega\tau}$$

$$\boxed{\mathrm{Prob}_{210} = |A_{210}|^2 = \frac{2^{13}}{3^9} \frac{a_0^2}{\pi} \frac{\mathcal{E}_0^2 \tau^2}{\hbar^2(1+\omega^2\tau^2)}}$$

$$\omega = \frac{E_{210} - E_{100}}{\hbar} = \frac{3}{8} \frac{m\alpha^2}{\hbar^3}$$

b) $\langle 200 | r\cos\theta | 100 \rangle = 0$ by spherical symmetry, so at this order no fraction of atoms are excited to $|200\rangle$.

3. *Quantum Mechanics*

Consider a three dimensional harmonic oscillator

$$H_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2$$

a) Find the energy and degeneracy of the first excited state of H_0.

b) The perturbation

$$H_1 = \lambda(xy + yz + zx)$$

is added. Find the corrections to the energy and to the states, to first order in λ, for all the states consider in (a).

QM#3

3 QM Solution

(3a)

a) $H_0 = \frac{\vec{p}^2}{2m} + \frac{1}{2} m\omega^2 \vec{r}^2$

energies : $E = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega$

1st excited: $\boxed{\begin{array}{l} |100\rangle, \; |010\rangle, \; |001\rangle \\ E = \frac{5}{2}\hbar\omega, \; \text{degeneracy } 3 \end{array}}$

b) $H_1 = \lambda(xy + yz + zx)$

Degenerate pert theory. Need matrix elements of H_1

First consider matrix elements of xy

$\langle n_x' n_y' n_z' | xy | n_x n_y n_z \rangle$

$= \langle n_x' | x | n_x \rangle \langle n_y' | y | n_y \rangle \delta_{n_z n_z'}$

Also, $\langle n_x' | x | n_x \rangle \propto \delta_{n_x, n_x' \pm 1}$

So only matrix element is between $|100\rangle$, $|010\rangle$

$\langle 010 | xy | 100 \rangle = \langle 0 | x | 1 \rangle \langle 1 | y | 0 \rangle$

$= |\langle 0 | x | 1 \rangle|^2$

$\left[x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \right]$ $= \hbar / 2m\omega$

$= \langle 100 | xy | 010 \rangle$

Similarly, yz has matrix element between $|001\rangle$, $|010\rangle$

xz " " $|100\rangle$, $|001\rangle$

[redacted]

So matrix elements of H_1 are

$$H_1 = \frac{\lambda \hbar}{2m\omega} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Diagonalize $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ eig'val 2

$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ eig'val -1

$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ eig'val -1

Result

1) $\frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$ $\quad E = \frac{5}{2}\hbar\omega + \frac{\lambda\hbar}{m\omega}$

2) $\frac{1}{\sqrt{2}}\left(|100\rangle - |001\rangle\right)$ $\quad E = \frac{5}{2}\hbar\omega - \frac{\lambda\hbar}{2m\omega}$

3) $\frac{1}{\sqrt{2}}\left(|100\rangle - |010\rangle\right)$ $\quad E = \frac{5}{2}\hbar\omega - \frac{\lambda\hbar}{2m\omega}$

or any linear combination of states (2), (3)

4. *Quantum Mechanics*

We consider a quantum system of 3 distinguishable spin 1/2 particles (for example one may be an electron, the second a proton, and the third a neutron), whose spin operators are denoted by $\vec{S}_1, \vec{S}_2, \vec{S}_3$, with components S_1^a, S_2^a, S_3^a for $a = 1, 2, 3$. The Hamiltonian is,

$$H = \alpha \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_3)$$

where α is a real constant.

(a) Show that the total angular momentum operator $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ commutes with H.

(b) Calculate the energy and multiplicity of the states with total angular momentum 3/2.

(c) Determine the remaining eigenvalues of H and their respective multiplicities.

QM #4

[redacted]

4

Question 2

We consider a quantum system of 3 distinguishable spin 1/2 particles (for example one may be an electron, the second a proton, and the third a neutron), whose spin operators are denoted by $\vec{S}_1, \vec{S}_2, \vec{S}_3$, with components S_1^a, S_2^a, S_3^a for $a = 1, 2, 3$. The Hamiltonian is,

$$H = \alpha \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_3) \tag{0.19}$$

where α is a real constant.
(a) Show that the total angular momentum operator $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ commutes with H.
(b) Calculate the energy and multiplicity of the states with total angular momentum 3/2.
(c) Determine the remaining eigenvalues of H and their respective multiplicities.

Solution to Question 2

(a) Total spin $\vec{S}$ rotates each $\vec{S}_i$ as a vector. Since the dot-cross product on three vectors is rotation invariant, so is H.

(b) The system of three spin 1/2 particles has 8 independent states, denoted by $|\gamma_1\gamma_2\gamma_3\rangle$ with $\gamma_i = \pm$. We shall use standard notation for spin 1/2 operators, define raising and lowering operators by $S_i^\pm = (S_i^1 \pm i S_i^2)/\sqrt{2}$, and denote their action on states by,

$$S_i^3|\gamma_1\gamma_2\gamma_3\rangle = \gamma_i \frac{\hbar}{2}|\gamma_1\gamma_2\gamma_3\rangle \qquad\qquad S^\pm|\mp\rangle = \frac{\hbar}{\sqrt{2}}|\pm\rangle \tag{0.20}$$

It will be convenient to express the Hamiltonian in determinant form,

$$H = \alpha \begin{vmatrix} S_1^1 & S_2^1 & S_3^1 \\ S_1^2 & S_2^2 & S_3^2 \\ S_1^3 & S_2^3 & S_3^3 \end{vmatrix} = i\alpha \begin{vmatrix} S_1^+ & S_2^+ & S_3^+ \\ S_1^- & S_2^- & S_3^- \\ S_1^3 & S_2^3 & S_3^3 \end{vmatrix} \tag{0.21}$$

Since $[H, \vec{S}] = 0$, the eigenvalue of H is constant throughout every irreducible representation of angular momentum. For the spin 3/2 representation, the highest weight state is $|+++\rangle$. It is straightforward to evaluate H on this state, as H consists of 6 terms, each of which contains one raising operator which kills $|+++\rangle$. Hence, all four states of the spin 3/2 representation have vanishing eigenvalue for H. These normalized states are,

$$\begin{array}{ll} |+++\rangle & \frac{1}{\sqrt{3}}\left(|++-\rangle + |+-+\rangle + |-++\rangle\right) \\ |---\rangle & \frac{1}{\sqrt{3}}\left(|--+\rangle + |-+-\rangle + |+--\rangle\right) \end{array} \tag{0.22}$$

(c) The remaining 4 states transform as the direct sum of two spin 1/2 representations. They are orthogonal to the states of the spin 3/2 representation, and may be chosen to be mutually orthonormalized,

$$
\begin{aligned}
|a+\rangle &= \frac{1}{\sqrt{2}}\left(|++-\rangle - |+-+>\right) \\
|b+\rangle &= \frac{1}{\sqrt{6}}\left(|++-\rangle + |+-+> -2|-++\rangle\right)
\end{aligned}
\tag{0.23}
$$

as well as the states $|a-\rangle$ and $|b-\rangle$ obtained by reversing all signs inside the kets in the above expressions. Evaluating H on the tensor product states, we find,

$$
\begin{aligned}
H|++-\rangle &= 2i\alpha\frac{\hbar^3}{4}\left(|-++\rangle - |+-+\rangle\right) \\
H|+-+\rangle &= 2i\alpha\frac{\hbar^3}{4}\left(|++-\rangle - |-++\rangle\right) \\
H|-++\rangle &= 2i\alpha\frac{\hbar^3}{4}\left(|+-+\rangle - |++-\rangle\right)
\end{aligned}
\tag{0.24}
$$

along with the equations obtained by reversing all signs inside the kets, so that we find,

$$
\begin{aligned}
H|a\pm\rangle &= -i\sqrt{3}\alpha\frac{\hbar^3}{4}|b\pm\rangle \\
H|b\pm\rangle &= +i\sqrt{3}\alpha\frac{\hbar^3}{4}|a\pm\rangle
\end{aligned}
\tag{0.25}
$$

As a result, the normalized eigenvectors of H, and corresponding eigenvalues are,

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\left(|a\pm\rangle - i|b\pm\rangle\right) &\qquad& +\sqrt{3}\alpha\frac{\hbar^3}{4} \\
&\frac{1}{\sqrt{2}}\left(|a\pm\rangle + i|b\pm\rangle\right) &\qquad& -\sqrt{3}\alpha\frac{\hbar^3}{4}
\end{aligned}
\tag{0.26}
$$

Each eigenvalue has double degeneracy. Note that the sum of all eigenvalues vanishes, consistently with the fact that $\mathrm{tr}(H) = 0$.

5. *Quantum Mechanics*

A non-relativistic particle of mass m is scattered by the central potential

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(r/a)}$$

where a is a constant.

a) Reduce the Schrödinger equation for the s-wave to the following equation,

$$\frac{d^2\phi(x)}{dx^2} + \frac{2}{\cosh^2 x}\phi(x) + \alpha^2\phi(x) = 0$$

For each value of α^2, this equation has the following two solutions,

$$\phi_\pm(x) = e^{\pm i\alpha x}(\tanh x \mp i\alpha)$$

b) Calculate the s-wave contribution to the total scattering cross section.

c) Does this system possess any bound states ? Justify your answer.

<u>**Useful Formulas**</u>:

$$\begin{aligned}
\psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikr}}{r}\right] \\
e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \\
f(\mathbf{k}',\mathbf{k}) &= \sum_{l=0}^{\infty}(2l+1)\frac{e^{2i\delta_l}-1}{2ik}P_l(\cos\theta)
\end{aligned}$$

QM #5

[redacted]

5

QUANTUM PHYSICS

Fall 2011 Comprehensive Exam Questions

Question 1

A non-relativistic particle of mass m is scattered by the central potential

$$V(r) = -\frac{\hbar^2}{ma^2}\frac{1}{\cosh^2(r/a)}$$

where a is a constant.

a) Reduce the Schrödinger equation for the s-wave to the following equation,

$$\frac{d^2\phi(x)}{dx^2} + \frac{2}{\cosh^2 x}\phi(x) + \alpha^2\phi(x) = 0$$

For each value of α^2, this equation has the following two solutions,

$$\phi_\pm(x) = e^{\pm i\alpha x}(\tanh x \mp i\alpha)$$

b) Calculate the s-wave contribution to the total scattering cross section.

c) Does this system possess any bound states ? Justify your answer.

Solution to Question 1

a) The Schrödinger equation for the system is

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi \tag{0.1}$$

We decompose the wave function in spherical harmonics Y_{lm}, and recast the energy E in terms of momentum $\hbar k$ by the customary relation,

$$\psi = \sum_{lm} R_{lm}Y_{lm} \qquad\qquad E = \frac{\hbar^2k^2}{2m} \tag{0.2}$$

The radial wave function $R_{00}(r)$ for the s-wave satisfies the equation,

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2}{a^2}\frac{1}{\cosh^2(r/a)} + k^2\right)R_{00} = 0 \tag{0.3}$$

Setting $R_{00}(r) = \phi(x)/x$, with $x = r/a$, we find the desired equation for the reduced wave function $\phi(x)$, where $\alpha = ka$.

b) The following formulas should be provided to the students,

$$\begin{aligned}\psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikr}}{r}\right] \\ e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta) \\ f(\mathbf{k}',\mathbf{k}) &= \sum_{l=0}^{\infty}(2l+1)\frac{e^{2i\delta_l}-1}{2ik}P_l(\cos\theta)\end{aligned} \tag{0.4}$$

The s-wave part of the scattering function is

$$\begin{aligned}\psi_{\mathbf{k}}(\mathbf{r}) &= \frac{1}{(2\pi)^{3/2}}\left[\frac{\sin(kr)}{kr} + \frac{e^{2i\delta_0}-1}{2ik}\frac{e^{ikr}}{r}\right] \\ &= \frac{1}{2ikr(2\pi)^{3/2}}\left[e^{2i\delta_0}e^{ikr} - e^{-ikr}\right]\end{aligned} \tag{0.5}$$

We know the solutions to the radial equation are $\phi_\pm(x)$

$$\begin{aligned}\phi_\pm(x) &= e^{\pm i\alpha x}(\tanh x \mp i\alpha) \\ \psi_\pm(r) &= \frac{\tanh x \mp i\alpha}{x}e^{\pm i\alpha x}\end{aligned} \tag{0.6}$$

The radial part of the wave function is a linear superposition of $\phi_\pm$,

$$\begin{aligned}\phi(x) &\propto A_+\phi_+(x) + A_-\phi_-(x) \\ &= A_+e^{ikr}\left(\tanh\frac{r}{a} - i\alpha\right) + A_-e^{-ikr}\left(\tanh\frac{r}{a} + i\alpha\right)\end{aligned} \tag{0.7}$$

where $A_\pm$ are two constants. The wave function $\phi(x)$ behaves as a free wave at ∞. It must also satisfy a regularity condition at $r = 0$. All together, we must impose,

$$\phi(0) = 0 \qquad\qquad \phi(r \gg a) \propto e^{2i\delta_0}e^{ikr} - e^{-ikr} \tag{0.8}$$

which gives rise to the conditions,

$$\begin{aligned}\alpha(A_+ - A_-) &= 0 \\ A_+e^{ikr}(1 - i\alpha) + A_-e^{-ikr}(1 + i\alpha) &\propto e^{2i\delta_0}e^{ikr} - e^{-ikr}\end{aligned} \tag{0.9}$$

From these equations, we deduce,

$$e^{2i\delta_0} = -\frac{1 - i\alpha}{1 + i\alpha} = -\frac{1 - ika}{1 + ika} \tag{0.10}$$

Finally, the s-wave contribution to the total cross section is

$$\sigma_s^{tot} = \frac{4\pi}{k^2} sin^2\delta_0 = \frac{4\pi}{(1+k^2a^2)k^2} \tag{0.11}$$

c) To analyze the existence of bound states, we need $E < 0$. This is achieved by analytically continuing α to imaginary values, $\alpha \rightarrow -i\alpha$, and then taking the new α to be real. The corresponding solutions are still given by the general form with $\alpha \rightarrow -i\alpha$,

$$\phi_\pm(x) = e^{\pm\alpha x}(\tanh x \mp \alpha) \tag{0.12}$$

The general solution is thus allowed to be of the form,

$$\phi_\alpha(x) = A_+\phi_+(x) + A_-\phi_-(x) \tag{0.13}$$

Imposing regularity at the origin, as well the bound state asymptotics at ∞,

$$\phi_\alpha(0) = 0 \qquad\qquad \phi_\alpha(+\infty) = 0 \tag{0.14}$$

The condition $\phi_\alpha(0) = 0$ requires $A_+ = A_-$. The solution then becomes proportional to,

$$\phi_\alpha(x) = e^{\alpha x}(\tanh x - \alpha) + e^{-\alpha x}(\tanh x + \alpha) \tag{0.15}$$

This expression is symmetric under $\alpha \rightarrow -\alpha$, so we assume $\alpha \geq 0$ without loss of generality. As $x \rightarrow +\infty$, we have $\phi_\alpha(x) \sim A_+e^{\alpha x}(1-\alpha)$. The bound state boundary condition thus requires $\alpha = 1$. It is manifest, however, that $\phi_{\alpha=1}(x) = 0$ identically. This looks surprising, since the two solutions $\phi_\pm(x)$ would seem linearly independent. Computing the Wronskian,

$$\left(\phi'_+\phi_- - \phi'_-\phi_+\right)(x) = 2\alpha(1-\alpha^2) \tag{0.16}$$

we see that ϕ_+ and ϕ_- are linearly dependent for $\alpha = 1$. The missing linearly independent solution may be obtained by taking the derivative in α,

$$\hat{\phi}(x) = \frac{\partial\phi_\alpha(x)}{\partial\alpha}\bigg|_{\alpha=1} = -\frac{2x}{\cosh x} - 2\sinh x \tag{0.17}$$

While this function satisfies the regularity condition at $x = 0$, it diverges exponentially as $x \rightarrow \infty$, and can thus not obey the bound state asymptotic condition. Therefore, there are no bound states for this problem. Note that a generalization of this equation, given by

$$\left(\frac{d^2}{dx^2} + \frac{m(m+1)}{\cosh^2 x} - \alpha^2\right)\phi(x) = 0 \tag{0.18}$$

for m integer, has $m-1$ bound states. The case at hand here is $m = 1$.

Question # 6 Name

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a) The energy at height z is $H = \frac{p^2}{2m} + mgz$. So

$$Z_1 = \int \frac{d^3p\,d^3x}{h^3} e^{-H/T} = \int \frac{d^3p\,d^3x}{h^3} e^{-\frac{p^2}{2mT} - \frac{mgz'}{T}}$$

$$= \frac{1}{h^3}(2\pi mT)^{3/2} A \int_z^{z+dz} e^{-mgz'/T} dz'$$

$$= \left(\frac{2\pi mT}{h^2}\right)^{3/2} A \left(\frac{T}{mg}\right) e^{-mgz/T} \left(1 - e^{-\frac{mg\,dz}{T}}\right)$$

$$= \left(\frac{2\pi mT}{h^2}\right)^{3/2} \frac{AT}{mg} e^{-mgz/T} \frac{mg}{T} dz \quad \text{at the thin layer.}$$

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For N indistinguishabe particles, we have

$$Z = \frac{1}{N!} Z_1^N = \frac{1}{N!}\left(\frac{2\pi mT}{h^2}\right)^{3N/2} A^N e^{-mgzN/T} dz^N$$

6 So $\boxed{F = -T \log Z = T \log N! - TN \log\left(\frac{A\,dz}{\lambda^3} e^{-mgz/T}\right)}$

where $\lambda = \left(\frac{h^2}{2\pi mT}\right)^{1/2}$. So

$$\boxed{F = T \log N! - TN \log\left(\frac{A\,dz}{\lambda^3}\right) + mgzN}$$

b) $$\mu = \left(\frac{\partial F}{\partial N}\right)_{S,V} = \frac{\partial}{\partial N}\left(TN\log N - NT - TN\log\left(\frac{A\,dz}{\lambda^3}\right) + mgzN\right)$$

$$= T\log N + T - T - T\log\left(\frac{A\,dz}{\lambda^3}\right) + mgz$$

3

$$= T \log\left(\underbrace{\frac{N}{A\,dz}}_{n(z)} \lambda^3\right) + mgz = \boxed{T \log\left(n(z)\lambda^3\right) + mgz}$$

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c) $\mu(z+dz) - \mu(z) = 0$ at equilibrium. So

$$T\log\left(n(z+dz)\lambda^3\right) + mg(z+dz) = T\log\left(n(z)\lambda^3\right) + mgz$$

$$n(z+dz)\lambda^3 \; e^{mg\,dz/T} = n(z)\lambda^3$$

$$n(z+dz)\lambda^3 \left(1 + \frac{mg\,dz}{T}\right) = n(z)\lambda^3$$

$$\left(n(z+dz) - n(z)\right)\cancel{\lambda^3} = -\frac{\cancel{\lambda^3}\,mg}{T} n(z+dz)\,dz$$

$$\frac{n(z+dz) - n(z)}{dz} = -\frac{mg}{T} n(z+dz)$$

Let $dz \to 0$ =>

$$\text{S} \quad \frac{dn}{dz}(z) = -\frac{mg}{T} n(z)$$

$$\Rightarrow \boxed{n(z) = n(0)\exp\left(-\frac{mgz}{T}\right)}$$

7. *Statistical Mechanics*

Consider an idealized crystal that has N lattice points, and the same number of interstitial positions (places between the lattice points where excited atoms can reside). Let E be the energy necessary to displace an atom from a lattice site to an interstitial position and let n be the number of atoms occupying interstitial sites in equilibrium (at a finite temperature).

a) What is the entropy S of this crystal?

b) What is the temperature T of this crystal?

Assume that $n \gg 1$ and $N - n \gg 1$.

Q7.
Stat Mech

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3. *SM*

Consider an idealized crystal which has N lattice points and the same number of interstitial positions (places between the lattice points where excited atoms can reside). Let E be the energy necessary to displace an atom from a lattice site to an interstitial position and let n be the number of atoms occupying interstitial sites in equilibrium (at a finite temperature).

(a) What is the entropy S of this crystal?

(b) What is the temperature T of this crystal?

Assume that $n \gg 1$ and $N - n \gg 1$.

Solution:

(a) There are C_n^N ways of selecting n (indistingsuishable) atoms from N (distingsuishable) lattice sites, and C_n^N ways to place them into N interstitial sites. The microscopic state number is thus $\Omega = (C_n^N)^2$ and the entropy

$$S \equiv k_B \ln \Omega = 2k_B \ln \frac{N!}{n!(N-n)!} .$$

Using Stirling's formula, $\ln n! \approx n \ln n - n$, then:

$$S \approx 2k_B \left[N \ln N - n \ln n - (N-n) \ln(N-n) \right] .$$

(b) Minimizing the free energy $F = nE - TS$ as $\partial_n F = 0$, we get

$$E = 2k_B T \ln(N/n - 1) \quad \Rightarrow \quad T = \frac{E}{2k_B \ln(N/n - 1)} .$$

8. *Statistical Mechanics*

Consider two identical particles in a box of volume V, under conditions such that classical (Boltzmann) statistics applies. The particles interact through the potential:

$$U(r) = \begin{cases} -\varepsilon\ , & r \le b \\ 0\ , & r > b \end{cases}$$

where r is the distance between the (point-like) particles.

a) Calculate the average pressure at temperature T (ignore the fact that the fluctuations in this quantity are large due to the presence of only two particles).

b) For $\varepsilon > 0$ comment on the physical meaning of your result in the limits $\varepsilon/T \gg 1$ and $\varepsilon/T \ll 1$.

Q8 Stat Mech

[redacted]

Solutions to [redacted] problems

Comp. Exam 2011

GZ1
$$H = \sum_{i=1}^{2} \frac{\vec{p}_i^{\,2}}{2m} + U(|\vec{r}_1 - \vec{r}_2|)$$

is the Hamiltonian and the partition sum:

$$Z = \frac{1}{2} \int \frac{d^3p_1\, d^3p_2}{(2\pi\hbar)^6} e^{-\frac{\vec{p}_1^{\,2} + \vec{p}_2^{\,2}}{2mT}} \int d^3r_1\, d^3r_2\, e^{-U/T}$$

$$= \frac{1}{2} Z_2 \int d^3r_1\, d^3r_2\, e^{-U/T}$$

where Z_2 is the momentum part of the partition sum of the ideal gas of 2 particles:

$$Z_2 = \left(\frac{1}{\lambda^3}\right)^2, \quad \lambda = \left(\frac{h^2}{2\pi m T}\right)^{1/2}$$

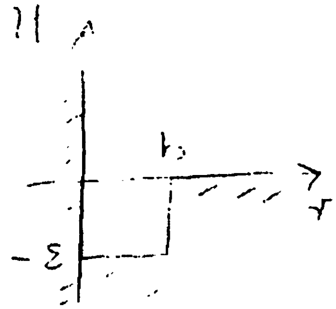
Changing variables to the C.M. and relative coord.:

$$\begin{cases} \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \\ \vec{r} = \frac{1}{2}(\vec{r}_1 - \vec{r}_2) \end{cases}$$

(Jacobian $= 1/2$)

The integral becomes :

$$\frac{1}{2}\int d^3R\, d^3r\; e^{-U(r)/T} = \frac{1}{2}\, V \int_0^{\text{size of the box}} 4\pi r^2 dr\; e^{-U(r)/T}$$



$$\text{integral} = \int_0^b + \int_b^{\text{size of the box}}$$

$$= e^{\varepsilon/T}\,\frac{4\pi}{3} b^3 + \left(V - \frac{4\pi}{3} b^3\right)$$

$$= \frac{4\pi}{3} b^3 \left(e^{\varepsilon/T} - 1\right) + V$$

$$\Rightarrow \quad Z = \frac{1}{4}\left(\frac{V}{\lambda^3}\right)^2 \left[1 + \frac{4\pi}{3}\frac{b^3}{V}\left(e^{\varepsilon/T} - 1\right)\right]$$

$$\Rightarrow \quad F = -T \ln Z = -2T \ln\left(\frac{V}{2\lambda^3}\right) - T \ln[\;\;]$$

$$P = -\left.\frac{\partial F}{\partial V}\right|_T = 2\frac{T}{V} - \frac{T}{V}\;\frac{\frac{4\pi}{3}\frac{b^3}{V}\left(e^{\varepsilon/T} - 1\right)}{1 + \frac{4\pi}{3}\frac{b^3}{V}\left(e^{\varepsilon/T} - 1\right)}$$

So for $\varepsilon < 0$ the pressure is increased with respect to the ideal gas value $\left(2\frac{T}{V}\right)$ while for $\varepsilon > 0$ it is decreased.

For $\varepsilon > 0$ and $\varepsilon/T \gg 1$:

$$P \to 2\frac{T}{V} - \frac{T}{V} = \frac{T}{V}$$

i.e. it's as if there was only one particle in the box (since the particles are bound).

For $\varepsilon/T \ll 1$:

$$P \to 2\frac{T}{V}\left[1 - \frac{1}{2}\frac{4\pi}{3}\frac{b^3}{V}\frac{\varepsilon}{T}\right]$$

or $$\left(P + \frac{1}{V^2}\frac{4\pi}{3}b^3\varepsilon\right)V = 2T$$

which is essentially the Van der Waals eq of state (without excluded volume because we did not put it in the potential).

9. *Statistical Mechanics*

A gas that deviates slightly from ideal behavior exhibits an equation of state given by

$$p\nu = RT - \frac{a}{\nu}$$

where p is the pressure, T is the absolute temperature, R is the gas constant, "a" is a small constant coefficient, and ν is the volume per unit mole, *i.e.* $\nu = V/N_m$. The system consists of N particles corresponding to N_m moles.

a) Deduce the dependence of the partition function Z on volume for this gas.

b) Use your knowledge of the perfect ideal gas to identify the fully normalized partition function for this system. The answer should include the proper quantum normalization for phase-space and account for indistinguishability. The particle mass is m.

c) If the average energy for this system is given by $\overline{E} = \frac{3}{2}N_m RT + a\frac{N_m^3}{V}$, find the specific heat at constant pressure, *i.e.* c_p . Give your answer in terms of T and ν.

Q9
Stat Mech

[redacted] SM - Answer [redacted]

a) given an Eq. of state $p = \frac{RT}{v} - \frac{a}{v^2}$

with $v = \frac{V}{\nu}$ and $\nu \equiv \frac{N}{N_A}$ ← Avogadro's #

Thermo + Stat Mech is related by:

$$p = \frac{1}{\beta}\frac{\partial}{\partial V}\ln Z \quad \text{with} \quad \beta \equiv \frac{1}{kT}$$

↑ Boltzmann's const

~~For this case~~ $$\frac{RT}{v} - \frac{a}{v^2} = \frac{1}{\beta\nu}\frac{\partial}{\partial v}\ln[Z(v)]$$

which can be integrated over v

$$\ln\left(\frac{Z}{Z_0}\right) = RT\beta\nu \ln v + \frac{a\beta\nu}{v}$$

↑ a Normalization constant.

but $R = N_A k \Rightarrow \frac{R\nu}{k} = \frac{N_A k}{k}\frac{N}{N_A} = N$

$$\Rightarrow \ln\left(\frac{Z}{Z_0}\right) = N\ln v + \frac{a\beta\nu^2}{V}$$

$$\Rightarrow \boxed{Z = Z_0 (V)^N e^{(a\beta\nu^2)/V}}$$

b) In the limit $a \to 0$ this system is exactly an ideal gas for which $Z_{indistinguishable} = \frac{(Z_{single})^N}{N!}$

where the single particle quantum normalization

$$Z_{single} = \left(\frac{2\pi m kT}{h^2}\right)^{3/2}$$ for particle mass m.

. and . . .

b) ~~[illegible]~~ Connecting the ideal behavior to the ~~actual Z for this system~~ identifies Z_0

$$\Rightarrow \boxed{Z = \frac{1}{N!}\left[\left(\frac{2\pi m}{h^2\beta}\right)^{3/2}\right]^N V^N e^{(a\beta\nu^2)/V}}$$

c) Given now that $pv = RT + \frac{a}{v}$

and $\bar{E} = \frac{3}{2}\nu RT + \frac{a\nu^2}{V}$ one needs to find

$$C_p = \frac{1}{\nu}\left(\frac{dQ}{dT}\right)_p$$

but $dQ = dE + p\,dV$ and $dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV$

$$\Rightarrow dQ = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV + p\,dV$$

both are now known from

$$\Rightarrow \frac{dQ}{\nu} = \frac{3R}{2}dT + \left(-\frac{a\nu^2}{V^2}\right)dv + p\,dv$$

but $p\,dv = R\,dT - \frac{a}{v^2}dv$

or $dv = \frac{R\,dT}{p + \frac{a}{v^2}}$

& ~~collecting~~ terms: $\frac{dQ}{\nu} = \frac{3R}{2}dT + \left(p - \frac{a\nu^2}{V^2}\right)dv$

$$= \frac{3R}{2}dT + \frac{(RT/v)}{\frac{RT}{v} + \frac{a}{v^2} + \frac{a}{v^2}}R\,dT$$

[redacted] cont...

[redacted]

c) Resulting in

$$\boxed{C_p = R\left[\frac{3}{2} + \frac{1}{1 + \frac{2a}{RTv}}\right]}$$

+ as $a \to 0$

$$C_p \to \frac{5}{2} R$$

Name:

Gaussian Units

(cgs)

10. *Electromagnetism*

A small sphere of polarizability α and radius a is placed at a great distance from a conducting sphere of radius b, which is maintained at a potential V relative to infinity. Find an approximate expression for the force on the dielectric sphere valid for $r \gg a$.

Assumption: Field over small sphere is essentially uniform. Field induced on sphere does not affect the field of the conducting sphere. is α is small.

$$\Phi(\vec{r}=R)=V=\frac{Q}{R} \quad \therefore \quad Q=VR$$

$$\therefore \Phi=\frac{VR}{|\vec{r}|} \quad \therefore \vec{E}=-\vec{\nabla}\Phi \qquad \vec{\nabla}=\left(-\hat{r}\,\partial_r\right)$$

$$\therefore \vec{E}=+\frac{VR}{|\vec{r}^2|}\hat{r} \qquad \therefore \vec{p}=\alpha\vec{E}$$

$$\therefore \vec{p}=\frac{\alpha VR}{r^2}\hat{r}$$

$$\therefore \vec{P}=\vec{p}\left(\tfrac{4\pi}{3}a^3\right)$$

$$\therefore \vec{P}=\tfrac{4\pi}{3}a^3\left(\frac{\alpha VR}{r^2}\right)\hat{r}$$

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Potential energy of a Polarization in an Electric field

$$U=-\frac{1}{2}\vec{P}\cdot\vec{E}$$

$$\therefore U=-\frac{1}{2}\left(\frac{4\pi a^3\alpha VR}{3\,r^2}\hat{r}\right)\cdot\left(\frac{VR}{r^2}\hat{r}\right)$$

$$U=-\frac{\frac{2\pi a^3}{3}\alpha V^2R^2}{r^4}$$

$$\vec{F}=-\vec{\nabla}U$$

$$\therefore =\hat{r}\,\partial_r\left(\frac{2\pi a^3}{3}\alpha V^2R^2\,\frac{1}{r^4}\right)$$

$$\vec{F}=-\left(\frac{8\pi a^3}{3}\alpha V^2R^2\,\frac{\hat{r}}{r^5}\right);$$ The force is attractive,

Name:

11. *Electromagnetism*

A thin spherical shell of radius R has a constant surface charge density σ and is rotating with angular frequency ω. Find the magnetic field inside and outside the shell.

The rotating surface density gives rise to a current density $\vec{K} = \sigma\vec{v} = \sigma r\omega\sin\theta\,\hat{\phi}$.

From Maxwell's equation $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$, since $\vec{J}=0$ and $\dot{\vec{D}}=0$, we have that $\vec{\nabla}\times\vec{H}=0$ both inside and outside, so that $\vec{H}$ can be rewritten as $\vec{H} = -\nabla\Phi_M$, where Φ_M, the scalar magnetic potential, satisfies $\vec{\nabla}\times(-\vec{\nabla}\Phi_M) = 0 \Rightarrow \nabla^2\Phi_M = 0$. Since we have spherical symmetry in this setup, the general solution to this is $\Phi_M = \sum_{l=0}^{\infty}\left(A_l r^l + B_l \frac{1}{r^{l+1}}\right)P_l(\cos\theta)$ subject to the boundary conditions at $r=R$:

(1) $(\vec{B}_{out} - \vec{B}_{in})\cdot\hat{r} = 0 \Rightarrow -\mu_0\frac{\partial\Phi_{out}}{\partial r}\Big|_R + \mu_0\frac{\partial\Phi_{in}}{\partial r}\Big|_R = 0 \Rightarrow \frac{\partial\Phi_{in}}{\partial r}\Big|_R = \frac{\partial\Phi_{out}}{\partial r}\Big|_R$

(2) $(\vec{H}_{out} - \vec{H}_{in})\times\hat{r} = \underset{\substack{\uparrow \\ \text{due to} \\ \text{spherical} \\ \text{symmetry}}}{-\vec{K}} \Rightarrow -\frac{\partial\Phi_{out}}{\partial\theta}\Big|_R + \frac{\partial\Phi_{in}}{\partial\theta}\Big|_R = -\sigma R\omega\sin\theta$

($\vec{H}$ in $\hat{z}$ direction
$\hat{z}\times\hat{r} = (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\times\hat{r}$
$= \sin\theta\,\hat{\phi}$

We next note that since there is no external magnetic field, $\Phi_{out}(r\to\infty) = 0 \Rightarrow A_l = 0$
and since the potential must be defined at the origin, $B_l = 0$ in Φ_{in}.

$$\Rightarrow \begin{cases} \Phi_{in} = \sum A_l r^l P_l(\cos\theta) \\ \Phi_{out} = \sum B_l \frac{1}{r^{l+1}} P_l(\cos\theta) \end{cases}$$

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and we now apply the BC's at $r=R$:

(1) $\Rightarrow \sum \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -\sum(\ell+1) B_\ell \frac{1}{R^{\ell+2}} P_\ell(\cos\theta)$

orthogonality of Legendre polynomials

$$\Rightarrow \ell A_\ell R^{\ell-1} = -(\ell+1)\frac{B_\ell}{R^{\ell+2}}$$

$$\Rightarrow A_\ell = -\frac{\ell+1}{\ell}\frac{1}{R^{2\ell+1}} B_\ell$$

$$\Rightarrow \Phi_{in} = -\sum \frac{\ell+1}{\ell}\frac{1}{R^{2\ell+1}} B_\ell r^\ell P_\ell(\cos\theta)$$

(2) $-\sum B_\ell \frac{1}{R^{\ell+1}} P_\ell'(\cos\theta) - \sum \frac{\ell+1}{\ell}\frac{R^\ell}{R^{2\ell+1}} B_\ell P_\ell'(\cos\theta) = -\sigma R \omega \sin\theta$

Due to the $\sin\theta$ term and the orthogonality of Legendre polynomials, all terms in the series except for $\ell=1$ drop out ($\ell=1 \Rightarrow P_1'(\cos\theta) = (\cos\theta)' = -\sin\theta$).

For $\ell=1$ we have:

$$B_1\frac{1}{R^2} + 2\frac{R}{R^{3}}B_1 = -\sigma R\omega$$

$$\Rightarrow 3B_1 = -\sigma R^3\omega \Rightarrow B_1 = -\frac{\sigma R^3 \omega}{3}$$

$$\Rightarrow A_1 = -2\frac{1}{R^3}B_1 = \frac{2\sigma\omega}{3}$$

$$\Rightarrow \begin{cases} \Phi_{in} = \frac{\cancel{2}\sigma\omega}{3} r\cos\theta \\ \Phi_{out} = -\frac{\sigma R^3\omega}{3}\frac{1}{r^2}\cos\theta \end{cases}$$

$$\Rightarrow \begin{cases} \vec{B}_{in} = -\mu_0 \nabla\Phi_{in} = -\frac{2\sigma\omega\mu_0}{3}\nabla(r\cos\theta) \\ \vec{B}_{out} = -\mu_0\nabla\Phi_{out} = \frac{\mu_0\sigma R^3\omega}{3}\nabla\left(\frac{1}{r^2}\cos\theta\right) \end{cases}$$

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$$\nabla(r\cos\theta) = \cos\theta\,\hat{r} + \frac{1}{\cancel{r}}\cancel{r}(-\sin\theta)\,\hat{\theta} = \hat{z}$$

$$\nabla\left(\frac{1}{r^2}\cos\theta\right) = \cos\theta\left(-2\frac{1}{r^3}\right)\hat{r} + \frac{1}{r}\cdot\frac{1}{r^2}(-\sin\theta)\,\hat{\theta}$$

$$= \frac{-1}{r^3}\left(2\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$

$$= -\frac{1}{r^3}\left(\hat{z} + \cos\theta\,\hat{r}\right)$$

$$\Rightarrow \begin{cases} \vec{B}_{in} = + \dfrac{2\mu_0\sigma\omega}{3}\hat{z} \\ \vec{B}_{out} = -\dfrac{\mu_0\sigma\omega}{3}\dfrac{R^3}{r^3}\left(\hat{z} + \cos\theta\,\hat{r}\right) \end{cases}$$

✓ ✗

$\vec{H}_{in} = -\vec{\nabla}\varphi_{in}$

$\cancel{\vec{B}_{in} = \mu_0(\vec{H}_{in} + \vec{M})}$

$\vec{B}_{in} = \mu_0(\vec{H}_{in} + \vec{M})$

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12. *Electromagnetism*

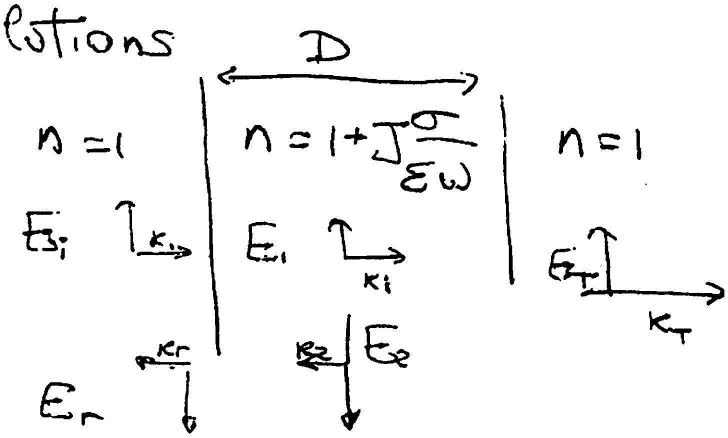
A plane polarized electromagnetic wave $E = E_0 e^{i(kz-\omega t)}\hat{\mathbf{x}}$ is incident normally on a flat uniform sheet of an excellent conductor ($\sigma \gg \omega\varepsilon_0$) having a thickness D.

a) Assuming that in space and in the conduction sheet $\mu/\mu_0 = \varepsilon/\varepsilon_0 = 1$, calculate the amplitude of the transmitted wave and of the reflected wave

b) Calculate the amount of light absorbed by the metal in the approximation $D \to \infty$

Hint: for the complex index of refraction of a metal you can use $n^2 = 1 + i\sigma/\omega\varepsilon_0$

Solutions



Using boundary conditions

(1)
$$E_i + E_r = E_1 + E_2$$
$$E_i - E_r = n(E_1 - E_2)$$

(2)
$$e^{ikD}E_1 + E_2 e^{-ikD} = E_t$$
$$n\left(E_1 e^{ikD} - E_2 e^{-ikD}\right) = E_t$$

Using the second ones

$$E_1 = \frac{E_t(1+n)e^{-ikD}}{2n}, \qquad E_2 = \frac{-E_t(1-n)e^{ikD}}{2n}$$

first ones

$$E_i = \frac{E_1(1+n) + E_2(1-n)}{2}$$
$$E_r = \frac{E_1(1-n) + E_2(1+n)}{2}$$

Then
$$E_i = \frac{E_t\left[(1+n)^2 e^{-ikD} - (1-n)^2 e^{ikD}\right]}{4n}$$
$$E_r = E_t\,\frac{(1-n^2)e^{-ikD} - (1-n^2)e^{ikD}}{4n} = E_i\,\frac{(1-n^2)e^{-ikD} - (1-n^2)e^{ikD}}{(1+n)^2 e^{-ikD} - (1-n)^2 e^{ikD}}$$

$$n=\sqrt{1+i\frac{\sigma}{\varepsilon\omega}}\sim\sqrt{i\frac{\sigma}{\varepsilon\omega}}=\pm e^{i\pi/4}\sqrt{\frac{\sigma}{\varepsilon\omega}}=\pm(1+i)\sqrt{\frac{\sigma}{2\varepsilon\omega}}$$

$$kD=\frac{\omega nD}{c}$$

$$\frac{E_r}{E_i}=\frac{(1-n^2)\left(e^{-i\frac{\omega dn}{c}}-e^{i\frac{\omega dn}{c}}\right)}{e^{-i\frac{\omega dn}{c}}(1+n)^2-(1-n)^2e^{i\frac{\omega dn}{c}}}=\frac{(1-n^2)\left(1-\cancelto{0}{e^{2i\omega Dn/c}}\right)}{(1+n)^2-\cancelto{0}{(1-n)^2e^{2i\omega Dn/c}}}$$

for $D\to\infty$

$$T\to 0$$

$$R=\left|\frac{1-n^2}{(1+n)^2}\right|=\frac{\left|i\frac{\sigma}{\varepsilon\omega}\right|}{\left|\left(2+\frac{i\sigma}{\varepsilon\omega}\right)^2\right|}=\frac{1}{\sqrt{1+\frac{4\varepsilon^2\omega^2}{\sigma^2}}}$$

$$1-R=A\sim\frac{2\varepsilon^2\omega^2}{\sigma^2}$$

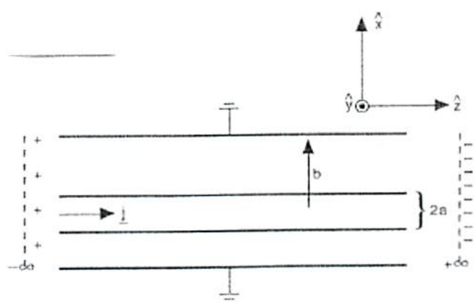
13. *Electromagnetism*

Consider a conducting sheet of thickness $2a$ along the x-direction and having conductivity σ. It is forced to carry a uniform current density j along the z-direction. The current sheet is surrounded by two parallel walls at a distance b from the center of the sheet. The walls can be considered to be perfect conductors at zero electrostatic potential. The sheet and the walls can be approximated as infinite in the other two dimensions (y, z). Here (x, y, z) refer to a Cartesian coordinate system with origin at the center of the sheet. The region between the current sheet and the walls is vacuum.

a) Find the electric field vector everywhere in this system.

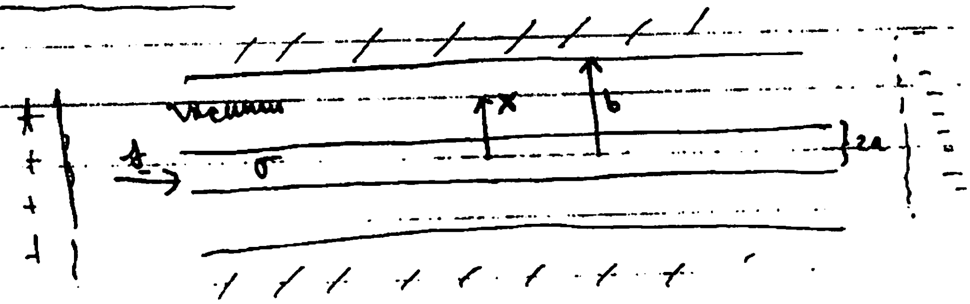
b) Find the Poynting vector inside and outside the current sheet for $x > 0$.

c) Use conservation of energy to deduce the value of the perpendicular component of the Poynting vector at the surface of the sheet, *i.e.* at $x = a$.



EM #1 - Answers

Given system



a) Find the electric field everywhere -

- Inside current sheet : $J = \sigma E \Rightarrow \boxed{E = \frac{J}{\sigma}\hat{z}}$

This implies that for $|x| < a$ the electrostatic potential is $\boxed{\phi = -\frac{J}{\sigma} z}$

Outside the current sheet it is vacuum $\Rightarrow \nabla^2 \phi = 0$

- System is independent of y & ϕ must match behavior at surface of current sheet

$\Rightarrow$ Try $\phi(x, z) = -E_0 z \underbrace{f(x)}_{\text{To be determined}}$ for $|x| > a$

From Laplace's Eq. $\frac{\partial^2}{\partial x^2} f(x) = 0$ subject to the

conditions: $f(x = \pm a) = 1$ and $f(x = \pm b) = 0$

which is satisfied by $f(x) = \frac{b - |x|}{b - a}$

$$\Rightarrow \phi(x, z) = -\frac{J}{\sigma}\left(\frac{b - |x|}{b - a}\right) z \quad \text{for } |x| > a$$

#1 cont --

a) Electric field for $|x| > a$ - outside sheet is

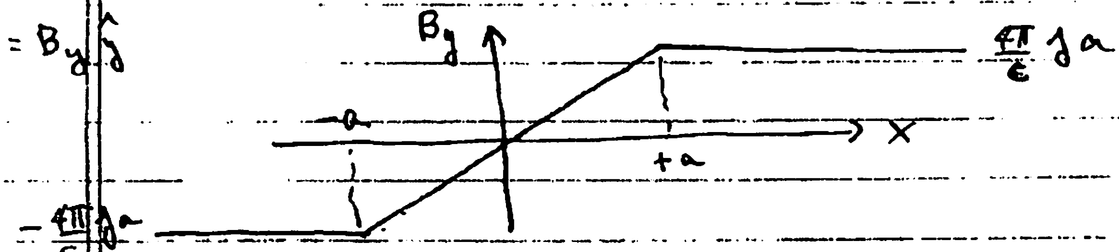
$$\underline{E} = -\nabla\phi = \frac{j}{\sigma}\left[\hat{x}\frac{\partial}{\partial x} + \hat{z}\frac{\partial}{\partial z}\right]\left[\left(\frac{b-x}{b-a}\right)z\right]$$

for $x > a$ $\Rightarrow$ $\boxed{\underline{E} = \frac{j}{\sigma(b-a)}\left[-z\hat{x} + (b-x)\hat{z}\right]}$

& for $x < -a$ $\Rightarrow$ $\boxed{\underline{E} = \frac{j}{\sigma(b-a)}\left[+z\hat{x} + (b+x)\hat{z}\right]}$

b) Poynting vector $S = \frac{c}{4\pi}\underline{E}\times\underline{B}$ requires knowledge of B-field. ~~For a uniform current sheet~~

$\underline{B} = B_y\hat{y}$



$$B_y = \frac{4\pi}{c} j \begin{cases} x & \text{for } |x| < a \\ +a & \text{for } x > a \\ -a & \text{for } x < -a \end{cases}$$

$$\underline{S} = \frac{c}{4\pi}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ E_x & 0 & E_z \\ 0 & B_y & 0 \end{vmatrix} = \frac{c}{4\pi}\left[\hat{x}(-E_z B_y) + \hat{z}\,E_x B_y\right]$$

$$\underline{S} = \frac{B_y c}{4\pi}\left[-E_z\hat{x} + E_x\hat{z}\right]$$

M # 1 cont ...

b) Inside the current sheet $|x| < a$ $E_x = 0$

$$\Rightarrow \quad \underline{S} = \frac{B_y c}{4\pi}\left[-E_z \hat{x}\right]$$

Say for $0 < x < a$

$$\boxed{\underline{S} = -\,j x \left(\frac{j}{\sigma}\right)\hat{x} = -\frac{j^2}{\sigma} x \, \hat{x}}$$

and for $x > a$

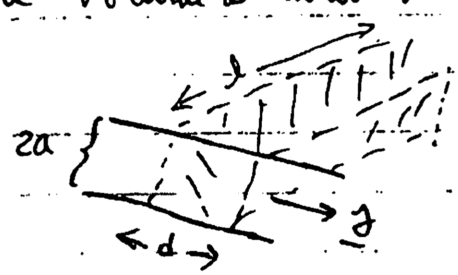
$$\cancel{\underline{S} = jb\left[-\frac{j(-z)}{\sigma(b-a)}\hat{x} + \ldots\right]}$$

$$\underline{S} = ja\left[-\hat{x}\,\frac{j(b-x)}{\sigma(b-a)} + \hat{z}\,\frac{(j)(-z)}{\sigma(b-a)}\right]$$

$$\boxed{\underline{S} = \frac{j^2 a}{\sigma(b-a)}\left[-(b-x)\hat{x} - z\hat{z}\right]}$$

c). Consider a Volume element within the current sheet

$V = (2a)(l)(d)$



~~inside this ... is Power absorbed~~

$$P = (\underline{E}\cdot\underline{j})V = \left(\frac{j}{\sigma}\right) j (2a)(l)(d)$$

but $l\,d =$ surface area A $\Rightarrow$ $\dfrac{P}{A} = \dfrac{j^2 2a}{\sigma}$

but Power comes in from 2-sides $\Rightarrow$ $\boxed{S_\perp(|x| = a) = \dfrac{j^2 a}{\sigma}}$ $\leftarrow$ agrees!