Consider a system of three spin-1/2 moments, $\vec{S}_1, \vec{S}_2, \vec{S}_3$. The permutation operator P_{12} exchanges spin 1 and 2:

$$P_{12}\,|\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_1, \sigma_3\rangle$$

where $\sigma_{1,2,3} = \pm 1/2$ are the eigenvalues of S_1^z, S_2^z, S_3^z. The permutation operator P_{123} performs a cyclic permutation on spins 1, 2, and 3 so that $2\to 1$, $1\to 3$, $3\to 2$

$$P_{123}\,|\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_3, \sigma_1\rangle$$

(a) Express P_{12} in terms of the spin operators $\vec{S}_1, \vec{S}_2$.

(b) Express P_{123} in terms of the spin operators $\vec{S}_1, \vec{S}_2, \vec{S}_3$.

See Prof. Chakravarty's lecture notes p. 38–41:

- $\uparrow = \begin{pmatrix}1\\0\end{pmatrix}$; $\downarrow = \begin{pmatrix}0\\1\end{pmatrix}$
- $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$; $\sigma_x \uparrow = \downarrow$; $\sigma_x \downarrow = \uparrow$
- $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$; $\sigma_y \uparrow = i\downarrow$; $\sigma_y \downarrow = -i\uparrow$
- $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$; $\sigma_z \uparrow = \uparrow$; $\sigma_z \downarrow = -\downarrow$

so $\vec{\sigma}_1\cdot\vec{\sigma}_2 = (\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2)$

and $\vec{\sigma}_1\cdot\vec{\sigma}_2 \uparrow\uparrow = (\cancel{\downarrow\downarrow} + \underbrace{(i\downarrow)(i\downarrow)}_{-\cancel{\downarrow\downarrow}} + \uparrow\uparrow) = \uparrow\uparrow$

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 \uparrow\downarrow = (\downarrow\uparrow + \underbrace{(i\downarrow)(-i\uparrow)}_{\downarrow\uparrow} + \uparrow(-\downarrow)) = 2\downarrow\uparrow - \uparrow\downarrow$$

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 \downarrow\uparrow = (\uparrow\downarrow + \underbrace{(-i\uparrow)(i\downarrow)}_{\uparrow\downarrow} - \downarrow\uparrow) = 2\uparrow\downarrow - \downarrow\uparrow$$

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 \downarrow\downarrow = (\cancel{\uparrow\uparrow} + \underbrace{(-i\uparrow)(-i\uparrow)}_{-\cancel{\uparrow\uparrow}} + (-\downarrow)(-\downarrow)) = \downarrow\downarrow$$

In summary:

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 \begin{Bmatrix} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{Bmatrix} = \begin{Bmatrix} \uparrow\uparrow \\ 2\downarrow\uparrow - \uparrow\downarrow \\ 2\uparrow\downarrow - \downarrow\uparrow \\ \downarrow\downarrow \end{Bmatrix}$$

Try $\underline{P_{12} = \frac{1}{2}\left(\mathbb{1} + \vec{\sigma}_1\cdot\vec{\sigma}_2\right) = \frac{1}{2}\left(\mathbb{1} + \frac{1}{4}\vec{S}_1\cdot\vec{S}_2\right)}$ or $2\left(\frac{\mathbb{1}}{4} + \vec{S}_1\cdot\vec{S}_2\right)$

$$P_{12}\ \uparrow\uparrow = \frac{1}{2}\left(\uparrow\uparrow + \uparrow\uparrow\right) = \uparrow\uparrow$$

$$P_{12}\ \uparrow\downarrow = \frac{1}{2}\left(\uparrow\downarrow + 2\downarrow\uparrow - \uparrow\downarrow\right) = \downarrow\uparrow$$

$$P_{12}\ \downarrow\uparrow = \frac{1}{2}\left(\downarrow\uparrow + 2\uparrow\downarrow - \downarrow\uparrow\right) = \uparrow\downarrow$$

$$P_{12}\ \downarrow\downarrow = \frac{1}{2}\left(\downarrow\downarrow + \downarrow\downarrow\right) = \downarrow\downarrow$$

(b) The effect of P_{123} can be gotten by applying P_{12} and then P_{23} on $|\sigma_1,\sigma_2,\sigma_3\rangle$

$$P_{12}\,|\overset{1}{\sigma_1},\overset{2}{\sigma_2},\overset{3}{\sigma_3}\rangle = |\overset{1}{\sigma_2},\overset{2}{\sigma_1},\overset{3}{\sigma_3}\rangle \Rightarrow P_{23}\,|\overset{1}{\sigma_2},\overset{2}{\sigma_1},\overset{3}{\sigma_3}\rangle = |\overset{1}{\sigma_2},\overset{2}{\sigma_3},\overset{3}{\sigma_1}\rangle$$

which is what $P_{123}|\sigma_1,\sigma_2,\sigma_3\rangle = |\sigma_2,\sigma_3,\sigma_1\rangle$

so $\underline{P_{123} = P_{23}P_{12} = \frac{1}{4}\left(\mathbb{1} + \frac{1}{4}\vec{S}_2\cdot\vec{S}_3\right)\left(\mathbb{1} + \frac{1}{4}\vec{S}_1\cdot\vec{S}_2\right)}$

or $4\left(\frac{\mathbb{1}}{4} + \vec{S}_2\cdot\vec{S}_3\right)\left(\frac{\mathbb{1}}{4} + \vec{S}_1\cdot\vec{S}_2\right)$

p_z dispersions are degenerate, and they disperse the same way as the s state (Why?). Of course, atoms could contain both s- and p-orbitals, in which case we have to include them both in our model. These states can also mix to form a more complex dispersion.

The generalization to three dimensions is simple. The equation for the amplitudes are

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}C(x,y.z,t) = E_0 C(x,y.z,t) &- A_x C(x+b,y.z,t) - A_x C(x-b,y.z,t) \\ &- A_y C(x,y+b,z,t) - A_y C(x,y-b,z,t) \\ &- A_z C(x,y,z+b,t) - A_y C(x,y,z-b,t), \end{aligned} \tag{1.172}$$

where we have assumed a cubic lattice with a lattice spacing of b, but have assumed for generality that the matrix elements are different for the electron hopping in different directions. The energy spectrum is given by

$$E_k = E_0 - 2A_x \cos k_x b - 2A_y \cos k_y b - 2A_z \cos k_z b, \tag{1.173}$$

while the amplitudes are given by

$$C(x,y,z,t) = e^{-E_k t/\hbar} e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{1.174}$$

1.5.2 Spin Waves

A magnetic Hamiltonian that can describe ferromagnetism is the ferromagnetic spin-1/2 Heisenberg model, where the nearest spins interact via a spin-spin interaction

$$H = -J\sum_n \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}. \tag{1.175}$$

For simplicity, I have absorbed the factor $(\hbar/2)^2$ in the coupling J, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector made of the Pauli matrices. The Hamiltonian is for a one-dimensional chain of spins, but you can easily generalize it to higher dimensions. First, define the raising and the lowering operators

$$\sigma_n^+ = \frac{\sigma_n^x + i\sigma_n^y}{2} \tag{1.176}$$

$$\sigma_n^- = (\sigma_n^+)^\dagger = \frac{\sigma_n^x - i\sigma_n^y}{2}. \tag{1.177}$$

Remember that the Pauli matrices are Hermitian and that $\sigma^+|+\rangle = 0$, $\sigma^+|-\rangle = |+\rangle$, $\sigma^-|-\rangle = 0$, and $\sigma^-|+\rangle = |-\rangle$. Now, the interaction for a pair of spins can be written as

$$\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1} = 2[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \sigma_n^z \sigma_{n+1}^z, \tag{1.178}$$

where we have used the fact that the Pauli matrices belonging to distinct sites commute. The interaction can also be written in terms of a permutation operator $P_{n,n+1}$ that permutes the spins on the sites n and $n+1$. To check this, define the kets for two spins as $|\pm, \pm\rangle$, where the first entry is for the first spin and the second entry is for the second spin. Then,

$$\left(2[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \sigma_n^z \sigma_{n+1}^z\right) |{+}{+}\rangle = |{+}{+}\rangle \tag{1.179}$$
$$\left(2[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \sigma_n^z \sigma_{n+1}^z\right) |{-}{-}\rangle = |{-}{-}\rangle \tag{1.180}$$
$$\left(2[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \sigma_n^z \sigma_{n+1}^z\right) |{+}{-}\rangle = 2\,|{-}{+}\rangle - |{+}{-}\rangle\,, \tag{1.181}$$
$$\left(2[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \sigma_n^z \sigma_{n+1}^z\right) |{-}{+}\rangle = 2\,|{+}{-}\rangle - |{-}{+}\rangle\,. \tag{1.182}$$

Therefore, as announced earlier,

$$\boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1} = 2P_{n,n+1} - 1. \tag{1.183}$$

What is the ground state of the ferromagnetic Heisenberg model? Since the coupling constant J, also called the exchange constant, is positive, a pair of nearest neighbor spins like to be parallel to the each other. So, perhaps, the groundstate is that state in which they are all lined up parallel to each other. This is clearly an infinitely degenerate state because it does not matter which direction in space they point as long as they are parallel to each other. Let us check that the assumed state is the lowest energy state. Note that the Hamiltonian acting on the presumed ground state is

$$-J \sum_n (2P_{n,n+1} - 1)\, |{+}{+}{+}\ldots\rangle = -JN\, |{+}{+}{+}\ldots\rangle\,. \tag{1.184}$$

The state $|{+}{+}{+}\ldots\rangle$ is definitely an eigenstate; physically it is clear that it is also the lowest energy state, but, with a little bit more effort, you can also show that there are no other eigenstates of energy lower than $-JN$, where N is the total number of spins in the lattice. As the temperature is raised, thermal fluctuations will create excited states, which will disorder the spins. There will be a temeperature T_c at which the system will loose its average magnetization and a phase transition will take place. It can be rigorously shown that $T_c = 0$ for dimensions $d \le 2$, but it is finite at $d = 3$. This proof is slightly off our track, so I won't give it to you here.

What do the excited states look like? Let us redefine the zero of energy by subtracting the ground state energy, so that

$$H - E_0 = -2J \sum_n (P_{n,n+1} - 1). \tag{1.185}$$

It is easy to guess that the first excited state would be one where one of the spins is flipped. We need to invent a nice notation to denote this. For example, if the $4th$ spin is flipped, we will label that state as

$$|x_4\rangle = |+ + + - + + + \ldots\rangle \,. \tag{1.186}$$

What is the action of the Hamiltonian on this state? If the permutation operator does not involve the $4th$ spin, the state is unchanged. If it involves the $4th$ spin, it will either permute it with the spin on the right, or on the left, so that

$$P_{34}\,|x_4\rangle = |x_3\rangle \,, \tag{1.187}$$
$$P_{45}\,|x_4\rangle = |x_5\rangle \,. \tag{1.188}$$

The terms in the Hamiltonian that survive are

$$[-2J(P_{34} - 1) - 2J(P_{45} - 1)]\,|x_4\rangle = 4J\,|x_4\rangle - 2J\,|x_3\rangle - 2J\,|x_5\rangle \,. \tag{1.189}$$

In general,

$$H\,|x_n\rangle = 4J\,|x_n\rangle - 2J\,|x_{n+1}\rangle - 2J\,|x_{n-1}\rangle \,. \tag{1.190}$$

This is identical to the problem we solved for an electron in a periodic lattice. The schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}C_n(t) = \sum_{n'} \langle n|H|n'\rangle\, C_{n'}(t), \tag{1.191}$$

where the only matrix elements of the Hamiltonian are

$$H_{n,n} = 4J, \tag{1.192}$$
$$H_{n,n+1} = H_{n-1,n} = -2J. \tag{1.193}$$

The set of linear difference equations can once again be solved by the choice

$$C_n(t) = \frac{1}{\sqrt{N}}e^{-ikx_n}e^{-iEt/\hbar}. \tag{1.194}$$

Then, the energy spectrum is given by

$$E_k = 4J(1 - \cos kb). \tag{1.195}$$

The definite energy solutions correspond to waves of a flipped spin whose amplitude at a given site n is determined by the wavevector k lying within the first Brillouin zone between $\frac{-\pi}{b}$ and $\frac{\pi}{b}$. The energy dispersion at long wavelengths is that of a free particle, a magnon, of an effective mass $m_{\text{eff}} = \hbar^2/(4Jb^2)$.

Once we start examining the problem of two flipped spins, we discover that the spin waves interact when they approach each other. The interaction may in fact give rise to bound states. Although the two spin wave problem can still be solved eaxctly with some effort, we may argue that if there is a small density of such excited states, or spin waves, at low temperatures, they can be approximated to be independent. Such an independent particle approximation reproduces many low temperature properties of ferromagnets. In the independent particle approximation, the excited state energy $\varepsilon(k_1, k_2, \ldots)$ is then given by

$$\varepsilon(k_1, k_2, \ldots) \approx E_{k_1} + E_{k_2} + \ldots \tag{1.196}$$

Spr. 2001 #1:

$\vec{S}_1, \vec{S}_2, \vec{S}_3$ all spin $\frac{1}{2}$ $P_{12}|\sigma_1,\sigma_2,\sigma_3\rangle = |\sigma_2,\sigma_1,\sigma_3\rangle$

$P_{123}|\sigma_1,\sigma_2,\sigma_3\rangle = |\sigma_2,\sigma_3,\sigma_1\rangle$

$\sigma_{1,2,3} = \pm\frac{1}{2}$ are the eigenvalues of S_1^z, S_2^z, S_3^z

(a) express P_{12} in terms of S_1 & S_2

recall the old trick $S^2 = S_1^2 + 2S_1\cdot S_2 + S_2^2$

$2\vec{S}_1\cdot\vec{S}_2 = 2S_1^z S_2^z + S_1^+ S_2^- + S_1^- S_2^+$

the effect of this operator on $|\sigma_1,\sigma_2,\sigma_3\rangle$ is

(I) $2\vec{S}_1\cdot\vec{S}_2|\sigma_1,\sigma_2,\sigma_3\rangle = 2S_1^z S_2^z|\sigma_1,\sigma_2,\sigma_3\rangle = 2S_1^z S_2^z|\sigma_2,\sigma_1,\sigma_3\rangle = \frac{1}{2}|\sigma_2,\sigma_1,\sigma_3\rangle$, if $\sigma_1=\sigma_2$

(II) $2\vec{S}_1\cdot\vec{S}_2|\sigma_1,\sigma_2,\sigma_3\rangle = -\frac{1}{2}|\sigma_1,\sigma_2,\sigma_3\rangle + |\sigma_2,\sigma_1,\sigma_3\rangle$ if $\sigma_1 \neq \sigma_2$

in (I), can write $\frac{1}{2}|\sigma_2,\sigma_1,\sigma_3\rangle = |\sigma_2,\sigma_1,\sigma_3\rangle - \frac{1}{2}|\sigma_1,\sigma_2,\sigma_3\rangle$

so the expression in (II) always works

then have $|\sigma_2,\sigma_1,\sigma_3\rangle = P_{12}|\sigma_1,\sigma_2,\sigma_3\rangle$, so

$2\vec{S}_1\cdot\vec{S}_2 = -\frac{1}{2} + P_{12} \Rightarrow \boxed{P_{12} = \frac{1}{2} + 2\vec{S}_1\cdot\vec{S}_2}$

(b) express P_{123} as S_1, S_2, & S_3

write $P_{123} = P_{12}P_{13} = \left(\frac{1}{2}+2\vec{S}_1\cdot\vec{S}_2\right)\left(\frac{1}{2}+2\vec{S}_1\cdot\vec{S}_3\right) =$

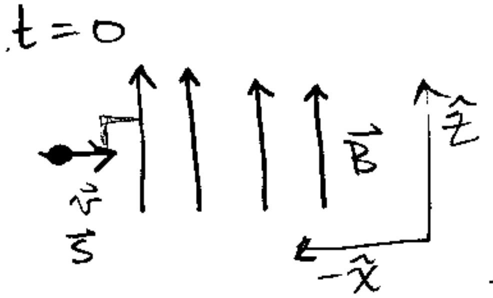
$\frac{1}{4} + \vec{S}_1\cdot\vec{S}_3 + \vec{S}_1\cdot\vec{S}_2 + 4(\vec{S}_1\cdot\vec{S}_2)(\vec{S}_1\cdot\vec{S}_3)$

Lea Fredrickson

Solution Problem # 2

Spring 2001 Comp

An electron is injected into a region where there is a constant magnetic field $|\vec{B}|$.



Let Θ be the angle between the electron momentum and the expectation value of its spin.

At $t=0$, $\Theta=0$ what is $\Theta(t)$? [Calculate the time-dependence of the momentum classically.] Express your answer in terms of the gyromagnetic ratio g of the electron. Leave g arbitrary – don't set it exactly equal to 2. At $t=0$, the direction of the electron's motion is perpendicular to the magnetic field, and it is completely polarized so that its spin is definitely along the direction of the beam.

To solve for the momentum of the electron,

$$\frac{d\vec{p}}{dt} = \vec{F} = q\frac{\vec{v}}{c}\times\vec{B} = m\frac{d\vec{v}}{dt}$$

$$m\frac{dv_x}{dt} = q\left(\frac{\vec{v}}{c}\times\vec{B}\right)_x = -q\frac{v_y B}{c} = e\frac{v_y B}{c}$$

$$m\frac{dv_y}{dt} = q\left(\frac{\vec{v}}{c}\times\vec{B}\right)_y = q\frac{v_x B}{c} = -e\frac{v_x B}{c}$$

$$m\frac{dv_z}{dt} = q\left(\frac{\vec{v}}{c}\times\vec{B}\right)_z = 0$$

$\Rightarrow \quad v_z = C_z = 0$

and we have two coupled differential equations

Differentiate them to uncouple them

$$m\frac{d^2 v_x}{dt^2} = eB\frac{dv_y}{dt} \qquad m\frac{d^2 v_y}{dt^2} = -eB\frac{dv_x}{dt}$$

$\Rightarrow$

$$\begin{cases} \dfrac{d^2 v_x}{dt^2} = -\dfrac{e^2B^2}{c^2m^2}v_x, & v_x(t) = A\cos\left(\frac{eB}{cm}t\right) + B\sin\left(\frac{eB}{cm}t\right) \\[2ex] \dfrac{d^2 v_y}{dt^2} = -\dfrac{e^2B^2}{c^2m^2}v_y, & v_y(t) = C\cos\left(\frac{eB}{cm}t\right) + D\sin\left(\frac{eB}{cm}t\right) \end{cases}$$

at $t=0$, all of the velocity is in the x direction

Therefore $\quad B = C = 0$

$$v_x(t) = A\cos\left(\frac{eB}{cm}t\right), \qquad v_y(t) = D\sin\left(\frac{eB}{cm}t\right)$$

let the magnitude of the initial velocity be v_i

$$\vec{v}(t) = v_i\left(\cos\left(\frac{eB}{cm}t\right)\hat{x} - \sin\left(\frac{eB}{cm}t\right)\hat{y}\right)$$

$$\vec{p}(t) = mv_i\left(\cos\left(\frac{eB}{cm}t\right)\hat{x} - \sin\left(\frac{eB}{cm}t\right)\hat{y}\right)$$

Therefore the angle that the momentum vector makes with the x axis is $\boxed{\theta_p = \frac{eBt}{mc}}$

Now to solve for the angle that the spin makes

The Hamiltonian is $H = -\vec{\mu}_s \cdot \vec{B}$

$$\vec{\mu}_s = -\frac{eg}{2mc}\vec{S}$$

$$\vec{H} = \frac{eg}{2mc}\vec{S}\cdot\vec{B} = \frac{eg}{2mc} B S_z$$

The initial state of the spin is

$$|\psi_0\rangle = |+x\rangle = \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right)$$

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle$$

$$= e^{-\frac{iegBt}{2mc\hbar}S_z}\left\{\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right]\right\}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left\{e^{-\frac{iegBt}{4mc}}|+z\rangle + e^{\frac{iegBt}{4mc}}|-z\rangle\right\}$$

Now to find the angle w.r.t. the $+x$ axis

$$\langle +x|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\langle +z| + \langle -z|\right]|\psi(t)\rangle$$

$$\langle +x|\Psi(t)\rangle = \frac{1}{2}\left(e^{-\frac{iegBt}{4mc}} + e^{\frac{iegBt}{4mc}}\right)$$

$$\left|\langle +x|\Psi(t)\rangle\right|^2 = \cos^2\left(\frac{egBt}{4mc}\right)$$

One might think that we now have the angle of precession, but there is a catch. Remember that the spin must rotate around twice to get back to the origional state.

$$\Rightarrow \left[\theta_s = \frac{egBt}{2mc}\right] = \omega_0 t$$

another way to see this is to take the expectation value of the spin

$$\langle\Psi(t)|\hat{S}_x|\Psi(t)\rangle = \langle\hat{S}_x\rangle$$

$$\langle\hat{S}_x\rangle = \frac{1}{\sqrt{2}}\left(e^{i\omega_0 t/2}, e^{-i\omega_0 t/2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2}\\ e^{i\omega_0 t/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\cos\omega_0 t \qquad \Rightarrow \theta_s = \omega_0 t = \frac{egBt}{2mc}$$

$$\Rightarrow \theta = \theta_s - \theta_p = \left(\frac{g}{2} - 1\right)\frac{eBt}{2mc} = (g-2)\frac{\omega_0 t}{2}$$

$$\boxed{\theta = (g-2)\frac{\omega_0 t}{2}}$$

Spring 2001 #3

$$V(r) = -V_0 \frac{e^{-\mu r}}{\mu r} \qquad U(r) = 2m V(r) = 2m V_0 \frac{e^{-\mu r}}{\mu r}$$

Then from abers eqn 8.33

$$f(\theta,\phi) = -\frac{(2\pi)^{3/2}}{4\pi} \int \underbrace{e^{-i k' \cdot r'}}_{\phi(r')} U(r') \psi(r') d^3 r' \qquad \vec{k}' = \frac{k \vec{r}}{r} \qquad \star$$

$$\boxed{= -\frac{(2\pi)^{3/2}}{4\pi} \int e^{-i k' \cdot r'} 2mV_0 \frac{e^{-\mu r'}}{\mu r} \psi(r) d^3 r}$$

b) First born approximation $\psi(r) \to \phi(r) = \frac{e^{i k \cdot r}}{(2\pi)^{3/2}}$

for spherically symetric cases

$$f^{(1)}(\theta,\phi) = -\frac{1}{4\pi} \int e^{i(k-k')\cdot r} U(r) d^3 r \qquad \vec{q} = \vec{k} - \vec{k}'$$

$$q = 2k \sin\frac{\theta}{2}$$

$$\vec{q}\cdot\vec{r} = qr\cos\theta$$

$$\star \quad = -\frac{1}{q}\int_0^\infty r \sin(qr) U(r) dr$$

$$= \frac{1}{q}\int_0^\infty r \sin(qr) 2mV_0 \frac{e^{-\mu r}}{\mu r} dr = \frac{-2mV_0}{q\mu}\int_0^\infty \sin(qr) e^{-\mu r} dr$$

$$\int_0^\infty e^{-\mu x} \sin kx \, dx = \frac{k}{\mu^2 + k^2}$$

$$\Rightarrow = \frac{2mV_0}{\cancel{q}\mu} \frac{\cancel{q}}{\mu^2 + q^2} = \frac{2mV_0}{\mu} \frac{1}{q^2 + \mu^2}$$

c) $\sigma_{tot} = \int |f(\cos\theta)|^2 \, d\Omega$

$$= \frac{4m^2V_0^2}{\mu^2} \int \frac{1}{(q^2+\mu^2)^2} \, d\Omega$$

Scattering nectron having 0 kinetic energy means

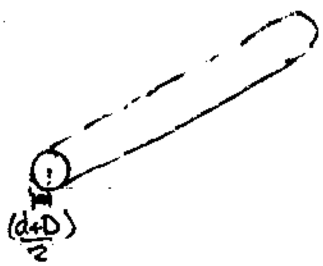
$$k \to 0 \;, \Rightarrow k' \to 0 \Rightarrow q \to 0$$

$$\sigma_{tot} = \frac{4m^2V_0^2}{\mu^2} \frac{1}{\mu^4} \cdot 4\pi = \frac{16\pi m^2 V_0^2}{\mu^6} \quad \text{in } k \to 0 \text{ case}$$

Spring 2001 #6

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D. The molecules have an average diameter d. The gas has a temperature T.

- Collisions will occur when centers of 2 molecules within $d+D$ of each other
  - $\Rightarrow$ equivilant to one particle of radius $d+D$ colliding w/ pt. particles (collisions just as likely)



$\leftarrow$ cylinder carved out as particle travels

$$\frac{V}{N} \approx 4\pi\left(\frac{d+D}{2}\right)^2 \ell$$

average volume/molecule

volume of cylinder

$\rightarrow \ell$ mean free path btwn collisions

$$\Rightarrow \ell \approx \frac{4}{4\pi(d+D)^2 n} \qquad n = N/V$$

- Now, $\nu = \dfrac{\bar{V}_{avg}}{\ell}$

  $\rightarrow$ need $\bar{V}_{avg}$

* This is not V_{rms} though! V_{rms} comes from the average of the squares of the velocities of the particles $V_{rms} = \sqrt{\overline{v^2}}$.

$\Rightarrow \bar{V}_{avg}$ (or average of the velocities not squared) will be more accurate here

Maxwell Dist. function $\quad D(v) = \left(\dfrac{m}{2\pi kT}\right)^{3/2} \underbrace{4\pi v^2}\, \underbrace{e^{-mv^2/2kT}}$ $\quad$ [units $(\frac{m}{s})^{-1}$]

$D(v) \propto$ (# of vectors $\vec{v}$ corresponding to speed v) $\cdot$ (probability of a molecule having $\vec{v}$)

$\rightarrow \left(\frac{m}{2\pi kT}\right)^{3/2}$ comes from normalization cond'n

$$1 = \int_0^\infty D(v)\,dv \qquad \left(\text{total prob} = 1\right)$$

· Now $\bar{v}_{avg}$ given by:

$$\bar{v} = \sum_{\text{all } v} v\, D(v)\,dv \overset{\text{or}}{=} \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^3 e^{-mv^2/2kT}\,dv$$

set $x = \frac{mv^2}{2kT}$, $dx = \frac{2mv\,dv}{2kT}$ so $dv = \frac{kT}{mv}dx$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right)^2 \underbrace{\int_0^\infty x^2 e^{-x}\,dx}$$

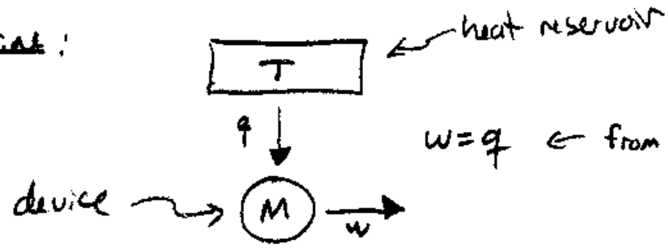
$$= \cancel{-x^2 e^{-x}}\Big|_0^\infty + 2\int x e^{-x}\,dx = \cancel{-2x e^{-x}}\Big|_0^\infty + 2\int_0^\infty e^{-x} = -2e^{-x}\Big|_0^\infty = 2$$

$$= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\frac{kT}{m}\right)^2 \cdot 2 = \sqrt{\frac{8kT}{m\pi}}$$

$$\rightarrow \nu = \sqrt{\frac{8kT}{m\pi}} \cdot \frac{\pi}{4}(d+D)^2 n = (d+D)^2 n \sqrt{\frac{kT\pi}{2m}} \quad ✓$$

(a) You are asked about the 2^{nd} law of thermodynamics, and you give one of the formulations, that there is no process the sole effect of which is the conversion of heat into work. The inquirer then points out that a steam engine converts heat into work. Explain how this is not a violation of the 2^{nd} law of thermodynamics. Your explanation should include an analysis of the steam engine, and a discussion of heat engines in general. (See Reif section 5.11)

perfect engine:



$w = q$ ← from 1^{st} law where $\Delta \bar{E} = 0$.

→ working in a cycle, the perfect engine extracts heat from a reservoir and performs an equivalent amount of work without producing any other effect on the environment

→ this is impossible due to the 2^{nd} law of thermodynamics. That is, it would require the spontaneous occurence of a process which goes from an initial situation, where a certain amount of energy is distributed randomly over the many degrees of freedom of a heat reservoir, to a much more special and enormously less probable final situation, where the energy is all associated with the motion of a single degree of freedom capable of performing macroscopic work; in short, because it would require a process where the entropy S decreases.

Reif pp 186-187

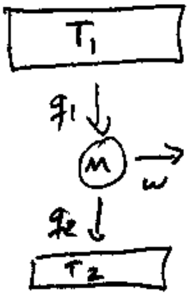
→ the entropy change of the heat reservoir at absolute temperature T_1 is

$$\Delta S_R = \frac{\Delta Q}{T_1} = \frac{-q}{T_1}$$

but we know that $\Delta S \geq 0$ (2^{nd} law), where ΔS is the total entropy change and

$$\Delta S = \Delta S_R + \Delta S_M$$

where $\Delta S_M = 0$ since M is back in the same macrostate after a cycle real engine:

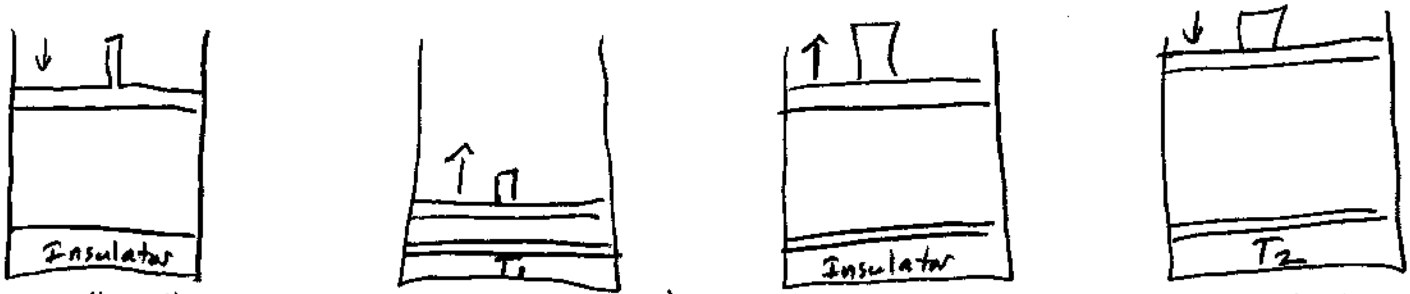


$w = q_1 - q_2$

→ Now, $\Delta S = \frac{-q_1}{T_1} + \frac{q_2}{T_2} \geq 0$ can be satisfied with positive work performed by the engine on the outside world. The efficiency of the engine is given by

$$\eta \equiv \frac{w}{q_1} = \frac{q_1 - q_2}{q_1} < 1$$ as since some heat does not get transformed into work but is instead rejected to some other heat reservoir.

consider a Carnot Engine:



engine is thermally isolated. Its external parameters changed slowly until the engine temperature reaches T_1

engine now in thermal contact with the heat reservoir at T_1. Engine absorbs heat q_1 from reservoir

engine again thermally insulated Temperature goes back to T_2

engine back in thermal contact w/ heat reservoir at $T = T_2$ and rejects heat q_2 into the reservoir (Now engine back in original state)

→ A steam engine is more complicated than a Carnot engine. In a steam engine the two reservoirs are represented by a boiler and a condenser.

→ The above explanation of real engines illustrates how a heat engine converts heat into work without violating the 2nd law of thermodynamics.

(b) You read an article in a physics journal in which a group of researchers announce that they have cooled a system to absolute zero. Discuss why one ought to be skeptical of this claim. Invoke the appropriate laws of thermodynamics. (Ref section 5.7)

3 laws of thermodynamics are:

1st law – $\Delta E = -W + Q$

2nd law – $dS = \frac{dQ}{T}$, $\Delta S \geq 0$

3rd law – as $T \to 0_+$, $S \to S_0$

→ The basic idea is that the 3rd law states that at $T=0$, $S=0$ (usual convention is to call the constant zero). This means that each element's entropy at $T=0$ is 0. So all the elements at $T>0$ have a finite, positive entropy.

→ But, the second law tells us that no system can reach absolute zero because entropy cannot be reduced to zero by finite means

→ Absolute zero was 1st calculated using the ideal gas law and can be defined as the temperature at which an ideal gas has no volume and exerts no pressure. ... but an ideal gas does not exist. A real gas will liquefy befor attaing absolute zero.

→ The same principle that tell's us no system may be 100% efficient, tells us the temperature can never be exactly absolute zero. Implies a definite position and velocity of a particle which violates uncertainty principle.

→ See the Nernst heat theorem: if one could reach absolute zero, all bodies would have the same entropy. Absolute zero can exist only in one possible state, which would possess a definite energy (zero point energy)

Summary:

2nd law states suggests the existence of an absolute temperature scale with absolute zero on it. 3rd law states that absolute zero cannot be reached in a finite number of steps. That is, 2nd law states that heat can never spontaneously move from a colder body to a hotter body. So, as a system approaches absolute zero, it will eventually have to draw energy from whatever systems are nearby. If it draws energy, it can never obtain absolute zero.

→ to make something cooler, you must put it in an environment that is colder than it. So, heat will flow from the object to the environment. No environment can be colder than absolute zero .... So, no object put in it can reach it either.

→ it would require an infinite amount of cycles from the device to reach absolute zero.

(c) Explain, using the laws of thermodynamics, why a substance cannot have negative heat capacity.

→ heat capacity of a system is defined as the ratio

$$c_y \equiv \left(\frac{đQ}{dT}\right)_y$$

in the limit as $đQ \to 0$ (or $dT \to 0$). Since $dS = \frac{đQ}{T}$, we have

$$C_y = T\left(\frac{dS}{dT}\right)_y$$

From the 2nd Law, we know that the entropy of a system must increase if irreversible or stay the same if reversible. So there is no way $\frac{dS}{dT} < 0$. But, this questions asks about a substance, not a system. um... you can get $\frac{\Delta S}{\Delta T} < 0$ if $\Delta T < 0$...

→ Heat capacity is also defined (in words) as the amount of heat required to raise unit mass of substance by one degree of temperature.

So, a negative heat capacity would imply that a substance gave off heat in order to raise its temperature. This would imply that $dQ < 0$. That, in turn, would imply that $dS < 0$ from the relation

$$dS = \frac{dQ}{T}.$$

assuming, of course, that $T > 0$ (which seems reasonable on the absolute scale ☺).

If $dS < 0$, then the 2nd law is violated... oh crap, I already said this in the previous part. I still have the problem that the substance is not necessarily the system. If energy is taken from the "system", the 2nd law is not violated.

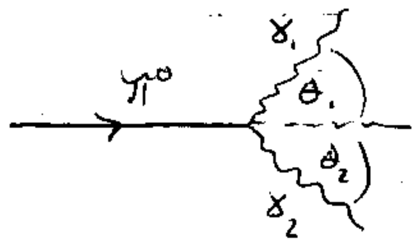
→ I can only assume that heat capacity is defined such that it treats the substance as the system. I cannot confirm this, however.

→ negative heat capacity means that if you put a hot thing by a cold thing than heat would go to the cold thing. As the change in temperature increases, then the cold thing would approach absolute value faster... then it seems reaching absolute zero is possible. We concluded this is impossible from part (b).

Spring 2001 # 10

A π^0 of velocity v_0 decays in flight into two photons $\pi^0 \to 2\gamma$. Compute the minimum & maximum values of the energies of the produced photons as a function of v_0.

In Lab frame:



- energy conservation: $\gamma m_\pi c^2 = E_1 + E_2$ (I) where 1, 2 refers to γ_1 & γ_2

- momentum conservation: $\gamma m_\pi v_0 = \frac{E_1}{c}\cos\theta_1 + \frac{E_2}{c}\cos\theta_2$ (II)

$$\frac{E_1}{c}\sin\theta_1 = \frac{E_2}{c}\sin\theta_2 \qquad \text{(III)}$$

$$\left(w/ \quad m_\gamma = 0 \qquad E_\gamma = P_\gamma c\right)$$

(II): $\left(\gamma m v_0 - \frac{E_1}{c}\cos\theta_1\right)^2 = \left(\frac{E_2}{c}\cos\theta_2\right)^2$

$$= \left(\frac{E_2}{c}\right)^2 - \left(\frac{E_2}{c}\sin\theta_2\right)^2$$

$$\left(\gamma m v_0 - \frac{E_1}{c}\cos\theta_1\right)^2 + \left(\frac{E_1}{c}\sin\theta_1\right)^2 = \left(\frac{E_2}{c}\right)^2$$

used (III) $\qquad = \left(\gamma m c - \frac{E_1}{c}\right)^2$

$$\to \gamma^2 m^2 v_0^2 - \frac{2\gamma m v_0 E_1 \cos\theta_1}{c} + \frac{E_1^2}{c^2}\cancel{\cos^2\theta_1} + \frac{E_1^2 \cancel{\sin^2\theta_1}}{c^2} = \gamma^2 m^2 c^2 - 2\gamma m E_1 + \cancel{E_1^2/c}$$

$\rightarrow \gamma^2 m^2 (v^2 - c^2) - 2\gamma m E_1 \left(\frac{v_0}{c} \cos\theta_1 - 1 \right) = 0$

$$\Rightarrow E_1 = \frac{\gamma m (v_0^2 - c^2)}{2\left(\frac{v_0}{c}\cos\theta - 1\right)} = \frac{\gamma m c^2 \left(\frac{v_0^2}{c^2} - 1\right)}{2\left(\frac{v_0}{c}\cos\theta - 1\right)}$$

now w/ $\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}$, ...

$$E_1 = \frac{mc^2}{2\gamma\left(1 - \frac{v_0}{c}\cos\theta_1\right)}$$

- In lab frame $\cos\theta_1$ can range from -1 to 1 corresponding to minimum & maximum energies. Photons can fly backward even in the lab frame because they travel at c and thus cannot be Lorentz boosted to the forward direction when they are travelling very-backward in the CM frame.

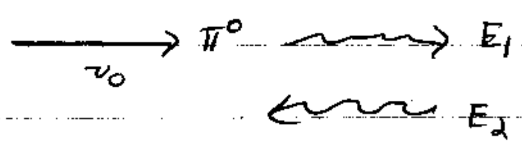
$$\rightarrow E_{max} = \frac{mc^2}{2\gamma\left(1 - \frac{v_0}{c}\right)} \quad \& \quad E_{min} = \frac{mc^2}{2\gamma\left(1 + \frac{v_0}{c}\right)}$$

$\rightarrow$ similar expressions can be derived for the min & max energies of the 2nd photon

A π^0 of velocity v_0 decays in flight into two photons $\pi^0 \to 2\gamma$. Compute the minimum and maximum values of the energies of the produced photons as a function of v_0.

Three ways of doing it:

(1) If you can accept that the maximum energy comes from the photon traveling in the same direction as the π^0 and the minimum energy when traveling in the exact opposite direction:



then from energy conservation & momentum conservation

$$E_{\pi^0} = E_1 + E_2 \quad ; \quad |\vec{P}_{\pi^0}| = |\vec{p}_1| - |\vec{p}_2| = E_1 - E_2 \quad \text{with } c = 1$$

combining $\quad E_{\pi^0} + |\vec{P}_{\pi^0}| = 2E_1$

so $\qquad$ as $E = \gamma m \,;\, |p| = \gamma\beta m$

$$E_1 = \frac{1}{2}\left(E_{\pi^0} + |\vec{P}_{\pi^0}|\right) = \frac{1}{2}\left(\gamma m_{\pi^0} + \gamma\beta m_{\pi^0}\right)$$

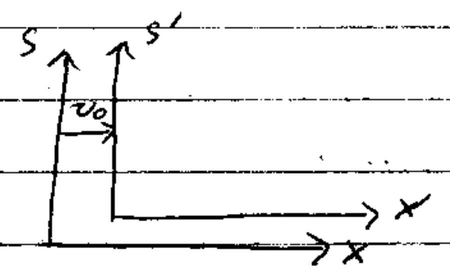
$$= \frac{\gamma m_{\pi^0}}{2}(1+\beta) \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

and the minimum energy would then be:

$$E_2 = \frac{\gamma m_{\pi^0}}{2}(1-\beta)$$

(2) Can be done via a Lorentz transformation from the rest frame of the π^0:

Just like
$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta c t')$$
$$y = y'$$
$$z = z'$$



a similar thing can be done with the energy - momentum four vector:

$$E/c = \gamma(E'/c + \beta p_x') = \gamma mc \Rightarrow E = \gamma mc^2$$
$$p_x = \gamma(p_x' + \beta E'/c) = \gamma\beta mc \rightarrow p_x = \gamma\beta mc^2$$

now in S' (π^0 rest frame) from momentum conservation $E_1' = \frac{M}{2} = p_{1,x}'$ (M of π^0)
$E_2' = \frac{M}{2} = -p_{2,x}'$. Then going into S (moving at v_0):

$$E_1 = \gamma(E_1' + \beta p_1') = \frac{M}{2}\gamma(1+\beta)$$
$$E_2 = \gamma(E_2' + \beta p_2') = \frac{M}{2}\gamma(1-\beta)$$

(3) $P = p_1 + p_2$ (4- vectors)   depends on metric used

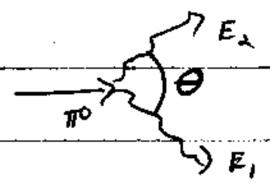
$$P^2 = M^2 = \cancel{p_1^2}^{\,0} + \cancel{p_2^2}^{\,0} + 2p_1 \cdot p_2 = 2\left[E_1E_2 \mp 2\,\underbrace{\vec{p}_1\cdot\vec{p}_2}_{E_1E_2\cos\theta}\right]$$

as $m_\gamma = 0$

so $M^2 = 2E_1E_2(1-\cos\theta) \Rightarrow E_1E_2 = \frac{M^2}{2(1-\cos\theta)}$ $\theta = 0$ $E_1E_2 = \infty$

$\theta = \pi$ $E_1E_2 = \frac{M^2}{2}$

from energy conservation



$$E = E_1 + E_2 \qquad E_2 = \frac{1}{E_1}\frac{M^2}{2(1-\cos\theta)}$$
$$= E_1 + \frac{1}{E_1}\frac{M^2}{2(1-\cos\theta)} \qquad ax^2 + bx + c = 0 \qquad x = \frac{-b}{2a} \pm \frac{1}{2}\sqrt{b^2 - 4ac}$$

$$\Rightarrow E_1E = E_1^2 + \frac{M^2}{2(1-\cos\theta)} \Rightarrow E_1^2 - EE_1 + \frac{M^2}{2(1-\cos\theta)} = 0$$

$$\Rightarrow E_1 = -\frac{(-E)}{2} \pm \frac{1}{2}\sqrt{E^2 - \frac{4M^2}{2(1-\cos\theta)}} = \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - \frac{M^2}{2(1-\cos\theta)}} = \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - \frac{M^2}{4}}$$

smallest value $\theta = \pi$

So $E_1 = \frac{E}{2} \pm \frac{1}{2} \underbrace{\sqrt{E^2 - m^2}}_{|\vec{p}|} = \frac{E}{2} \pm \frac{1}{2} |\vec{p}|$

$$= \frac{\gamma m_{\pi^0}}{2} \pm \frac{1}{2} \gamma \beta m_{\pi^0} = \frac{\gamma m_{\pi^0}}{2} (1 \pm \beta)$$

hence the maximum energy is $\frac{\gamma m_{\pi^0}}{2}(1+\beta)$ and minimum $\frac{\gamma m_{\pi^0}}{2}(1-\beta)$

consider the penetration of a magnetic field B into a conducting medium by diffusion and convection.
The medium obeys an Ohm's law of the form $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J}$, where $\vec{E}$ is the electric field, $\vec{J} = -ne\vec{v}$ is the current density of electrons of velocity $\vec{v}$ and number density n, and η is a constant uniform scalar resistivity.

(a) Using Faraday's law and Ampere's law (neglect displacement current) obtain a differential equation for the magnetic field.

Ampère's law with the displacement current is given by

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \qquad (1)$$

apply "$\nabla \times$" to both sides

$$\nabla \times (\nabla \times \vec{B}) = \frac{4\pi}{c}\nabla \times \vec{j} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{E}) \qquad (2)$$

From Faraday's law, we know $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$. So, eq (2) becomes

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \frac{4\pi}{c}(\nabla \times \vec{j}) - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

where we used the vector relation

$$\nabla \times (\nabla \times \vec{C}) = \nabla(\nabla \cdot \vec{C}) - \nabla^2 \vec{C}$$

we can use since dealing w/ cartesian coord.

Since $\nabla \cdot \vec{B} = 0$, we have

$$\boxed{\frac{\partial^2 \vec{B}}{\partial t^2} - c^2 \nabla^2 \vec{B} = 4\pi c\,(\nabla \times \vec{j})} \qquad (3)$$

This is a general result.

For our problem, we are told to ignore the displacement current in eq (1) and are given that $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j}$ is valid. So ignoring the displacement current, eq (3) becomes

$$\nabla^2 \vec{B} = -\frac{4\pi}{c}(\nabla \times \vec{j}) \qquad (4)$$

where we can get an expression for $\nabla \times \vec{j}$ from Ohm's law eq. That is,

$$\eta \nabla \times \vec{j} = \nabla \times \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \qquad (5)$$

substituting eq (5) into eq (4) yields

$$\nabla^2 \vec{B} = -\frac{4\pi}{c\eta} \nabla\times \left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) = \frac{-4\pi}{c\eta}\left[\nabla\times\vec{E} + \nabla\times\left(\frac{\vec{v}}{c}\times\vec{B}\right)\right]$$

From what we are given, we now that $\vec{j} = -ne\vec{v} \Rightarrow \vec{v} = -\frac{1}{ne}\vec{j}$, using this result and Faraday's law, we get

$$\nabla^2\vec{B} = \frac{-4\pi}{c\eta}\left[\frac{-1}{c}\frac{\partial\vec{B}}{\partial t} - \frac{1}{e}\nabla\times\left(\frac{\vec{j}\times\vec{B}}{cn}\right)\right] \qquad (6)$$

Now, from Ampère's law (ignoring the displacement current), we know that

$$\vec{j} = \frac{c}{4\pi}\nabla\times\vec{B}$$

so, making this substitution into eq (6) we get

$$\nabla^2\vec{B} = \frac{+4\pi}{c^2\eta}\frac{\partial\vec{B}}{\partial t} + \frac{4\pi}{c}\frac{\eta}{\eta ec}\frac{c}{4\pi}\nabla\times\left[\frac{(\nabla\times\vec{B})\times\vec{B}}{n}\right]$$

the position of the parentheses is important since $\nabla\times(\vec{B}\times\vec{B}) = 0$. Rearranging terms, we get

$$\boxed{\frac{\partial\vec{B}}{\partial t} - \frac{c^2\eta}{4\pi}\nabla^2\vec{B} = \frac{-1}{e}\nabla\times\left[\frac{(\nabla\times\vec{B})\times\vec{B}}{n}\right]} \qquad (7)$$

(b) Now consider a simple boundary problem: the conducting medium is located in the half-space $x>0$. There exists a density gradient of scale length $L = \frac{n}{\left(\frac{dn}{dy}\right)}$. At $t=0$ a uniform field B_0 along z is applied in the space $x<0$. Write down the differential equation for the field $B_z(x,t)$. Describe which terms describe field diffusion ($t \propto x^2$) and convection ($t\propto x$).

we just want the D.E. for $B_z(x,t)$ so, let $\vec{B} = B_z(x,t)\,\hat{z}$. Then

$$\nabla^2\vec{B} = \frac{\partial^2 B_z}{\partial x^2}\hat{z}$$

and

$$(\nabla\times\vec{B})\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_z(x,t) \end{vmatrix} \times B_z(x,t)\,\hat{z} = -\frac{\partial B_z}{\partial x}\hat{y}\times B_z\hat{z}$$

$$= -B_z\frac{\partial B_z}{\partial x}\hat{x}$$

So, we get

$$\boxed{\frac{\partial B_z}{\partial t}\hat{z} - \frac{c^2\eta}{4\pi}\frac{\partial^2 B_z}{\partial x^2}\hat{z} = \frac{+1}{e}\nabla\times\left[\frac{B_z}{n}\frac{\partial B_z}{\partial x}\hat{x}\right]} \quad (8)$$

where $n = L\frac{dn}{dy}$. That is, n has a dependence on y.

(c) Show that the solution $B = [1-(\frac{KB_0}{D})x]^{-1}$ satisfies the differential equation in steady-state where $K = (\mu_0 neL)^{-1}$ and $D = \frac{\eta}{\mu_0}$.

$\frac{K}{D} = \frac{1}{neL\eta}$. note that steady-state $\Rightarrow \frac{\partial B}{\partial t} \doteq 0$.

So, eq (8) becomes

$$\frac{c^2\eta}{4\pi}\frac{\partial^2}{\partial x^2}\left[1-\left(\frac{KB_0}{D}\right)x\right]^{-1}\hat{z} = \frac{-1}{e}\nabla\times\left\{\frac{[1-(\frac{KB_0}{D})x]^{-1}}{n(y)}\frac{\partial[1-(\frac{KB_0}{D})x]^{-1}}{\partial x}\hat{x}\right\}$$

$$\Rightarrow \frac{c^2\eta}{4\pi}\left[\frac{-2(\frac{KB_0}{D})^2}{[1-(\frac{KB_0}{D})x]^3}\right]\hat{z} = \frac{-1}{e}\nabla\times\left[\frac{1}{n(y)}\frac{1}{(1-(\frac{KB_0}{D})x)}\frac{(\frac{-KB_0}{D})}{[1-(\frac{KB_0}{D})x]^2}\hat{x}\right]$$

$$\Rightarrow \frac{-2ec^2\eta}{4\pi}\left(\frac{KB_0}{D}\right)\frac{\hat{z}}{[1-(\frac{KB_0}{D})x]^3} = \nabla\times\left[\frac{1}{n(y)}\frac{\hat{x}}{[1-(\frac{KB_0}{D})x]^3}\right]$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ [\;] & 0 & 0 \end{vmatrix} = \hat{z}\frac{\partial}{\partial y}[\;] = \frac{\hat{z}}{[1-(\frac{KB_0}{D})x]^3}\frac{\partial}{\partial y}\left[\frac{1}{n(y)}\right]$$

So, this becomes

$$\frac{-ec^2\eta}{2\pi}\left(\frac{KB_0}{D}\right) = \frac{\partial}{\partial y}\left[\frac{1}{n(y)}\right] = \frac{-1}{[n(y)]^2}\frac{\partial n}{\partial y}$$

$$\Rightarrow \frac{-ec^2\eta B_0}{2\pi}\frac{1}{neL\eta} = \frac{-L}{\left[L\frac{dn}{dy}\right]^2}\frac{d}{dy}\left(\frac{dn}{dy}\right) = -\frac{1}{L}$$

$$\Rightarrow \frac{c^2 B_0}{\cancel{2}\pi n} = 1$$

→ this does not seem satisfied :o ... i think it is because i went from mks units to cgs units ... oh well..

(d) Show that in the absence of diffusion ($\eta = 0$) a propagating field $B_z(x-vt)$ satisfies the differential equation. Find the propagation velocity in terms of B_0 and ∇n.

if $\eta = 0$, then

$$\frac{\partial B_z}{\partial t} = \frac{-1}{e}\nabla\times\left[\frac{B_z}{n}\frac{\partial B_z}{\partial x}\hat{x}\right]$$

$$\Rightarrow -v B_z = -\frac{1}{e}\nabla\times\left[\frac{B_z}{n}B_z\hat{x}\right] = -\frac{B_z^2}{e}\left(-\frac{1}{[n(y)]^2}\frac{\partial n}{\partial y}\right)$$

$$\Rightarrow v = -\frac{B_z}{e}\frac{1}{n^2}\frac{\partial n}{\partial y}$$

$$\therefore \boxed{v = \frac{-B_z}{en^2}\frac{\partial n}{\partial y}} \Rightarrow v \text{ is in the } z\text{-direction}$$

The "official" answer claims that

$$v \propto \vec{B}\times\nabla n$$

But if $\vec{v}\perp\vec{B}$ and $\vec{B}$ is in the z-direction, how is v in the z-direction given above ....

Solution Problem #12
Spring 2001 Comp

Lea Fredrickson (1)

A magnetic field is given by

$$\vec{B} = (B_x, B_y, B_z) = ((1-\gamma)x, (-1+\gamma)y, -2\gamma z)$$

where γ is a constant

a) show that this field satisfies Maxwell's equations and may be derived from a scalar potential

I will work in cartesian coordinates

$$\vec{\nabla}\cdot\vec{B} = \frac{dB_x}{dx} + \frac{dB_y}{dy} + \frac{dB_z}{dz}$$

$$= (1-\gamma) + (-1+\gamma) - 2\gamma = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \checkmark$$

$$\vec{\nabla}\times\vec{B} = \left(\frac{dB_z}{dy} - \frac{dB_y}{dz}\right)\hat{x} + \left(\frac{dB_x}{dz} - \frac{dB_z}{dx}\right)\hat{y} + \left(\frac{dB_y}{dx} - \frac{dB_x}{dy}\right)\hat{z}$$

$\vec{\nabla}\times\vec{B} = 0$ (there are no cross terms)

therefore the field may be derived from a scalar potential $\vec{B} = -\vec{\nabla}\Phi_m$

b) For $\gamma = 0$, find the equation for the field lines, the vector potential, and show that the field lines are lines of constant vector potential.

for $\gamma = 0$

$$\vec{E} = (x, -y, 0)$$

The equation for the field lines is

$$\vec{B} = x\hat{x} - y\hat{y}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B_x = x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

$$B_y = -y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$

$$B_z = 0 = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

let $A_x = 0$

$$\begin{cases} x = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ -y = -\dfrac{\partial A_z}{\partial x} \\ 0 = \dfrac{\partial A_y}{\partial x} \end{cases}$$

$$A_y = C_y(y, z)$$

$$A_z = xy + C_z(y, z)$$

$$A_x = 0$$

$\vec{A} = A_z\hat{z} = xy\,\hat{z}$

$\vec{\nabla}\times\vec{A} = \vec{B} = \left(\frac{\partial A_z}{\partial y}\right)\hat{x} + \left(-\frac{\partial A_z}{\partial x}\right)\hat{y} = x\hat{x} + y\hat{y}$

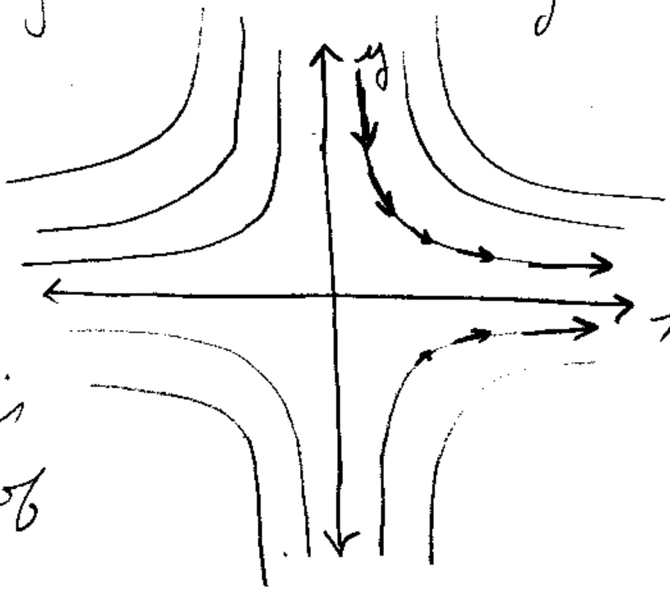
Let $A = \text{const} = xy$

$\Rightarrow \quad x = \frac{\text{const}}{y} \qquad y = \frac{\text{const}}{x}$

$\ln x = \ln \text{const} - \ln y$

$\int \frac{dx}{x} = \ln x = -\int \frac{dy}{y} \Rightarrow \frac{dx}{x} = -\frac{dy}{y}$

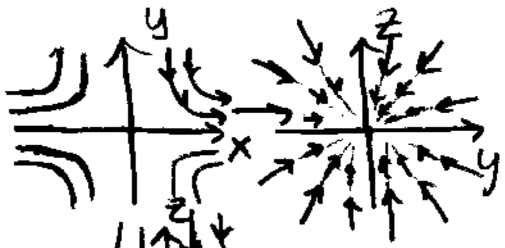
$\frac{dy}{dx} = \frac{-y}{x} = \frac{B_y}{B_x}$



So, the slope of the lines that define a equipotential surface is the same as the slope of the $\vec{B}$ field.

c) $\gamma = 0$ see above

$\gamma = \frac{1}{3} \quad \vec{B} = \left(\frac{4}{3}x, -\frac{2}{3}y, -\frac{2}{3}z\right) \rightarrow$



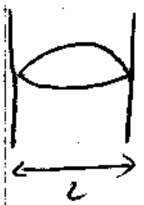
$\gamma = 1 \quad \vec{B} = (2x, 0, -2z) \longrightarrow$

Stat. Mech. S'04 #6; S'01 # 13

For relativistic bosons

$$E = |\vec{p}|c$$

a) First we need the density of states $D(E)$ for 3-D:



$$\frac{L}{\lambda/2} = n \Rightarrow \frac{2L}{n} = \lambda; \quad p = \frac{h}{\lambda} = \frac{h}{2L}\, n$$

$$E = c|\vec{p}| = c\left(p_x^2 + p_y^2 + p_z^2\right)^{1/2} = \frac{hc}{2L}\left(n_x^2 + n_y^2 + n_z^2\right)^{1/2}$$

$$= \frac{hc}{2L}\, n \Rightarrow n = \frac{2L}{hc}\, E \Rightarrow dn = \frac{2L}{hc}\, dE$$

$$\frac{(2s+1)}{8}\int_0^\infty 4\pi n^2\, dn = \frac{(2s+1)}{8}\int 4\pi \left(\frac{2L}{hc}\right)^3 E^2\, dE = \int_0^\infty \underbrace{\frac{(2s+1)\,4\pi \overset{L^3}{V}}{(hc)^3}\, E^2}\, dE$$

$$D(E) = \frac{(2s+1)\,4\pi V}{(hc)^3}\, E^2$$

The condition for BEC is determined by the boson temperature T_B, which can be derived followingly:

$$\int_0^\infty \frac{1}{e^{E/kT} - 1}\, D(E)\, dE = N \quad \text{for } T = T_B$$

$$\Rightarrow \frac{(2s+1)\,4\pi V}{(hc)^3}\int_0^\infty \frac{E^2}{e^{E/kT} - 1}\, dE = \frac{(2s+1)\,4\pi V (kT)^3}{(hc)^3}\underbrace{\int_0^\infty \frac{x^2}{e^x - 1}\, dx}_{= 2.404}$$

$x = E/kT \Rightarrow E = kTx \Rightarrow dE = kT\,dx$

hence

$$\frac{(2s+1)\,4\pi V}{(hc)^3}(kT_B)^3\, 2.404 = N$$

$$\Rightarrow \quad (kT_B)^3 = \frac{N}{V}\,\frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}$$

$$\Rightarrow \quad \boxed{T_B = \left(\frac{N}{k^3 V}\,\frac{(hc)^3}{(2s+1)4\pi\cdot 2.404}\right)^{1/3}}$$

b) yes it does occur - just derive $D(E)$ for 2-D case and repeat above steps:

$$E = \frac{hc}{2L}\left(n_x^2 + n_y^2\right) = \frac{hc}{2L}\,n \quad \Rightarrow \quad n = \frac{2L}{hc}E \quad \Rightarrow d n = \frac{2L}{hc}\,dE$$

$$\frac{(2s+1)}{4}\int_0^\infty 2\pi n\,dn = \frac{(2s+1)}{4}\int_0^\infty 2\pi\left(\frac{2L}{hc}\right)E\,dE = \int_0^\infty \underbrace{\frac{(2s+1)\,2\pi A}{(hc)^2}}\,E\,dE$$

L^2

$$D(E) = \frac{(2s+1)\,2\pi A}{(hc)^2}\,E$$

$$\int_0^\infty \frac{(2s+1)\,2\pi A}{(hc)^2}\,\frac{E}{e^{E/kT}-1}\,dE = \frac{(2s+1)\,2\pi A}{(hc)^2}(kT)^2\underbrace{\int_0^\infty \frac{x}{e^x-1}\,dx}_{\frac{\pi^2}{6}} = N$$

$$x \equiv E/kT \Rightarrow E = kTx \Rightarrow dE = kT\,dx$$

so

$$(kT_B)^2 = \frac{N}{A}\,\frac{3(2s+1)}{\pi^3 (hc)^2} \quad \Rightarrow \quad \boxed{T_B = \left(\frac{N}{k^2 A}\,\frac{3(2s+1)}{\pi^3 (hc)^2}\right)^{1/2}}$$

Stat. Mech. S'04 #6

c) BEC does not occur in 1-D case.

$$E = c|\vec{p}| = \frac{hc}{2L} n \Rightarrow n = \frac{2L}{hc} E \Rightarrow dn = \frac{2L}{hc} dE$$

$$\frac{(2s+1)}{2} \int_0^\infty dn = \int_0^\infty \underbrace{\frac{(2s+1)}{\cancel{2}} \frac{\cancel{2}L}{hc}}_{} dE$$

$$D(E) = \frac{(2s+1)L}{hc}$$

$$\int_0^\infty \frac{(2s+1)L}{hc} \frac{dE}{e^{-E/kT} - 1} = \frac{(2s+1)L}{hc} kT \underbrace{\int_0^\infty \frac{dx}{e^x - 1}}_{\text{undefined}} = \text{undefined.}$$

$x \equiv E/kT \Rightarrow dE = kT dx$

Bose-Einstein Condensation in 1, 2 and 3 dimensions for massive and massless bosons in a box

I. MASSIVE BOSONS

Consider a gas of massive, non-interacting, non-relativistic, identical, spin-0 bosons. The total number of bosons in a given state with energy ϵ is given by

$$N = \int_0^\infty \bar{n}(\epsilon) dN = \int_0^\infty \bar{n}(\epsilon) D(\epsilon) d\epsilon \tag{1}$$

where

$$\bar{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \tag{2}$$

is the quantum distribution function for bosons, and

$$D(\epsilon) = \frac{dN}{d\epsilon} \tag{3}$$

is the "density of states" function. For a particle in a box of side length L, the contained modes are quantized by the condition that the wave function vanish at the walls, Ψ(x,y,z=0)=Ψ(x,y,z=L)=0. Thus for each spatial dimension i, we have the condition

$$k_i = \frac{n_i \pi}{L} \tag{4}$$

In **3D**:

$$D_{3D}(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = 4\pi n^2 \frac{dn}{d\epsilon} \tag{5}$$

For a massive particle in the box the energy is quadratic in the momentum,

$$\epsilon = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) = \frac{\hbar^2}{2m}({k_x}^2 + {k_y}^2 + {k_z}^2) = \frac{\hbar^2\pi^2}{2mL^2}({n_x}^2 + {n_y}^2 + {n_z}^2) = \frac{\hbar^2\pi^2}{2mL^2} n^2 \tag{6}$$

thus

$$n = \frac{L}{\hbar\pi}\sqrt{2m\epsilon}, \qquad and \qquad dn = \frac{L}{2\hbar\pi}\sqrt{\frac{2m}{\epsilon}} d\epsilon \tag{7}$$

Combining terms into the density of states,

$$D_{3D}(\epsilon) = 4\pi \frac{2mL^2}{\hbar^2\pi^2}\epsilon\Big(\frac{L}{2\hbar\pi}\sqrt{\frac{2m}{\epsilon}}\Big) = 2\pi\Big(\frac{L}{\hbar\pi}\Big)^3 (2m)^{\frac{3}{2}}\sqrt{\epsilon} \tag{8}$$

The total number of particles can now be expressed as

$$\begin{aligned}
N_{3D} &= \frac{2\pi}{8}\Big(\frac{L}{\hbar\pi}\Big)^3 (2m)^{\frac{3}{2}} \int_0^\infty \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)} - 1} d\epsilon \\
&= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}} \frac{1}{1 - e^{-\beta(\epsilon-\mu)}} d\epsilon \\
&= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}} \Big(\sum_{l=0}^\infty e^{-\beta l(\epsilon-\mu)}\Big) d\epsilon \\
&= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \sqrt{\epsilon}\Big(\sum_{l=1}^\infty e^{-\beta l(\epsilon-\mu)}\Big) d\epsilon \\
&= \frac{2\pi V}{h^3}(2m)^{\frac{3}{2}} \int_0^\infty \sqrt{\epsilon}\Big(\sum_{l=1}^\infty e^{-\beta l\epsilon} e^{\beta l\mu}\Big) d\epsilon \\
&= \frac{V}{h^3}\Big(\frac{2m\pi}{\beta}\Big)^{\frac{3}{2}} \sum_{l=1}^\infty \frac{e^{\beta l\mu}}{l^{\frac{3}{2}}}
\end{aligned} \tag{9}$$

where the factor of 8 has been introduced since we are including only the positive values of n, and thus only the first quadrant of the 3D sphere in n-space. The last sum on the right is a polylogarithm function, also called the the weighted Zeta function (weighted by the exponential factor). We have used the expansion condition $e^{-\beta(\epsilon-\mu)} < 1$, which is validated by the physical mandate that we do not obtain negative values for $\bar{n}(\epsilon)$. It follows then that we posit the restriction $\epsilon > \mu$.

From the expression it can be seen that N_{3D} is a maximum at μ=0, which is therefore when the condensate occurs. The sum can be evaluated and we arrive at the condensate phase transition temperature,

$$N_{3D} = \frac{V}{h^3}\Big(\frac{2m\pi}{\beta}\Big)^{\frac{3}{2}} \underbrace{\sum_{l=1}^{\infty} \frac{1}{l^{\frac{3}{2}}}}_{\zeta(\frac{3}{2})} \approx \frac{V}{h^3}\Big(\frac{2m\pi}{\beta}\Big)^{\frac{3}{2}}(2.612) \qquad \Longrightarrow \qquad T_c \approx \frac{h^2}{2m\pi k_b}\Big(\frac{N_{3D}}{2.612V}\Big)^{\frac{2}{3}} \tag{10}$$

For massive bosons in **2D** we follow the same procedure. The density of states is

$$D_{2D}(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = 2\pi n \frac{dn}{d\epsilon}. \tag{11}$$

The energy has a similar form as previously

$$\epsilon = \frac{1}{2m}({p_x}^2 + {p_y}^2) = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{12}$$

Plugging into the integral for N_{2D}, we note that the density of states here does not depend on the energy. Consequently the integral is over only the distribution function, with a factor of 1/4 that comes from dealing with only the first quadrant of the 2D sphere in n-space.

$$N_{2D} = \frac{2\pi}{4}\Big(\frac{L}{\hbar\pi}\Big)^2 \frac{2m}{2}\int_0^{\infty} \frac{1}{e^{\beta(\epsilon-\mu)}-1}d\epsilon = \frac{2\pi m A}{h^2\beta}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l} \tag{13}$$

For the condensate to occur, μ=0 and the above expression diverges, $\zeta(1)\rightarrow\infty$. Hence, the condensate does not occur for massive bosons in 2D.

Lastly for the **1D** case, the density of states is simply

$$D_{1D}(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = (1)\frac{dn}{d\epsilon} \tag{14}$$

The energy is, again, the same as above

$$\epsilon = \frac{1}{2m}({p_x}^2) = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{15}$$

So the total number is

$$N_{1D} = \frac{L}{2h}\sqrt{2m}\int_0^{\infty}\frac{1/\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon = \frac{L}{2h}\sqrt{\frac{2\pi m}{\beta}}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l^{\frac{1}{2}}} \tag{16}$$

There is a factor of 1/2 from dealing with only positive n values. Again, this expression is non-physical for μ=0, so the condensate for massive bosons in 1D does not occur.

II. MASSLESS BOSONS

For massless bosons by contrast, we must express their energy relativistically. Thus from the relation

$$\epsilon = c|p| \tag{17}$$

it is evident that the energy is linear in momentum. This alters the conditions for the BEC to occur. In every case the energy is given as

$$\epsilon = c\hbar|k| = c\hbar\frac{n\pi}{L} \qquad \Longrightarrow \qquad n = \frac{\epsilon L}{c\hbar\pi} \tag{18}$$

For **3D** the density of states is

$$D_{3D}(\epsilon) = 4\pi n^2 \frac{dn}{d\epsilon} = 4\pi \Big(\frac{L}{c\hbar\pi}\Big)^3 \epsilon^2 \tag{19}$$

so the total number is

$$N_{3D} = \frac{4\pi}{8}\Big(\frac{L}{c\hbar\pi}\Big)^3 \int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1} d\epsilon = \frac{8\pi V}{(ch\beta)^3}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l^3} \underset{\mu\to 0}{=} \frac{8\pi V}{(ch\beta)^3}\zeta(3) \approx \frac{8\pi V}{(ch\beta)^3}1.1202 \tag{20}$$

Now we can calculate the phase transition temperature for massless bosons in **3D**:

$$T_c \approx \Big(\frac{N_{3D}}{8\pi V(1.1202)}\Big)^{\frac{1}{3}} \frac{ch}{k_b}. \tag{21}$$

In **2D** the density of states is

$$D_{2D}(\epsilon) = 2\pi n \frac{dn}{d\epsilon} = 2\pi \Big(\frac{L}{c\hbar\pi}\Big)^2 \epsilon \tag{22}$$

Note that now the 2D density of states *does* depend on the energy. The total number of particles is

$$N_{2D} = \frac{2\pi}{4}\Big(\frac{L}{c\hbar\pi}\Big)^2 \int_0^\infty \frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1} d\epsilon = \frac{2\pi A}{(ch\beta)^2}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l^2} \underset{\mu\to 0}{=} \frac{2\pi A}{(ch\beta)^2}\zeta(2) = \frac{2\pi A}{(ch\beta)^2}\Big(\frac{\pi^2}{6}\Big) \tag{23}$$

This result shows that massless bosons in 2D do indeed form a condensate, whereas massive bosons in 2D do not. The temperature of condensation here is

$$T_c = \Big(\frac{3N_{2D}}{A\pi^3}\Big)^{\frac{1}{2}} \frac{ch}{k_b} \tag{24}$$

Finally, for the 1D system of massless bosons we have a density of states that is independent of energy, just like the 2D massive boson system.

$$D_{1D}(\epsilon) = \frac{dn}{d\epsilon} = \frac{L}{c\hbar\pi}. \tag{25}$$

Again, the integral diverges for μ=0,

$$N_{1D} = \frac{L}{2c\hbar\pi}\int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}-1} d\epsilon = \frac{L}{ch}\sum_{l=1}^{\infty}\frac{e^{\beta l\mu}}{l} \underset{\mu\to 0}{\Rightarrow} \frac{L}{ch\beta}\zeta(1) \Rightarrow \infty \tag{26}$$

and thus in the 1D massless case we find the same condition as the massive bosons in 1D, i.e., the condensate is forbidden in this geometry.

III. CONCLUSION

For the particle-in-a-box model, BEC's occur for both massive and massless bosons in 3D. They occur only in the massless case for 2D, and never for 1D. This model can in principle be applied to higher spatial dimensions whereupon evaluation of the total number of particles would be an integral of the form

$$\begin{aligned} N_{qD} &\sim \int_0^\infty \frac{\epsilon^{(q-2)/2}}{e^{\beta(\epsilon-\mu)}-1} d\epsilon \underset{\mu\to 0}{\sim} \zeta\Big(\frac{q}{2}\Big) \qquad (Massive) \\ N_{qD} &\sim \int_0^\infty \frac{\epsilon^{(q-1)}}{e^{\beta(\epsilon-\mu)}-1} d\epsilon \underset{\mu\to 0}{\sim} \zeta(q) \qquad (Massless) \end{aligned} \tag{27}$$

where q is the dimension of the space.

Spring 2001 #14.

$F = -KT \ln Z$ $\qquad Z = \left[\sum_r e^{-\beta \varepsilon_r}\right]^N$ eq 9.4.4 Reif

$\approx z_1^N$

$F = -KTN \ln\left[\sum_r e^{-\beta\varepsilon_r}\right]$

page 344

$= -KTN \ln\left[\sum_r e^{-\frac{\varepsilon_r}{KT}}\right]$

$z_1 = \sum_r e^{-\varepsilon_r \beta} = \frac{1}{h_0^3} \int_{-\infty}^{\infty} e^{-\beta\frac{p^2}{2m}} d^3p \int_0^L d^3r$

$= \frac{V}{h_0^3} \int_{-\infty}^{\infty} e^{-\beta\frac{p^2}{2m}} d^3p = \frac{V}{h_0^3} \int e^{-(\beta \frac{p_x^2}{2m})} dp_x \int e^{-\frac{\beta p_y^2}{2m}} dp_y \ldots$

$= \left(\sqrt{\pi 2m\cancel{K}T}\right)^3 \frac{V}{h_0^3}$

b)

$= \frac{V (2\pi m KT)^{3/2}}{h_0^3}$ Reif 7.2.6

$F = -KTN \ln\left[\frac{V}{h_0^3}(2\pi m KT)^{3/2}\right]$

b) For the electronic motion (exicited statese)

$z_1 = \sum_r e^{-\varepsilon_r \beta} + \sum_d \Omega_0 e^{\beta \varepsilon_d}$ $\qquad \varepsilon_d$ is negative.

Reif 9.12.7

degeneracy of ground state.

$\Rightarrow F = -KTN \ln\left[z_1\right] \ldots^{3/2} \sum_d \Omega_0 e^{\beta\varepsilon_d}$ diverges since you have a positive exponent.

The cut off in a real gas is that the ground state is smaller than the next closest state by a wide energy gap so electrons have an overwhelming probability to be in the ground state

Problem #14 Spring 2001

a) $F = -KT \ln Z$

$$Z = \frac{1}{N!}\left(\frac{V}{(2\pi\hbar^2/MKT)^{3/2}}\right)^N$$

$$F = -KT\left[N \ln\left(\frac{V}{(2\pi\hbar^2/MKT)^{3/2}}\right) - \ln N!\right]$$

aside

$$Z = \sum_i e^{-\beta \epsilon_i} \xrightarrow{\text{classical}} \int e^{-\beta \frac{p^2}{2m}} \frac{d\vec{p}\, d\vec{r}}{h_0^{3N}}$$

$$F = -KT\left[N \ln V - \tfrac{3}{2} N \ln(2\pi\hbar^2) + \tfrac{3}{2} N \ln MKT - \underbrace{\ln N!}_{N\ln N - N}\right]$$

$$F = -NKT\left[\ln V - \tfrac{3}{2}\ln(2\pi\hbar^2) + \tfrac{3}{2}\ln(mKT) - \ln N - 1\right]$$

b)

$$Z = Z' Z_q = \frac{1}{N!}\left(\frac{V}{\left(2\pi\hbar^2/mKT\right)^{3/2}}\right)^N \sum_i e^{-\beta\epsilon_i}$$

where $\epsilon_i = -\dfrac{\alpha^2 m}{2n^2} = \dfrac{-me^4}{2\hbar^2 n^2}$

$$\epsilon_i = -\frac{me^4}{8\varepsilon_0^2 h^2 n^2} = \frac{-me^4}{32\pi^2\epsilon_0^2\hbar^2 n^2}$$

$$Z_q = \left(\sum_{n=1}^{\infty} e^{+\beta\frac{\alpha}{n^2}}\right)^N$$

$$F = -NKT\left[\text{Classical ln piece} + \ln\sum_{n=1}^{\infty} e^{+\beta\frac{\alpha}{n^2}}\right]$$

Selected Answers

Spring 2001

1) (a) $P_{12} = 2\left(\frac{1}{4} + \vec{S}_1 \cdot \vec{S}_2\right)$

(b) $P_{123} = \left[\frac{1}{4} + \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + 4(\vec{S}_1 \cdot \vec{S}_2)(\vec{S}_1 \cdot \vec{S}_3)\right]$

2) $\Theta = (g-2)\frac{\omega t}{2}$

3) (a) $f(\theta,\phi) = -2\pi^2 \int \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}'\cdot\vec{r}'}\, 2mV(r)\,\psi(r)\, d^3r$ (b) $f(\theta,\phi) = \frac{2mV_0}{\mu} \frac{1}{(q^2+\mu^2)}$

(c) $\sigma \xrightarrow[k\to 0]{} \frac{16\pi m^2 V_0^2}{\mu^6}$, $k^2 = 2mE$, $\vec{k}' = k\frac{\vec{r}}{r}$

4) (a) $E_n = \frac{-\alpha^2 mc^2}{2n^2}$ degeneracy is n^2 (b) $H_{LS} \sim \alpha^4$, $H_{\text{gross structure}} \sim \alpha^2$

$H_{kin} \sim \alpha^4$

(c) $n=2$: 2×3 P states + 2×1 S state

nL_j : 2 - $2S_{\frac{1}{2}}$ states, 2 - $2P_{\frac{1}{2}}$ states, 4 - $2P_{\frac{3}{2}}$ states

(d) $\vec{\mu}_e \cdot \vec{\mu}_p \sim \vec{S}_e \cdot \vec{S}_p$: 4 states w/ $n=1$ & $l=0$ are split into a spin zero singlet and spin zero triplet

H_{HFS} is same order in α as H_{FS} but has $\frac{m_e}{m_p}$ factor as well

5) $E = \frac{\hbar^2 \pi^2}{2m(b-a)^2}$, $\psi(r) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{b-a}}\ \frac{1}{r} \sin\left[\frac{\pi(r-a)}{(b-a)}\right]$ ← use radial wave eqn $l=0$

6) see spring 2002 #6

9) (a) $F_\phi = \frac{q a_0}{2c}\left|\frac{\partial \vec{B}}{\partial t}\right|$ (b) $\frac{p^2}{B} = \frac{q^2 a_0^2 B}{c^2}$, $\frac{dJ}{dt} = 0$, $J = \oint p\cdot dl + \frac{q}{c}\int d\vec{\sigma}\cdot\vec{B} = \frac{\pi c}{a_0}\frac{p^2}{B}$

11) (a) $\frac{\partial \vec{B}}{\partial t} = \frac{-\eta}{\mu_0}\nabla\times(\nabla\times\vec{B}) - \frac{1}{\mu_0 e}\nabla\times\left[\frac{(\nabla\times\vec{B})\times\vec{B}}{n}\right]$

$\vec{v} = \frac{1}{\mu_0 n^2 e}(\vec{B}\times\nabla n)$

(b) $\frac{\partial B_z}{\partial t} = \underbrace{\frac{\eta}{\mu_0}\frac{\partial^2 B_z}{\partial x^2}}_{\text{diffusion } t \propto x^2} + \underbrace{\frac{1}{\mu_0 e}\nabla\times\left[\frac{B_z \frac{\partial B_z}{\partial x}}{n(y)}\hat{x}\right]}_{\text{convection } t\propto x}$

12) (a) $\nabla\times\vec{B} = 0$, $\nabla\cdot\vec{B} = 0$ potential field

13) see

(b) $A_x = xy = \text{const} \Rightarrow y = \frac{\text{const}}{x}$

WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Thursday, March 29, and Friday, March 30, 2001

PART I - THURSDAY, MARCH 29

Important – please read carefully.

The exam (8 hours) is in two parts:

Part 1 Quantum Mechanics, Thermodynamics, Statistical Mechanics

March 29 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

March 30 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

1) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. <u>Use one side only.</u>

3) Print your name and problem number on EACH **AND EVERY** page. (Note: Pages without names may not be counted.)

4) Return the problem page as the first page of your answers.

5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

1. *Quantum Mechanics.*

Consider a system of three spin-1/2 moments, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$. The permutation operator P_{12} exchanges spins 1 and 2:

$$P_{12}\ |\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_1, \sigma_3\rangle$$

where $\sigma_{1,2,3} = \pm\frac{1}{2}$ are the eigenvalues of S_1^z, S_2^z, S_3^z. The permutation operator P_{123} performs a cyclic permutation on spins 1,2, and 3 so that $2 \rightarrow 1$, $1 \rightarrow 3$, $3 \rightarrow 2$.

$$P_{123}\ |\sigma_1, \sigma_2, \sigma_3\rangle = |\sigma_2, \sigma_3, \sigma_1\rangle$$

(a) Express P_{12} in terms of the spin operators $\mathbf{S}_1$, $\mathbf{S}_2$.
(b) Express P_{123} in terms of the spin operators $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$.

2. *Quantum Mechanics.*

An electron is injected into a region where there is a constant magnetic field of magnitude B. At $t = 0$, the direction of the electron's motion is perpendicular to the magnetic field, and it is completely polarized so that its spin is definitely along the direction of the beam.

Let Θ be the angle between the electron's momentum and the expectation value of its spin. At $t = 0$, $\Theta = 0$. What is Θ as a function of the time t? [Calculate the time-dependence of the momentum classically.] Express your answer in terms of the gyromagnetic ratio g of the electron. Leave g arbitrary – don't set it exactly equal to 2.

3. *Quantum Mechanics.*

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_o \frac{e^{-\mu r}}{\mu r}$$

(a) Imagine you could find the solution $\psi(\mathbf{r})$ to the time-independent Schrödinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\mathbf{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

(b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

(c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

4. *Quantum Mechanics.*

The simplest approximation for the Hamiltonian of an electron in a hydrogen atom is

$$H_o = \frac{\mathbf{p}^2}{2m} - \frac{\alpha \hbar c}{r}$$

where $\alpha \approx 1/137.036$ is a dimensionless constant. In cgs units, the electric charge e is related to α by $e^2 = \alpha \hbar c$.

(a) In this approximation, what are the energy levels of the hydrogen atom and what is the degeneracy of each level?

(b) There are some corrections to H_o that give rise to a small correction to the energy levels called the fine structure. What are the effects that give rise to the fine structure? Just describe them briefly – don't try to remember the formulas. What is the order of magnitude of the fine structure splitting compared to the splitting between the eigenvalues of H_o? Why?

(c) Consider the states in the first excited level with the approximation H_o above. Into how many levels are these states split, and what is the degeneracy of each level? What are the quantum numbers of the states in each level?

(d) There is a further splitting called the hyperfine structure. What is the effect that causes the hyperfine structure? Here too just describe it briefly. Into how many levels is the ground state level (i.e. all the states in the lowest energy level when hyperfine structure is ignored) split by the hyperfine effect, and what is the degeneracy of each level? Why is the hyperfine splitting small compared to the fine structure splitting?

5. *Quantum Mechanics.*

A particle of mass m is constrained to move between two concentric impermeable spheres of radii $r = a$ and $r = b$. There is no other potential. Find the ground state energy and normalized wave function.

6. *Statistical Mechnics and Thermodynamics*

Calculate the collision frequency for the collisions between the molecules of a gas and a fixed sphere of diameter D. The molecules have an average diameter d. The gas has a temperature T.

7. *Statistical Mechnics and Thermodynamics*

This is an essay question. Answer **two** of the following three questions.

(a) You are asked about the second law of thermodynamics, and you give one of the formulations, that there is no process the sole effect of which is the conversion of heat into work. The inquirer then points out that a steam engine converts heat into work. Explain how this is not a violation of the second law of thermodynamics. Your explanation should include an analysis of the steam engine, and a discussion of heat engines in general.

(b) You read an article in a physics journal in which a group of researchers announce that they have cooled a system to absolute zero. Discuss why one ought to be skeptical of this claim. Invoke the appropriate laws of thermodynamics.

(c) Explain, using the laws of thermodynamics, why a substance cannot have a negative heat capacity.

WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Thursday, March 29, and Friday, March 30, 2001

PART II - FRIDAY, MARCH 30

Important – please read carefully.

The exam (8 hours) is in two parts:

Part 1 Quantum Mechanics, Thermodynamics, Statistical Mechanics

March 29 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

March 30 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

2) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. <u>Use one side only.</u>

6) Print your name and problem number on EACH **AND EVERY** page. (Note: Pages without names may not be counted.)

7) Return the problem page as the first page of your answers.

8) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

8. *Electricity and Megnetism*

One hemisphere of a metallic sphere of radius R is kept at a potential V while the other hemisphere is kept at a potential of $-V$.

(a) What is the approximate potential and electric field a far away distance r from the center of the sphere. Keep only the leading contribution in R/r.

(b) Suppose the the potential V varies in time as $V_0 e^{-i\omega t}$ where $\omega R \ll c$. What is the electric field far away from the sphere? Again keep only the leading contribution in R/r. (If you can't figure out an exact expression then explain the generic behavior.)

9. *Electricity and Megnetism*

A charged particle moves in a plane perpendicular to a magnetic field $\vec{B}$, which is uniform in space but varies very slowly with time.

(a) Find a relation between the momentum p, the magnetic field B, and the instantaneous cyclotron (or gyration) radius a_0 of the particle's trajectory. (The radius will change very slowly in time as the B field varies.)

(b) Using Faraday's law, derive an approximate relation between the magnitude of the induced electromotive force around the orbit, the time derivative of B and the intantaneous radius a_0.

(c) Utilizing your answer to the previous parts, or otherwise, show that p^2/B remains constant in time.

10. *Electricity and Megnetism*

A π^0 of velocity v_0 decays in flight into two photons $\pi^0 \to 2\gamma$. Compute the minimum and maximum values of the energies of the produced photons as a functions of v_0.

11. *Electricity and Megnetism*

Consider the penetration of a magnetic field $\mathbf{B}$ into a conducting medium by diffusion and convection.

The medium obeys an Ohm's law of the form $\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$, where $\mathbf{E}$ is the electric field, $\mathbf{J} = -ne\mathbf{v}$ is the current density of electrons of velocity $\mathbf{v}$ and number density n, and η is a constant uniform scalar resistivity.

(a) Using Faraday's law and Ampere's law (neglect displacement current) obtain a differential equation for the magnetic field.

(b) Now consider a simple boundary problem: the conducting medium is located in the half-space $x > 0$. There exists a density gradient of scale length $L = n/(dn/dy)$. At t=0 a uniform field B_0 along $\mathbf{z}$ is applied in the space $x < 0$. Write down the differential equation for the field $B_z(x,t)$. Identify which terms describe field diffusion ($t \propto x^2$) and convection ($t \propto x$).

(c) Show that the solution $B = [1 - (kB_0/D)x]^{-1}$ satisfies the differential equation in steady-state where $k = (\mu_0 neL)^{-1}$ and $D = \eta/\mu_0$.

(d) Show that in the absence of diffusion ($\eta = 0$) a propagating field $B_z(x - vt)$ satisfies the differential equation. Find the propagation velocity v in terms of B_0 and ∇n.

12. *Electricity and Megnetism*

A magnetic field is given by

$$\mathbf{B} = (B_x, B_y, B_z) = ((1+\gamma)x, (-1+\gamma)y, -2\gamma z)$$

where γ is a constant.

(a) Show that this field satisfies Maxwell's equations and may be derived from a scalar potential.

(b) For $\gamma = 0$, find the equation for field lines, the vector potential, and show that field lines are lines of constant vector potential.

(c) Sketch field lines for three parameter values $\gamma = 0$, $\gamma = 1/3$, $\gamma = 1$.

13. *Statistical Mechnics and Thermodynamics*

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\vec{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

14. *Statistical Mechnics and Thermodynamics*

(a) What is the free energy (as a function of temperature, T, volume, V, and particle number, N) of a ideal gas obeying Maxwell-Boltzmann statistics?

(b) Assume that the ideal gas is made up of hydrogen atoms. Now the free energy must include a contribution reflecting the different possible electronic excited states of the hydrogen atoms. Show that this contribution diverges. What cuts off this divergence in a real gas?