1. *Quantum Mechanics* (Fall 2003)

Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J(S_1^x S_2^x + S_1^y S_2^y + k S_1^z S_2^z) + \mu(S_1^z + S_2^z)B.$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wavefunction.

(b) Repeat for a symmetric spatial wavefunction.

2. *Quantum Mechanics* (Fall 2003)

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V = V_0 \left[\delta(\mathbf{r} - a\hat{\mathbf{z}}) - \delta(\mathbf{r} + a\hat{\mathbf{z}}) \right].$$

Compute the differential cross section, $d\sigma/d\Omega$ in the Born approximation.

3. *Quantum Mechanics* (Fall 2003)

Consider a particle moving in the potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{if } x > 0 \\ \infty & \text{otherwise} \end{cases}$$

(a) What is the lowest energy eigenvalue?

(b) What is $\langle x^2 \rangle$?

4. *Quantum Mechanics* (Fall 2003)

An operator A, corresponding to an observable α, has two normalized eigenfunctions ϕ_1 and ϕ_2, with distinct eigenvalues a_1 and a_2, respectively. An operator B, corresponding to an observable β, has normalized eigenfunctions χ_1 and χ_2, with distinct eigenvalues b_1 and b_2, respectively. The eigenfunctions are related by:

$$\phi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13}$$

$$\phi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}.$$

An experimenter measures α to be $42\hbar$. The experimenter proceeds to measure β, followed by measuring α again. What is the probability the experimenter will measure α to be $42\hbar$ again?

5. *Quantum Mechanics* (Fall 2003)

A sample of hydrogen atoms in the ground state is placed between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor at $t = 0$ to produce a homogeneous electric field, $\mathcal{E}$, between the plates of:

$$\begin{aligned} \mathcal{E} &= 0, && (t < 0) \\ \mathcal{E} &= \mathcal{E}_0 \exp(-t/\tau), && (t > 0), \end{aligned}$$

where τ is a constant. A long time compared to τ passes.

(a) To first order, calculate the fraction of atoms in the $2p$ $(m = 0)$ state.

(b) To first order, what is the fraction of atoms in the $2s$ state?

You may find the following helpful. The normalized radial wavefunctions of the hydrogen atom are:

$$\begin{aligned} R_{10}(r) &= 2\left(\frac{Z}{a}\right)^{3/2} \exp\left(-\frac{Zr}{a}\right) \\ R_{20}(r) &= \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2}\left(1 - \frac{Zr}{2a}\right) \exp\left(-\frac{Zr}{2a}\right) \\ R_{21}(r) &= \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r \exp\left(-\frac{Zr}{2a}\right) \end{aligned}$$

where r is the radial coordinate, a is the Bohr radius, and $Z = 1$ for a hydrogen atom. The first spherical harmonics are:

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\,\exp(\pm i\phi)$$

A useful integral may be:

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

6. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a gas of non-conserved Bosons in three dimensions. The energy-verses-momentum relationship for each of these exotic particles is $E = Ap^2$.

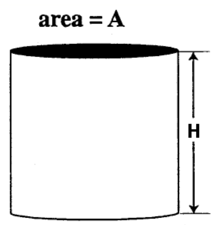
(a) Calculate the grand partition function of this gas. Assume a spin degeneracy factor of 1.

(b) Find the energy of this gas as a function of temperature. The energy goes as a simple power law in the temperature. What is the power?

(c) Find the pressure as a function of temperature. What power law describes the temperature dependence of the pressure?

7. *Statistical Mechanics and Thermodynamics* (Fall 2003)

A gas of noninteracting particles fills a cylindrical container that has cross-sectional area A and height H. Each particle has mass m, and is subject to the gravitational field at the surface of the Earth. The circular bottom and top of the container are parallel to the surface of the Earth. There are N particles in the container, and the temperature of the container is T.



(a) Find the partition function of the gas.

(b) What is the pressure of the gas at the top of the container?

(c) What is the pressure of the gas at the bottom of the container?

(d) Finally, what is the difference between the pressure at the bottom of the container and the pressure at the top of the container? Interpret the answer that you get.

8. *Electricity and Magnetism* (Fall 2003)

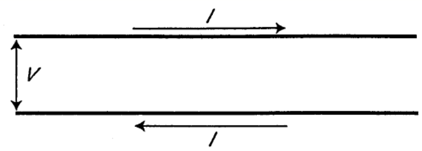
Consider a vacuum diode which is a parallel plate capacitor (in vacuum) with plate area A and plate separation d. The cathode plate, which is at $\phi = 0$, is heated as to thermionically emit electrons which then travel to the anode plate (at $\phi = V$) (this arrangement acts as a diode due to the fact that in reverse bias, no charges will flow). Assume a steady-state bias V and diode current I. You may model the electrons in the diode as a cold fluid with density $n(x)$ and velocity $v(x)$. You may assume that the electrons are born from the cathode with zero velocity.

(a) Find the 1-D potential distribution in the diode, $\phi(x)$. (Hint: Try a power law solution.)

(b) Find the diode current as a function of bias voltage V.

(c) What unphysical result is caused by the assumption that electrons are born from the cathode with zero velocity?

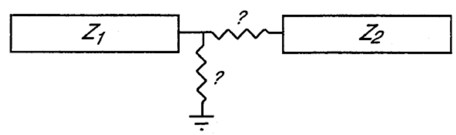
9. *Electricity and Magnetism* (Fall 2003)



(a) A two-wire transmission line has inductance L and capacitance C per unit length (and no resistance). Show that the impedance of this transmission line $Z = V/I$ is real and equal to $\sqrt{L/C}$.

Note: Assume AC signals are transmitted on the line, $I = I_0 \exp(ikx - i\omega t)$.

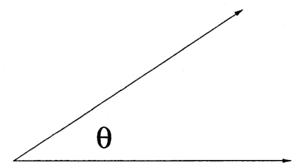
(b) Two long transmission lines are connected together. The first has impedance Z_1 and the second has impedance $Z_2 \neq Z_1$. A wave $V_i \exp(ikx - i\omega t)$ travels on the first transmission line and encounters the second. What are the relative amplitudes of the reflected and transmitted waves $(V_r/V_i, V_t/V_i)$?



(c) Reflection due to impedance mismatch between two transmission lines can be eliminated through adding series or parallel resistance between the lines. For the transmission lines in (b), how would you connect a resistor (and what is its values) in order to match the impedances and eliminate the reflected wave? (Consider both $Z_1 > Z_2$ and $Z_1 < Z_2$.)

10. *Electricity and Magnetism* (Fall 2003)

Consider a wedge formed by two conducting half-planes, as depicted in the figure. One plane is maintained at electrostatic potential V_1 while the other is at V_2. What is the electrostatic potential in the region between the two half-planes?



11. *Electricity and Magnetism* (Fall 2003)

The dispersion relation for a photon in an ionized plasma (in *CGS* units) is,

$$k^2c^2 = \omega^2 - 4\pi n e^2/m_e$$

where k is the photon wavenumber, $c = (3.0 \times 10^{10}\,\text{cm/s})$, and ω is the radiation frequency in radians/s. Here, n is the electron number density, $e = (4.8 \times 10^{-10}\,\text{esu})$ is the electron charge, and $m_e = (9.11 \times 10^{-28}\,\text{g})$ is the electron mass.

(a) *Explain* why electromagnetic waves with frequencies below about (10 MHz) can't be received from space on Earth.

(b) Pulsars are objects observed in our galaxy which regularly emit a short burst of electromagnetic waves containing a wide range of frequencies all at once. If a pulsar is located (1.0×10^{22} cm) away and the density of electrons in the space between us and the pulsar is a uniform ($0.01\ \text{cm}^{-3}$), what is the difference of the arrival times at Earth of the radiation emitted near (6 kHz) compared to (10 kHz)? (You may assume the measurement happens far enough above the Earth so that the effect in part (a) can be ignored. You may leave your answer as an expression without substituting the numbers.)

12. *Electricity and Magnetism* (Fall 2003)

(a) Consider and infinitely long electron beam with N electrons, a flat top radial profile with radius a, and velocity v_b. What is the force on an electron at the edge of the beam ($r = a$)?

(b) In reality no beam is infinitely long. Suppose the beam density has the form

$$n_b = \frac{N}{\pi^{3/2}\sqrt{2}\sigma_z a^2} \exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

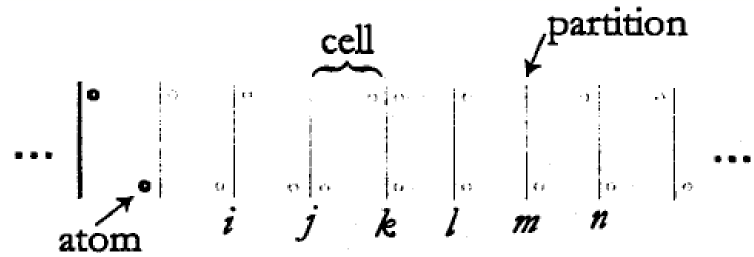
for $r < a$ and is 0 for $r > a$ and its velocity is $\mathbf{v_b} = v_b\hat{\mathbf{z}}$. In the relativistic limit, what is the force (both in r and in z) for an electron at $r = a$ and at $z = 0$ and at $z = \sqrt{2}\sigma_z$?

Hint: One way to solve this problem is to start with the wave equations for the scalar and vector potentials, ϕ and $\mathbf{A}$, in the Lorentz gauge. Rewrite them in terms of the variables x, y, $\xi = z - v_b t$. Then simplify them in the limit $v_b \to c$. Use these equations to solve the problem.

(c) For the electron beam at SLAC, $N = (2\times10^{10})$, $\sigma_z = (0.6 \text{ mm})$, $a = (25\ \mu\text{m})$, and the electrons have an energy of (50 GeV). Do the approximations used in part (b) hold for such a beam?

13. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a hypothetical system made up of N "partitions", a small section of which is shown in the figure below (the system is a closed ring, in order to eliminate end effects).



Each "cell" contains two atoms, one in the top half of the cell and one in the bottom half of the cell. Each atom occupies one of two positions in its half of the cell, to the left or to the right. The energies associated with an individual partition are given by the following rules: (i) Unless exactly two atoms are associated with a partition, the energy of that configuration is infinite (e.g., $\epsilon_k = \epsilon_m = +\infty$). (ii) If two atoms are on the same side of a partition, then the energy of that configuration is zero (e.g., $\epsilon_l = 0$). (iii) If two atoms are on opposite sides of a partition, then the energy of the configuration is ϵ (e.g., $\epsilon_i = \epsilon_n = \epsilon$).

(a) What are the energy levels possible for a system of N partitions and associated atoms? What is the degeneracy of each level? What is the canonical partition function for the system?

(b) Compute the free energy per particle in the thermodynamic limit and show that there is a discontinuity at a temperature T_c (i.e., the system exhibits a phase transition). Find T_c.

14. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a system of classical spins in d dimensions which are confined to point at angles $\theta = 0,\ 2\pi/3,\ 4\pi/3$ in a plane, i.e., $\mathbf{s}_i = (\cos\theta_i, \sin\theta_i)$, with θ_i taking the three values above. The spins interact according to the Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j$$

where $\langle i, j \rangle$ are nearest neighbors. Using mean-field theory, find the critical temperature, T_c, below which the spins order.

Selected Answers Fall 2003

4) $P = \frac{97}{169}$

5) (a) $|c_j|^2 = \frac{2^{15}}{3^{10}} \frac{a_o^2 e^2 \mathcal{E}_o^2}{\hbar^2(\omega^2 + \frac{1}{\tau^2})}$, $\omega^2 = \left[\frac{13eV(1-\frac{1}{4})}{\hbar} \right]^2$

(b) ϕ

8) (a) $\Phi(x) = V\left(\frac{x}{d}\right)^{4/3}$ (b) $j = \frac{-\phi''}{4\pi}\sqrt{\frac{2e}{m\phi}} = \frac{-V^{3/2}}{9\pi d^2}\sqrt{\frac{2e}{m}}$ (c) $j = env$ is infinite

9) (a) use δV, δQ, $\frac{V}{I}$ (b) $V_r = \frac{Z_2 - Z_1}{Z_2 + Z_1} V_i$ and $V_t = \frac{2Z_2}{Z_2 + Z_1} V_i$

(c) case: $Z_1 > Z_2$, R in series $R = Z_1 - Z_2$

$Z_2 > Z_1$, R in parallel $R = \frac{Z_1 Z_2}{Z_2 - Z_1}$

11) (a) ionosphere ($n \sim 5 \times 10^{13}\ e^-/cm^3$) (b) 5×10^3 years

13) (1) 2 energy levels: $0 \leftarrow$ 2-fold degenerate

$n\epsilon \leftarrow$ 2^n-fold degenerate

$Z = 2 + 2^N \exp[-N\beta\epsilon]$

(2) $\frac{A}{N} = -\frac{\beta}{N} \ln Z = -kT \ln 2 + \epsilon - \frac{kT}{N} \ln\left(1 + \frac{2}{[2e^{-\beta\epsilon}]^N}\right)$

$T_c = \frac{\epsilon}{k \ln 2}$

1. *Quantum Mechanics* (Fall 2003)

Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J(S_1^x S_2^x + S_1^y S_2^y + kS_1^z S_2^z) + \mu(S_1^z + S_2^z)B.$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wavefunction.

(b) Repeat for a symmetric spatial wavefunction.

The total wavefunction is the product of the spatial and spin wavefunctions. Since electrons are fermions, they must have antisymmetric total wavefunctions, which means the spin part and the spatial part have opposite parity.

$$H = J(S_1^x S_2^x + S_1^y S_2^y + kS_1^z S_2^z) + \mu(S_1^z + S_2^z)B$$
$$= J[\vec{S}_1\cdot\vec{S}_2 + (k-1)S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$
$$= J[\tfrac{1}{2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2) + (k-1)S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$
$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (k-1)S_1^z S_2^z] + \mu(S_1^z + S_2^z)B$$

Since $\vec{S}_i^2$ has eigenvalue $s_i(s_i+1)\hbar^2$ and $s_i = \frac{1}{2}$

$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (k-1)\tfrac{1}{2}(S_z^2 - S_{1z}^2 - S_{2z}^2)] + \mu S_z B$$
$$= J[\tfrac{1}{2}\vec{S}^2 - \tfrac{3}{4}\hbar^2 + (k-1)\tfrac{1}{2}(S_z^2 - \tfrac{\hbar^2}{2})] + \mu S_z B$$

because in an energy eigenstate $\vec{S}_1$ and $\vec{S}_2$ must be in the z-direction.

Now we see that the states $|s_1 s_2 s\, m\rangle$ are energy eigenstates:

$$|0\,0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \qquad |1\,1\rangle = |++\rangle$$
$$|1\,0\rangle = \tfrac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \qquad |1\,-1\rangle = |--\rangle$$

and the first is the only antisymmetric one.

a. Antisymmetric spatial wavefunction $\Rightarrow$ symmetric spin wavefunction

$$E_{|11\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (k-1)\tfrac{\hbar^2}{4}] + \mu\hbar B = \tfrac{1}{4}\hbar^2 Jk + \mu\hbar B$$
$$E_{|10\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (k-1)(-\tfrac{\hbar^2}{4})] + 0 = \tfrac{1}{4}\hbar^2 J(2-k)$$
$$E_{|1-1\rangle} = J[\tfrac{1}{2}(2\hbar^2) - \tfrac{3}{4}\hbar^2 + (k-1)(\tfrac{\hbar^2}{4})] - \mu\hbar B = \tfrac{1}{4}\hbar^2 Jk - \mu\hbar B$$

b. Symmetric spatial wavefunction $\Rightarrow$ antisymmetric spin wavefunction

$$E_{|00\rangle} = J[\tfrac{1}{2}(0) - \tfrac{3}{4}\hbar^2 + (k-1)(-\tfrac{\hbar^2}{4})] + 0 = -\tfrac{1}{4}\hbar^2 J(k+2)$$

2. *Quantum Mechanics* (Fall 2003)

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V = V_0 \left[\delta(\mathbf{r} - a\hat{\mathbf{z}}) - \delta(\mathbf{r} + a\hat{\mathbf{z}})\right].$$

Compute the differential cross section, $d\sigma/d\Omega$ in the Born approximation.

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\theta,\phi)|^2 \quad \text{where} \quad f^{(1)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{x}'}V(\vec{x}')\,d^3x'$$

$$\vec{k} = k\hat{z} \Rightarrow f^{(1)}(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int e^{ikz' - i\vec{k}'\cdot\vec{x}'}V_0\left[\delta^3(\vec{x}'-a\hat{z}) - \delta^3(\vec{x}'+a\hat{z})\right]d^3x'$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\left[e^{ika - ik_z'a} - e^{-ika + ik_z'a}\right]$$

$$= -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\left[2i\sin((k-k_z')a)\right]$$

Now $k_z' = k'\cos(\theta) = k\cos(\theta)$ since $k' = k$ by conservation of energy assuming the scattering body is much larger.

$$f^{(1)}(\theta,\phi) = -\frac{imV_0}{\pi\hbar^2}\sin(ak(1-\cos(\theta)))$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\theta,\phi)|^2 = \frac{m^2V_0^2}{\pi^2\hbar^4}\sin^2(ak(1-\cos(\theta)))$$

$$= \frac{m^2V_0^2}{\pi^2\hbar^4}\sin^2(2ak\sin^2(\theta/2))$$

3. *Quantum Mechanics* (Fall 2003)

Consider a particle moving in the potential

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2 & \text{if } x > 0 \\ \infty & \text{otherwise} \end{cases}$$

(a) What is the lowest energy eigenvalue?

(b) What is $\langle x^2 \rangle$?

See Griffiths Problem 2.42

a. We assume that the solutions to the half harmonic oscillator are a subset of the solutions to the full harmonic oscillator. Only the odd solutions are permissible.

To find the lowest odd solution we use the fact that $a|\psi_0\rangle = 0$

Recall $a = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x + ip)$

$$0 = a|\psi_0\rangle = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega x|\psi_0\rangle + \hbar\frac{\partial}{\partial x}|\psi_0\rangle\right)$$

$$\Rightarrow \frac{\partial}{\partial x}|\psi_0\rangle = -\frac{m\omega}{\hbar}x|\psi_0\rangle$$

$$\Rightarrow |\psi_0\rangle = Ae^{-\frac{m\omega}{2\hbar}x^2} \quad \text{which is even}$$

$$|\psi_1\rangle = a^+|\psi_0\rangle = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega x - ip)|\psi_0\rangle$$

$$\propto m\omega x e^{-\frac{m\omega}{2\hbar}x^2} - \hbar\frac{\partial}{\partial x}e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= m\omega x e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x e^{-\frac{m\omega}{2\hbar}x^2} \quad \text{which is odd}$$

So the lowest energy eigenvalue is $E_1 = (1+\frac{1}{2})\hbar\omega = \frac{3}{2}\hbar\omega$

b. $|\psi_1\rangle$ is odd so $|\langle\psi_1|\psi_1\rangle|^2$ is even which means the probability distribution for x is the same on both sides of zero for the full SHO. Therefore the standard deviation of x won't be affected by considering only the positive side. Therefore we can just calculate $\langle x^2\rangle$ for the full SHO.

$$\langle\psi_1|x^2|\psi_1\rangle = \langle\psi_1|\frac{\hbar}{2m\omega}(a^+ + a)^2|\psi_1\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_1|\cancel{(a^+)^2} + a^+a + aa^+ + \cancel{a^2}|\psi_1\rangle$$

$$= \frac{\hbar}{2m\omega}\langle\psi_1|1+2|\psi_1\rangle$$

$$= \frac{3\hbar}{2m\omega}$$

4. *Quantum Mechanics* (Fall 2003)

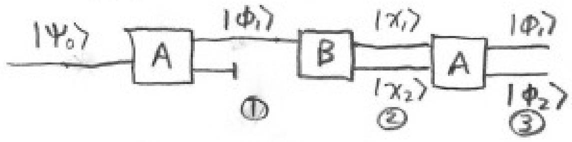
An operator A, corresponding to an observable α, has two normalized eigenfunctions ϕ_1 and ϕ_2, with distinct eigenvalues a_1 and a_2, respectively. An operator B, corresponding to an observable β, has normalized eigenfunctions χ_1 and χ_2, with distinct eigenvalues b_1 and b_2, respectively. The eigenfunctions are related by:

$$\phi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13}$$
$$\phi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}.$$

An experimenter measures α to be $42\hbar$. The experimenter proceeds to measure β, followed by measuring α again. What is the probability the experimenter will measure α to be $42\hbar$ again?

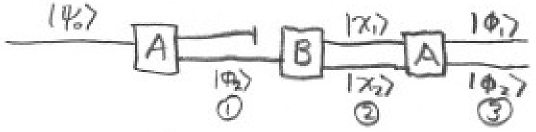
We know that the system starts in an eigenstate of A with eigenvalue $42\hbar$, but we don't know if this is $|\phi_1\rangle$ or $|\phi_2\rangle$ so we will check both cases.

Case $|\phi_1\rangle$:



$$P_2(\chi_1;\psi_1) = |\langle \chi_1|\psi_1\rangle|^2 = |\langle \chi_1|\phi_1\rangle|^2 = \frac{4}{13}$$
$$P_2(\chi_2;\psi_1) = |\langle \chi_2|\psi_1\rangle|^2 = |\langle \chi_2|\phi_1\rangle|^2 = \frac{9}{13}$$
$$P_3(\phi_1;\psi_2) = P_2(\chi_1;\psi_1)\,P_3(\phi_1;\chi_1) + P_2(\chi_2;\psi_1)\,P_3(\phi_1;\chi_2)$$
$$= \frac{4}{13}|\langle \phi_1|\chi_1\rangle|^2 + \frac{9}{13}|\langle \phi_1|\chi_2\rangle|^2$$
$$= \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2 = \frac{16}{169} + \frac{81}{169} = \frac{97}{169}$$

Case $|\phi_2\rangle$:



$$P_2(\chi_1;\psi_1) = |\langle \chi_1|\psi_1\rangle|^2 = |\langle \chi_1|\phi_2\rangle|^2 = \frac{9}{13}$$
$$P_2(\chi_2;\psi_1) = |\langle \chi_2|\psi_1\rangle|^2 = |\langle \chi_2|\phi_2\rangle|^2 = \frac{4}{13}$$
$$P_3(\phi_2;\psi_2) = P_2(\chi_1;\psi_1)\,P_3(\phi_2;\chi_1) + P_2(\chi_2;\psi_1)\,P_3(\phi_2;\chi_2)$$
$$= \frac{9}{13}|\langle \phi_2|\chi_1\rangle|^2 + \frac{4}{13}|\langle \phi_2|\chi_2\rangle|^2$$
$$= \left(\frac{9}{13}\right)^2 + \left(\frac{4}{13}\right)^2 = \frac{81}{169} + \frac{16}{169} = \frac{97}{169}$$

So in either case the probability is $\frac{97}{169}$

5. *Quantum Mechanics* (Fall 2003)

A sample of hydrogen atoms in the ground state is placed between the plates of a parallel plate capacitor. A voltage pulse is applied to the capacitor at $t = 0$ to produce a homogeneous electric field, ε, between the plates of:

$$\varepsilon = 0, \qquad (t < 0)$$

$$\varepsilon = \varepsilon_0 \exp(-t/\tau), \qquad (t > 0),$$

where τ is a constant. A long time compared to τ passes.

(a) To first order, calculate the fraction of atoms in the $2p$ $(m = 0)$ state.

(b) To first order, what is the fraction of atoms in the $2s$ state?

You may find the following helpful. The normalized radial wavefunctions of the hydrogen atom are:

$$R_{10}(r) = 2\left(\frac{Z}{a}\right)^{3/2} \exp\left(-\frac{Zr}{a}\right)$$

$$R_{20}(r) = \frac{1}{\sqrt{2}}\left(\frac{Z}{a}\right)^{3/2} \left(1 - \frac{Zr}{2a}\right) \exp\left(-\frac{Zr}{2a}\right)$$

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\left(\frac{Z}{a}\right)^{5/2} r \exp\left(-\frac{Zr}{2a}\right)$$

where r is the radial coordinate, a is the Bohr radius, and $Z = 1$ for a hydrogen atom. The first spherical harmonics are:

$$Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \qquad Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta) \qquad Y_{1\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)\exp(\pm i\phi)$$

A useful integral may be:

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

a. $P(f) = |\langle \Psi_f | \Psi \rangle|^2$ where $\langle \Psi_f | \Psi \rangle = \delta_{fi} - \frac{i}{\hbar}\int_0^t \langle \Phi_f | H'(t') | \Phi_i \rangle e^{i\omega_{fi} t'} dt'$

and $H'(t) = e\Phi = -eEz = -eE_0 e^{-t/\tau} z$ so

$$\langle 210 | H'(t') | 100 \rangle = -eE_0 e^{-t/\tau} \langle 210 | z | 100 \rangle$$

$$\langle 210 | z | 100 \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} R_{21}(r) Y_{10}(\theta,\phi)\, r\cos(\theta)\, R_{10}(r) Y_{00}(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{2\sqrt{6}} a^{-5/2} r e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos(\theta)\, r\cos(\theta)\, 2a^{-3/2} e^{-r/a} \frac{1}{\sqrt{4\pi}} r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= \frac{2\pi}{4\pi} \frac{1}{\sqrt{2}} a^{-4} \int_0^\pi \cos^2(\theta) \sin(\theta)\, d\theta \int_0^\infty r^4 e^{-3r/2a}\, dr$$

$$= \frac{a^{-4}}{2\sqrt{2}} \left(\frac{2}{3}\right) 4! = \frac{2^8}{\sqrt{2}\, 3^5} a$$

Therefore $P(|210\rangle) = \left| -\frac{i}{\hbar}\left(-eE_0 \frac{2^8}{\sqrt{2}\,3^5} a\right) \int_0^t e^{-t'/\tau} e^{i\omega_{fi} t'} dt' \right|^2$

$$= \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \left| \int_0^t e^{-t'/\tau + i\omega_{fi} t'} dt' \right|^2 = \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \left| \frac{e^{-t/\tau} e^{i\omega_{fi} t} - e^0}{-\frac{1}{\tau} + i\omega_{fi}} \right|^2$$

$$= \frac{2^{15}}{3^{10}} \frac{a^2 e^2 E_0^2}{\hbar^2} \frac{1}{\left(\frac{1}{\tau}\right)^2 + \omega_{fi}^2} \quad \text{where } \omega_{fi} = \frac{E_2 - E_1}{\hbar} = -\frac{3E_1}{4\hbar}$$

b. $P(|200\rangle) = 0$ to first order by the m-selection rule.

6. *Statistical Mechanics and Thermodynamics* (Fall 2003)

Consider a gas of non-conserved Bosons in three dimensions. The energy-verses-momentum relationship for each of these exotic particles is $E = Ap^2$.

(a) Calculate the grand partition function of this gas. Assume a spin degeneracy factor of 1.

(b) Find the energy of this gas as a function of temperature. The energy goes as a simple power law in the temperature. What is the power?

(c) Find the pressure as a function of temperature. What power law describes the temperature dependence of the pressure?

See Reif Page 347

a. $\mathcal{Z} \equiv \sum_{N'} Z(N') e^{-\alpha N'} \cong Z(N) e^{-\alpha N} \Delta^* N'$ since narrowly peaked

$\Rightarrow \ln(\mathcal{Z}) = \ln(Z(N)) - \alpha N$ ($\Delta^* N'$ Is not important if we take log)

$$\Rightarrow \mathcal{Z} = Z(N) e^{-\alpha N} = \left(\sum_R e^{-\beta E_R}\right) e^{-\alpha N}$$
$$= \sum_R e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \dots)} e^{-\alpha(n_1 + n_2 + \dots)}$$
$$= \sum_{n_1, n_2, \dots} e^{-(\alpha + \beta\epsilon_1)n_1 - (\alpha + \beta\epsilon_2) - \dots}$$
$$= \left(\sum_{n_1} e^{-(\alpha+\beta\epsilon_1)n_1}\right)\left(\sum_{n_2} e^{-(\alpha+\beta\epsilon_1)n_2}\right)\dots$$
$$= \left(\frac{1}{1-e^{-(\alpha+\beta\epsilon_1)}}\right)\left(\frac{1}{1-e^{-(\alpha+\beta\epsilon_2)}}\right)\dots$$
$$\Rightarrow \ln(\mathcal{Z}) = -\sum_r \ln\left(1 - e^{-(\alpha+\beta\epsilon_r)}\right)$$
$$= -\sum_r \ln\left(1 - e^{-\beta\epsilon_r}\right)$$ since $\alpha = 0$ for non-conserved particles

$$\Rightarrow \ln(\mathcal{Z}) = -\int_0^\infty \ln(1 - e^{-\beta\epsilon}) \rho(\epsilon) d\epsilon$$

$\rho(\epsilon) d\epsilon = \rho(\vec{n}) d^3 n = \frac{1}{8} 4\pi n^2 dn \qquad \epsilon = Ap^2 = A\hbar^2 \frac{\pi^2 n^2}{L^2}$

$$\Rightarrow n^2 = \frac{L^2}{A\hbar^2\pi^2}\epsilon \Rightarrow n = \frac{L}{\hbar\pi\sqrt{A}}\epsilon^{1/2} \Rightarrow dn = \frac{L}{\hbar\pi\sqrt{A}} \frac{\epsilon^{-1/2}}{2} d\epsilon$$
$$\Rightarrow \rho(\epsilon) = \frac{\pi}{2} n^2 dn = \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \epsilon^{1/2} d\epsilon$$
$$\Rightarrow \ln(\mathcal{Z}) = -\frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \epsilon^{1/2} \ln(1 - e^{-\beta\epsilon}) d\epsilon$$

b. Note that $\alpha = 0 \Rightarrow \ln(\mathcal{Z}) = \ln(Z)$ so

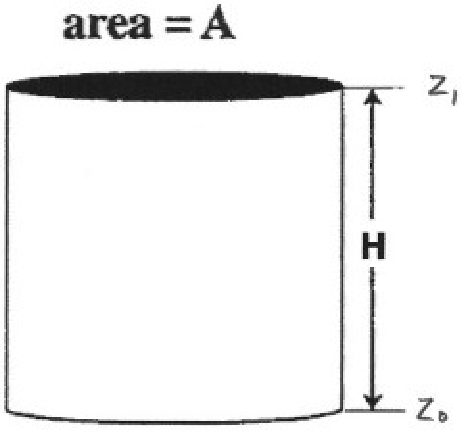
$$E = -\frac{\partial \ln(Z)}{\partial\beta} = \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \frac{\epsilon^{1/2}}{1-e^{-\beta\epsilon}} \epsilon e^{-\beta\epsilon} d\epsilon$$
$$= \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \int_0^\infty \frac{\epsilon^{3/2}}{e^{\beta\epsilon}-1} d\epsilon \qquad \text{Let } x = \beta\epsilon$$
$$= \frac{\pi}{4}\left(\frac{L}{\hbar\pi\sqrt{A}}\right)^3 \frac{1}{\beta^{5/2}} \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx \Rightarrow E \propto T^{5/2}$$

c. The pressure of a photon gas like this is $p = \frac{1}{3}\frac{E}{V}$

so $p \propto T^{5/2}$

7. *Statistical Mechanics and Thermodynamics* (Fall 2003)

A gas of noninteracting particles fills a cylindrical container that has cross-sectional area A and height H. Each particle has mass m, and is subject to the gravitational field at the surface of the Earth. The circular bottom and top of the container are parallel to the surface of the Earth. There are N particles in the container, and the temperature of the container is T.



(a) Find the partition function of the gas.

(b) What is the pressure of the gas at the top of the container?

(c) What is the pressure of the gas at the bottom of the container?

(d) Finally, what is the difference between the pressure at the bottom of the container and the pressure at the top of the container? Interpret the answer that you get.

a. $$\mathcal{Z} = \sum_r e^{-\beta E_r} \cong \frac{1}{h^3}\iint e^{-(\frac{p^2}{2m}+mgz)\beta}\, d^3p\, d^3x$$
$$= \frac{1}{h^3} A \iint e^{-(p^2/2m+mgz)\beta}\, 4\pi p^2 dp\, dz$$
$$= \frac{1}{h^3} 4\pi A \left(\frac{2m}{\beta}\right)^{3/2} \int_0^\infty u^2 e^{-u^2} du \int_{z_0}^{z_1} e^{-mg\beta z} dz$$
$$= \frac{1}{h^3} 4\pi A \left(\frac{2m}{\beta}\right)^{3/2} \frac{\sqrt{\pi}}{4} \left(-\frac{1}{mg\beta}\right)\left(e^{-mg\beta z_1} - e^{-mg\beta z_0}\right)$$
$$= \frac{A}{mg\beta}\left(\frac{2m\pi}{\beta h^2}\right)^{3/2}\left(e^{-mg\beta z_0} - e^{-mg\beta z_1}\right)$$
$$\Rightarrow \ln(Z) = \ln\left(\frac{z^N}{N!}\right) = N\ln\left(\frac{A}{mg\beta}\right) + \frac{3}{2}N\ln\left(\frac{2m\pi}{\beta h^2}\right) + N\ln\left(e^{-mg\beta z_0} - e^{-mg\beta z_1}\right) - N\ln(N)$$

b. $$P_{top} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V_{top}} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial z_1}\frac{\partial z_1}{\partial V} = \frac{1}{\beta A}\frac{\partial \ln(Z)}{\partial z_1} = \frac{N}{\beta A}\frac{mg\beta e^{-mg\beta z_1}}{e^{-mg\beta z_0} - e^{-mg\beta z_1}}$$
$$= \frac{mgN}{A}\frac{1}{e^{mg\beta(z_1 - z_0)} - 1} = \frac{mgN}{A}\frac{1}{e^{mg\beta H} - 1}$$

c. $$P_{bot} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V_{bot}} = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial z_0}\frac{\partial z_0}{\partial V} = -\frac{1}{\beta A}\frac{\partial \ln(Z)}{\partial z_0} = \frac{-N}{\beta A}\frac{-mg\beta e^{-mg\beta z_0}}{e^{-mg\beta z_0} - e^{-mg\beta z_1}}$$
$$= \frac{mgN}{A}\frac{1}{1 - e^{mg\beta(z_0 - z_1)}} = \frac{mgN}{A}\frac{1}{1 - e^{-mg\beta H}}$$

d. $$\Delta p = P_{bot} - P_{top} = \frac{mgN}{A}\left(\frac{1}{1 - e^{-mg\beta H}} - \frac{1}{e^{mg\beta H} - 1}\right)$$
$$= \frac{mgN}{A}\left(\frac{e^{mg\beta H}}{e^{mg\beta H} - 1} - \frac{1}{e^{mg\beta H} - 1}\right) = \frac{mgN}{A} = \frac{mgN}{V}H = nmgH$$

The increased pressure at the bottom of the container is due to the force from the weight of the gas above it.

8. *Electricity and Magnetism* (Fall 2003)

Consider a vacuum diode which is a parallel plate capacitor (in vacuum) with plate area A and plate separation d. The cathode plate, which is at $\phi = 0$, is heated as to thermionically emit electrons which then travel to the anode plate (at $\phi = V$) (this arrangement acts as a diode due to the fact that in reverse bias, no charges will flow). Assume a steady-state bias V and diode current I. You may model the electrons in the diode as a cold fluid with density $n(x)$ and velocity $v(x)$. You may assume that the electrons are born from the cathode with zero velocity.

(a) Find the 1-D potential distribution in the diode, $\phi(x)$. (Hint: Try a power law solution.)

(b) Find the diode current as a function of bias voltage V.

(c) What unphysical result is caused by the assumption that electrons are born from the cathode with zero velocity?

a. We use $I = \int \vec{J} \cdot d\vec{a}$, $\vec{J}(x) = \rho(x)\vec{v}(x)$, and $\rho(x) = en(x)$ to write $I(x) = \int \vec{J}(x) \cdot d\vec{a} = AJ(x) = A\rho(x)v(x) = Aen(x)v(x)$

By conservation of energy, $\frac{1}{2}mv^2(x) = e\Phi(x)$ where $\Phi(0)=0$, $\Phi(d)=V$

so by Gauss' Law $\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \Rightarrow$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{I}{\epsilon_0 A v(x)} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}}$$

Now assume a power law solution: $\Phi(x) = Cx^n$

$$\Phi(d) = V \Rightarrow V = Cd^n \Rightarrow C = Vd^{-n} \Rightarrow \Phi(x) = V\left(\frac{x}{d}\right)^n$$

Now we find n by matching exponents of x:

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(\frac{V}{d^n}x^n\right) = n\frac{V}{d^n}\frac{\partial}{\partial x}(x^{n-1}) = n(n-1)\frac{V}{d^n}x^{n-2}$$

$$\frac{\partial^2\Phi}{\partial x^2} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e}}V^{-1/2}d^{n/2}x^{-n/2}$$

$$\Rightarrow n-2 = -n/2 \Rightarrow \frac{3}{2}n = 2 \Rightarrow n = \frac{4}{3}$$

Therefore $\Phi(x) = V\left(\frac{x}{d}\right)^{4/3}$

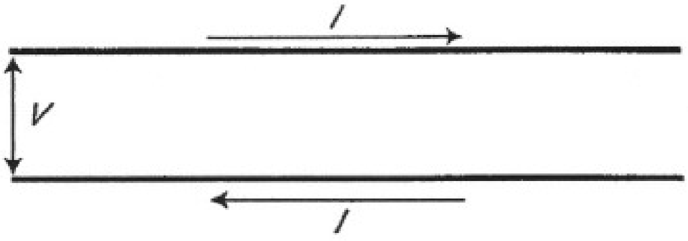
b. $$\frac{\partial^2\phi}{\partial x^2} = -\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2e\Phi(x)}} = n(n-1)\frac{V}{d^n}x^{n-2} = \frac{4}{9}\frac{V}{d^{4/3}}x^{-2/3}$$

At $x=d$, $-\frac{1}{\epsilon_0}\frac{I}{A}\sqrt{\frac{m}{2eV}} = \frac{4}{9}Vd^{-2}$

$$\Rightarrow I = -\frac{4}{9}\epsilon_0 V\frac{A}{d^2}\sqrt{\frac{2eV}{m}} = -\frac{4}{9}\epsilon_0 V^{3/2}\frac{A}{d^2}\sqrt{\frac{2e}{m}}$$

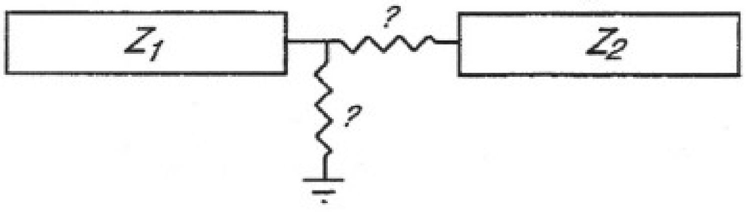
c. $I = Aen(x)v(x)$ must be independent of x, so it is nonzero everywhere if the current is nonzero, so if $v(x)=0$, then $n(x)=\infty$, which is unphysical.

9. *Electricity and Magnetism* (Fall 2003)



(a) A two-wire transmission line has inductance L and capacitance C per unit length (and no resistance). Show that the impedance of this transmission line $Z = V/I$ is real and equal to $\sqrt{L/C}$ (Note: Assume AC signals are transmitted on the line, $I = I_o \exp(ikx - i\omega t)$).

(b) Two long transmission lines are connected together. The first has impedance Z_1 and the second has impedance $Z_2 \neq Z_1$. A wave $V_i \exp(ikx - i\omega t)$ travels on the first transmission line and encounters the second. What are the relative amplitudes of the reflected and transmitted waves $(V_r/V_i, V_t/V_i)$?



(c) Reflection due to impedance mismatch between two transmission lines can be eliminated through adding series or parallel resistance between the lines. For the transmission lines in (b), how would you connect a resistor (and what is its values) in order to match the impedances and eliminate the reflected wave? (Consider both $Z_1 > Z_2$ and $Z_1 < Z_2$.)

a. $Q = CV \Rightarrow I = \frac{dQ}{dt} = C\frac{dV}{dt} = C\frac{d}{dt}\left(-\frac{d\Phi_B}{dt}\right) = -C\frac{d^2}{dt^2}(LI)$

$= -CL(-i\omega)^2 I \Rightarrow CL\omega^2 = -1 \Rightarrow \omega = \pm i\sqrt{\frac{1}{CL}}$

$\Rightarrow V = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} = i\omega L I = \pm\sqrt{\frac{L}{C}}\, I \Rightarrow Z = \frac{V}{I} = \sqrt{\frac{L}{C}}$

b. Recall $Z = \sqrt{\frac{\mu}{\epsilon}}$ and $v = \frac{c}{n}$ so $n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} \cong \frac{Z_0}{Z}$ since $\mu \cong \mu_0$

Then for normal incidence $\frac{E_0^r}{E_0} = \frac{n'-n}{n'+n}$ and $\frac{E_0^t}{E_0} = \frac{2n}{n'+n}$

$\Rightarrow \frac{V_r}{V_i} = \frac{n_2-n_1}{n_2+n_1} = \frac{\frac{1}{Z_2}-\frac{1}{Z_1}}{\frac{1}{Z_2}+\frac{1}{Z_1}} = \frac{Z_1-Z_2}{Z_1+Z_2}$, $\frac{V_t}{V_i} = \frac{2n_1}{n_2+n_1} = \frac{2\frac{1}{Z_1}}{\frac{1}{Z_2}+\frac{1}{Z_1}} = \frac{2Z_2}{Z_1+Z_2}$

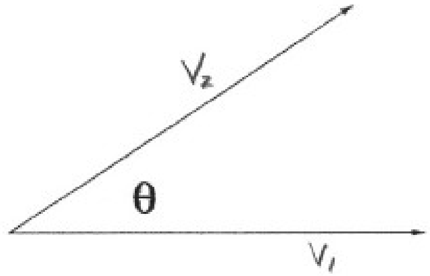
c. We want to make the total impedance after Z_1 equal to Z_1, and since Z is real this just means equating the resistances, where Z_1 and Z_2 are themselves resistances.

Case $Z_1 > Z_2$: Series resistor $R = Z_1 - Z_2$

Case $Z_1 < Z_2$: Parallel resistor $\frac{1}{R} + \frac{1}{Z_2} = \frac{1}{Z_1} \Rightarrow R = \frac{Z_1 Z_2}{Z_2 - Z_1}$

10. *Electricity and Magnetism* (Fall 2003)

Consider a wedge formed by two conducting half-planes, as depicted in the figure. One plane is maintained at electrostatic potential V_1 while the other is at V_2. What is the electrostatic potential in the region between the two half-planes?



We solve Laplace's equation in cylindrical coordinates

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \cancel{\frac{\partial^2 \Phi}{\partial z^2}} = 0$$

We seek solutions of the form $\Phi(r,\phi) = R(r)\,Q(\phi)$

$$\frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = 0 \quad \left(\text{by multiplying by } \frac{r^2}{RQ}\right)$$

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda R \quad \text{and} \quad \frac{\partial^2 Q}{\partial \phi^2} = -\lambda Q \quad (\text{by independence of variables})$$

When r ranges from 0 to ∞, only the $\lambda = 0$ eigenvalue is possible

$$\Rightarrow R(r) = A + B\ln(r) \quad \text{and} \quad Q(\phi) = C + D\phi$$

B.C. $\Phi(r=\infty) < \infty \quad \Rightarrow B = 0 \quad \Rightarrow \Phi(r,\phi) = C + D\phi$

$\Phi(r,0) = V_1 \quad \Rightarrow \quad C = V_1 \qquad \Phi(r,\theta) = V_2 \Rightarrow V_1 + D\theta = V_2$

$\Rightarrow D = \dfrac{V_2 - V_1}{\theta} \qquad$ Therefore $\Phi(r,\theta) = V_1 + \dfrac{V_2 - V_1}{\theta}\phi$

11. *Electricity and Magnetism* (Fall 2003)

The dispersion relation for a photon in an ionized plasma (in *CGS* units) is,

$$k^2c^2 = \omega^2 - 4\pi ne^2/m_e$$

where k is the photon wavenumber, $c = 3.0 \times 10^{10}$ cm/s, and ω is the radiation frequency in radians/s. Here, n is the electron number density, $e = 4.8 \times 10^{-10}$ esu is the electron charge, and $m_e = 9.11 \times 10^{-28}$ g is the electron mass.

(a) *Explain* why electromagnetic waves with frequencies below about 10 MHz can't be received from space on Earth.

(b) Pulsars are objects observed in our galaxy which regularly emit a short burst of electromagnetic waves containing a wide range of frequencies all at once. If a pulsar is located 1.0×10^{22} cm away and the density of electrons in the space between us and the pulsar is a uniform 0.01 cm^{-3}, what is the difference of the arrival times at Earth of the radiation emitted near 6 kHz compared to 10 kHz? (You may assume the measurement happens far enough above the Earth so that the effect in part (a) can be ignored. You may leave your answer as an expression without substituting the numbers.)

a. For low frequenencies the value of the dispersion relation will become negative, which means that K becomes imaginary, corresponding to evanescent waves which have an exponentially decaying amplitude.

b. Let $L = 1.0 \times 10^{22}$ cm, $f_1 = 6$ KHz, $f_2 = 10$ KHz, $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$, $n = 0.01$ cm^{-3}

$t_1 = \frac{L}{V_1} = \frac{L}{c}$ and $t_2 = \frac{L}{V_2} = \frac{L}{c} n_2$

where $n_i = \frac{c}{V} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}} \cong \sqrt{\frac{\epsilon(\omega)}{\epsilon_0}}$ since $\mu \cong \mu_0$

But $v = \frac{\omega}{K} \Rightarrow n_i = \sqrt{\frac{c^2}{v^2}} = \sqrt{\frac{K^2c^2}{\omega^2}} = \sqrt{1 - \frac{4\pi ne^2}{\omega^2 m_e}} =$

Therefore $\Delta t = |t_2 - t_1| = \left| \sqrt{1 - \frac{4\pi ne^2}{\omega_2^2 m_e}} - \sqrt{1 - \frac{4\pi ne^2}{\omega_1^2 m_e}} \right|$

Problem # 1 Fall 2003

$$H = J\left(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$$\vec{S}_1 \cdot \vec{S}_2 = S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z$$

$$H = J\left(\vec{S}_1 \cdot \vec{S}_2 + (K-1) S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$$\vec{S}^2 = \left(\vec{S}_1 + \vec{S}_2\right)^2 = \vec{S}_1^2 + 2\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2^2$$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - S_2^2\right)$$

$$H = J\left(\frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right) + (K-1) S_1^z S_2^z\right) + \mu\left(S_1^z + S_2^z\right)B$$

$\frac{1}{2}\left(\frac{1}{2}+1\right)$

(Sym) $1(1+1)$

Asym $0(0+1)$

a) antisymmetric spatial wave function ...

For fermions the total wave function must be anti-symmetric, therefore the spin piece must be symmetric

$\langle \text{sym} | H | \text{sym} \rangle$

$$\langle H \rangle_{sym} = \langle 1\, m | H | 1\, m \rangle$$

$$= J\left(\frac{\hbar^2}{2} \Big\{ \underbrace{1(1+1)}_{2} - \underbrace{2 \cdot \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}_{\frac{3}{4}}}_{\frac{3}{2}} \Big\} + (K-1)\hbar^2 m_1 m_2 \right) + \mu\hbar(m_1+m_2)B$$

$$\boxed{E = J\hbar^2\left(\frac{1}{4} + (K-1)m_1 m_2\right) + \mu B\hbar(m_1+m_2)}$$

$$E_{\uparrow\uparrow} = \langle \uparrow\uparrow | H | \uparrow\uparrow \rangle = J\hbar^2\left(\frac{1}{4} + \frac{(K-1)}{4}\right) + \mu B\hbar = \boxed{\frac{J\hbar^2 K}{4} + \mu B\hbar}$$

$$E_{\downarrow\downarrow} = \langle \downarrow\downarrow | H | \downarrow\downarrow \rangle = J\hbar^2\left(\frac{1}{4} + \frac{(K-1)}{4}\right) - \mu B\hbar = \boxed{\frac{J\hbar^2 K}{4} - \mu B\hbar}$$

$$E_{\downarrow\uparrow} = \frac{1}{2}\left[\langle \uparrow\downarrow | H | \uparrow\downarrow \rangle + \langle \downarrow\uparrow | H | \downarrow\uparrow \rangle\right] = \frac{2J\hbar^2}{2}\left(\frac{1}{4} - \frac{(K-1)}{4}\right)$$

$$E_{\downarrow\uparrow} = J\hbar^2\left(\frac{1}{4} - \frac{(K-1)}{4}\right) = \boxed{\frac{J\hbar^2(2-K)}{4}}$$

b) for antisymmetric state

$$E = J\hbar^2\left(-\frac{3}{4} + (K-1)m_1 m_2\right) + \mu\hbar B\underbrace{\cancel{(m_1+m_2)}}_{\to 0}$$

$$E_{\uparrow\downarrow} = \frac{1}{2}\left[\langle \uparrow\downarrow | H | \uparrow\downarrow \rangle - \langle \downarrow\uparrow | H | \downarrow\uparrow \rangle\right] = J\hbar^2\left(-\frac{3}{4} - \frac{(K-1)}{4}\right)$$

$$= \boxed{J\hbar^2\left(-\frac{(K-2)}{4}\right)}$$

Fall 2003 #1 (p 1 of 3)

Two identical spin-$\frac{1}{2}$ particles interact via the Hamiltonian

$$H = J\left(S_1^x S_2^x + S_1^y S_2^y + K S_1^z S_2^z \right) + \mu (S_1^z + S_2^z) B$$

(a) Find the energy levels of this system assuming that the particles are in an anti-symmetric spatial wave function.

The total wave function for electrons must be antisymmetric, so, if the spatial part is anti-symmetric, then the spin part must be symmetric to preserve the symmetry.

Now, let's rewrite the Hamiltonian into a more user friendly form.

$$S^2 = (S_1 + S_2)^2 = S_1^2 + S_2^2 + 2 S_1 \cdot S_2 = S_1^2 + S_2^2 + 2(S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z)$$

$$\Rightarrow S_1^x S_2^x + S_1^y S_2^y = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) - S_1^z S_2^z$$

So, then the Hamiltonian becomes

$$H = J\left[\frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right) + (K-1) S_1^z S_2^z \right] + \mu B (S_1^z + S_2^z)$$

note $S_i^2 = S_i(S_i + 1) = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}$

So,

$$H = J\left[\frac{1}{2} S^2 - \frac{3}{4} + (K-1) S_1^z S_2^z \right] + \mu B (S_1^z + S_2^z)$$

what are the possible values of S?

$$|S_1 - S_2| \leq S \leq |S_1 + S_2|$$

$$\left|\frac{1}{2} - \frac{1}{2}\right| \leq S \leq \left|\frac{1}{2} + \frac{1}{2}\right|$$

$$0 \leq S \leq 1$$

$$\Rightarrow S = 0 \text{ or } 1$$

Fall 2003 #1 (p 2 of 3)

For $S=0$, $m_s=0$ and the only state possible is

$$|00\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad (1)$$

note: $|00\rangle$ is anti-symmetric ... if you let $\uparrow \to \downarrow$ and $\downarrow \to \uparrow$, you get back $-|00\rangle$

For $s=1$, $m_s = -1, 0, 1$ and the 3 states are

$$|1,-1\rangle = |\downarrow\downarrow\rangle$$

$$|1,0\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \qquad (2)$$

$$|1,1\rangle = |\uparrow\uparrow\rangle$$

note: all these states are symmetric. let $\uparrow \to \downarrow$ and $\downarrow \to \uparrow$ in $|1,0\rangle$ and you get back $|1,0\rangle$ ... The other two are trivially symmetric. So, for part (c), we want to find the energy levels of the $S=1$ (symmetric spin part).

So, we have

$\underline{|1,-1\rangle}$ $\langle 1,-1|H|1,-1\rangle = J\left[\frac{1}{2}S(S+1) - \frac{3}{4} + (K-1)S_1^z S_2^z\right] + \mu B(S_1^z + S_2^z)\Big|_{S=1}$

$$= J\left[1 - \frac{3}{4} + (K-1)\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\right] + \mu B\left(-\tfrac{1}{2} + -\tfrac{1}{2}\right)$$

$$= J\left[\frac{1}{4} + \frac{K}{4} - \frac{1}{4}\right] - \mu B$$

$$\therefore \boxed{E_{1,-1} = \frac{JK}{4} - \mu \vec{B}}$$

$|1,0\rangle$ $\langle 10|H|10\rangle = \frac{1}{2}\Big[J\left(\frac{1}{2}\cdot 2 - \frac{3}{4} + (K-1)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right) + \mu B\left(\frac{1}{2}-\frac{1}{2}\right) +$

$$+ J\left(1 - \frac{3}{4} + (K-1)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right) + \mu B\left(-\frac{1}{2}+\frac{1}{2}\right)\Big]$$

$$= \frac{1}{2}\left[J\left(\frac{1}{4} - \frac{K}{4} + \frac{1}{4}\right) + J\left(\frac{1}{4} - \frac{K}{4} + \frac{1}{4}\right)\right]$$

$$= \frac{2J}{2}\left(\frac{1}{2} - \frac{K}{4}\right)$$

$$\therefore \boxed{E_{10} = J\left(\frac{1}{2} - \frac{K}{4}\right)}$$

$|1,1\rangle$ $\langle 1,1|H|1,1\rangle = J\left[\frac{1}{4} + (K-1)\left(\frac{1}{4}\right)\right] + \mu B\left(\frac{1}{2} + \frac{1}{2}\right)]$

$$= J\left(\frac{K}{4}\right) + \mu B$$

$$\Rightarrow \boxed{E_{11} = \frac{JK}{4} + \mu B}$$

b) repeat for a symmetric spatial wave function

Now, we use the anti-symmetric spin part (eq (1)).

$|0,0\rangle$ $\langle 00|H|00\rangle = \frac{J}{2}\left[-\frac{3}{4} + (K-1)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\right] + \frac{\mu B}{2}\cancelto{0}{\left(\frac{1}{2}-\frac{1}{2}\right)}$

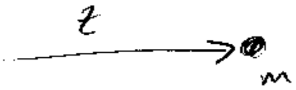
$$-\frac{J}{2}\left[-\frac{3}{4} + (K-1)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\right] - \frac{\mu B}{2}\cancelto{0}{\left(-\frac{1}{2}+\frac{1}{2}\right)}$$

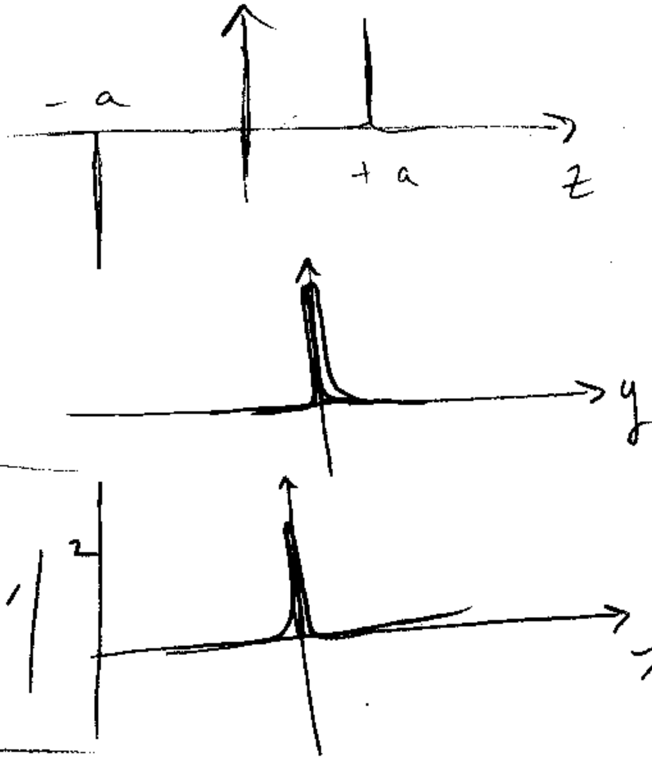
$$\Rightarrow \boxed{E_{00} = 0}$$

Problem #2 Fall 2003

$\delta(x)\,\delta(y)\,\delta(\cancel{z}-a)$

$$V = V_0\left[\delta(\vec{r}-a\hat{z}) - \delta(\vec{r}+a\hat{z})\right]$$





$$\boxed{\frac{d\sigma}{d\Omega} = \left|f(\theta,\phi)\right|^2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^4}\left|\int e^{i\vec{q}\cdot\vec{r}\,'}V(\vec{r}\,')\,d^3r'\right|^2}$$

$$f(\theta,\phi) = V_0\int e^{i\vec{q}\cdot\vec{r}\,'}\left(\delta(\vec{r}-a\hat{z}) - \delta(\vec{r}+a\hat{z})\right)d^3r'$$

$$= V_0\left(e^{i\vec{q}\cdot a\hat{z}} - e^{-i\vec{q}\cdot a\hat{z}}\right) = V_0\,2\sin(\vec{q}\cdot a\hat{z})$$

$q_z = q\sin\left(\frac{\theta}{2}\right)$

$q = 2K\sin\left(\frac{\theta}{2}\right)$

$$= V_0\,2\sin\left(2K\sin^2(\theta/2)\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2V_0^2}{\pi^2\hbar^4}\sin^2\left(2K\sin^2(\theta/2)\right)}$$

Fall 2003 # 2) (p 1 of 2)

A free particle of mass m, travelling with momentum p parallel to the z-axis, scatters off the potential

$$V = V_0 \left[\delta(\vec{r} - a\hat{z}) - \delta(\vec{r} + a\hat{z}) \right].$$

Compute the differential cross section, $\frac{d\sigma}{d\Omega}$ in the Born approximation.

(see Zettili problem 11.2, p 618)

First re-write the delta functions as

$$\delta(\vec{r} \pm a\vec{z}) = \delta(x)\,\delta(y)\,\delta(z \pm a)$$

Now, we recognize that this is not a spherically symmetric potential. So the first Born approximation scattering amplitude is then (Abers eq 8.36)

$$f^{(1)}(\theta,\phi) = -\frac{2m}{4\pi}\int d^3r\, V(r)\, e^{i\vec{q}\cdot\vec{r}} \qquad (1)$$

where $\vec{q}$ is the momentum transfer defined as

$$\vec{q} = \vec{k} - \vec{k}'$$

then

$$q^2 = |\vec{k}|^2 + |\vec{k}'|^2 - 2\vec{k}\cdot\vec{k}$$

Since this is an elastic collision, $|\vec{k}'| = |\vec{k}|$. So,

$$q^2 = 2k^2(1 - \cos\theta) = 4k^2 \sin^2\left(\frac{\theta}{2}\right) \qquad (2)$$

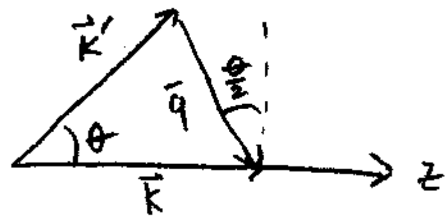
Substituting our expression for $V(r)$ into eq (1) yields

$$f^{(1)}(\theta,\phi) = \frac{-2m}{4\pi} V_0 \int dx\, \delta(x)\, e^{iq_x x} \int dy\, \delta(y)\, e^{iq_y y} \int dz \left(\delta(z-a) - \delta(z+a) \right) e^{iq_z z}$$

$$= -\frac{mV_0}{2\pi}\left[e^{iq_z a} - e^{-iq_z a} \right] = \frac{-mV_0}{\pi}\, i \sin(q_z a)$$

Fall 2003 #2 (p 2 of 2)

but, what is q_z ?



$\leftarrow q_z = q \sin\left(\frac{\theta}{2}\right)$

From eq (2), we have an expression for q. So,

$$q_z = 2k \sin^2\left(\frac{\theta}{2}\right)$$

Substituting this result into our expression for the scattering amplitude yields

$$f^{(1)}(\theta,\phi) = \frac{-mV_0}{\pi} i \sin\left[2ka \sin^2\left(\frac{\theta}{2}\right)\right]$$

Then, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left|f^{(1)}(\theta,\phi)\right|^2$$

$$\therefore \boxed{\frac{d\sigma}{d\Omega} = \frac{m^2 V_0^2}{\pi^2} \sin^2\left[2ka \sin^2\left(\frac{\theta}{2}\right)\right]}$$

Problem # 3 Fall 2003

$$V(x) = \begin{cases} \frac{1}{2} m \omega^2 x^2 , & x > 0 \\ \infty , & \text{otherwise} \end{cases}$$

a) $$E_n = \hbar\omega\left((2n+1) + \frac{1}{2}\right) = \hbar\omega\left(2n + \frac{3}{2}\right)$$

$$\boxed{E_0 = \hbar\omega\left(\frac{3}{2}\right) = \frac{3}{2}\hbar\omega}$$

b) $$x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a} + \hat{a}^{+})$$

$$x^2 = \frac{\hbar}{2m\omega}\left(aa + aa^{+} + \underbrace{a^{+}a}_{N} + a^{+}a^{+}\right)$$

$$[a, a^{+}] = aa^{+} - a^{+}a = 1$$

$$aa^{+} = N + 1$$

$$x^2 = \frac{\hbar}{2m\omega}\left(aa + 2N + 1 + a^{+}a^{+}\right)$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}\langle 2n+1|(2N+1)|2n+1\rangle = \frac{\hbar}{2m\omega}\left[2(2n+1)+1\right]$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}(4n+3) = \frac{\hbar}{m\omega}\left(2n + \frac{3}{2}\right)}$$

$$\langle T \rangle = \frac{3}{4}\hbar\omega = \frac{1}{2}E_0$$

$$\langle x^2 \rangle = \frac{2}{m\omega^2}\langle T \rangle = \frac{3}{2}\frac{\hbar}{m\omega}$$

Fall 2003 #3 (p 1 of 2)

Consider a particle moving in the potential

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & x > 0 \\ \infty & \text{otherwise} \end{cases}$$

(a) what is the lowest energy eigenvalue?

(See Zettili problem 4.9, p253)

This is an unsymmetric harmonic oscillator potential. So, we must have the wave function vanish at $x=0$. So, those solutions must be those of an ordinary (symmetric) harmonic oscillator that have odd parity since only odd solutions vanish at the origin.

So, since we already know the energies of a symmetric harmonic oscillator

$$E_n = (n + \tfrac{1}{2})\omega$$

Then the energies of this unsymmetric potential must be given by those corresponding to the odd n energy levels of ~~the~~ symmetric potential. That is,

$$E_n = \left[(2n+1) + \tfrac{1}{2}\right]\omega$$

$$\therefore \quad E_n = \left[2n + \tfrac{3}{2}\right]\omega$$

So, the lowest energy eigenvalue is

$$\boxed{E_0 = \frac{3}{2}\omega}$$

(b) what is $\langle x^2 \rangle$?

From the virial theorem for harmonic oscillators, we know that

$$\langle V \rangle = \frac{E_n}{2}$$

Since $\langle V(x) \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle$, Then $\langle x^2 \rangle = \frac{2}{m\omega^2} \langle V \rangle$

and thus,

$$\langle x^2 \rangle = \frac{2}{m\omega^2} \frac{E_n}{2} = \frac{E_n}{m\omega^2}$$

$$\Rightarrow \boxed{\langle x^2 \rangle = \frac{\left(2n + \frac{3}{2}\right)}{m\omega}}$$

For lowest energy, we have

$$\langle x^2 \rangle = \frac{3}{2m\omega}$$

Problem #4 Fall 2003

$$A(\phi_1 + \phi_2) = \overbrace{a_1\phi_1 + a_2\phi_2}^{\alpha}$$

$$B(\chi_1 + \chi_2) = \underbrace{b_1\chi_1 + b_2\chi_2}_{\beta}$$

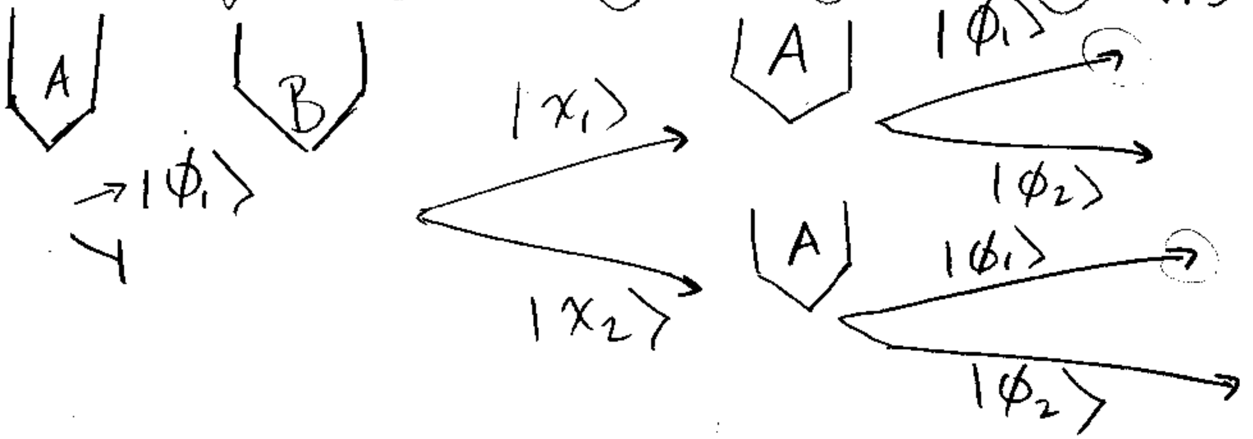
$$\phi_1 = (2\chi_1 + 3\chi_2)/\sqrt{13}$$

$$\phi_2 = (3\chi_1 - 2\chi_2)/\sqrt{13}$$

because we measure A first, then we know that after we measure A, we have an eigenstate of A

If $42\hbar = a_1$, then we have the state ϕ_1 when we get around to measure B

$$B|\phi_1\rangle = B\frac{(2\chi_1 + 3\chi_2)}{\sqrt{13}} = \frac{2b_1\chi_1 + 3b_2\chi_2}{\sqrt{13}}$$



B)

$$P_{b_1} = |\langle \chi_1 | I | \phi_1 \rangle|^2 = \frac{4}{13} \rightarrow \text{path } ①$$

$$P_{b_2} = |\langle \chi_2 | I | \phi_1 \rangle|^2 = \frac{9}{13} \longrightarrow \text{path } ②$$

A)

Path ① $P_{a_1} = |\langle \phi_1 | \chi_1 \rangle|^2 = \frac{4}{13}$

Path ② $P_{a_1} = |\langle \phi_1 | \chi_2 \rangle|^2 = \frac{9}{13}$

$$\boxed{P_{\phi_1} = \left(\frac{4}{13}\right)^2 + \left(\frac{9}{13}\right)^2}$$

$\longrightarrow$ what if start with $|\phi_2\rangle$?

$$P = ||\langle \chi_1 | \phi_2 \rangle|^2|^2 + |\langle \chi_2 | \phi_2 \rangle|^4$$

$$\boxed{P = \left(\frac{9}{13}\right)^2 + \left(\frac{4}{13}\right)^2}$$

SAME

$$\boxed{P = \frac{97}{169}} \checkmark$$

An operator A, corresponding to an observable α, has two normalized eigenfunctions ϕ_1 and ϕ_2, with distinct eigenvalues a_1 and a_2, respectively. An operator B, corresponding to an observable β, has normalized eigenfunctions χ_1 and χ_2, with distinct eigenvalues b_1 and b_2, respectively. The eigenfunctions are related by

$$\phi_1 = \frac{1}{\sqrt{13}}(2\chi_1 + 3\chi_2)$$

$$\phi_2 = \frac{1}{\sqrt{13}}(3\chi_1 - 2\chi_2)$$

An experimenter measures α to be $42\hbar$. The experimenter proceeds to measure β, followed by α again. What is the probability the experimenter will measure α to be $42\hbar$ again?

we are told that

$$A|\phi_1\rangle + A|\phi_2\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$$

however when the experimenter measures α to be $42\hbar$, we do not know if that corresponds to a_1 or a_2 ($|\phi_1\rangle$ or $|\phi_2\rangle$). So, we need to consider two cases:

case (1) $a_1 = 42\hbar$

in this case, the state is in $|\phi_1\rangle$ after the first measurement. Then the state immediately after is

$$|\psi_B\rangle = \sum_{n=1}^{2} |\chi_n\rangle\langle\chi_n|\phi_1\rangle$$

$$= \frac{2}{\sqrt{13}}|\chi_1\rangle + \frac{3}{\sqrt{13}}|\chi_2\rangle$$

So, now we have two possibilities. We can be in either of these two states

$$|\psi_{\chi_1}\rangle = \frac{2}{\sqrt{13}}|\chi_1\rangle \quad \text{or} \quad |\psi_{\chi_2}\rangle = \frac{3}{\sqrt{13}}|\chi_2\rangle$$

Fall 2003 #4 (p 2 of 2)

So, the probability of measuring $42\hbar$ again is

$$P = |\langle \phi_1 | \psi_{x_1} \rangle|^2 + |\langle \phi_1 | \psi_{x_2} \rangle|^2$$

$$= \left|\frac{4}{13}\right|^2 + \left|\frac{9}{13}\right|^2$$

$$\therefore \boxed{P = \frac{97}{169}}$$

case (ii) $q_2 = 42\hbar$

following the same procedure with

$$|\phi\rangle_2 = \frac{1}{\sqrt{13}} (3|\chi_1\rangle - 2|\chi_2\rangle)$$

we have

$$P = |\langle \phi_2 | \psi_{x_1} \rangle|^2 + |\langle \phi_2 | \psi_{x_2} \rangle|^2$$

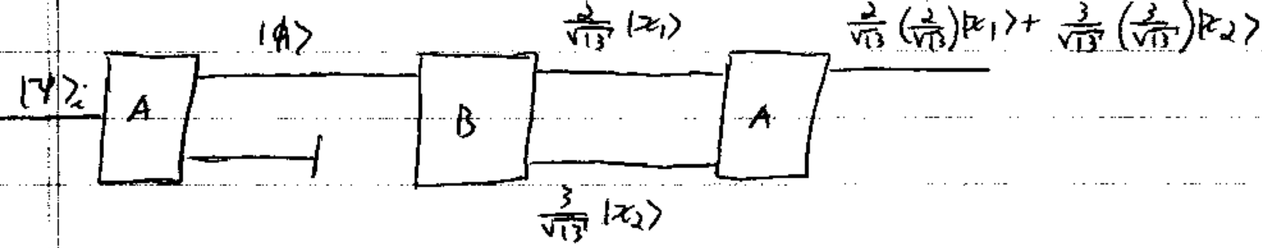
$$= \left|\frac{9}{13}\right|^2 + \left|\frac{4}{13}\right|^2$$

$$\therefore \boxed{P = \frac{97}{169}}$$

QM F'03 #4

$A|\psi\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$; $B|\psi\rangle = b_1|x_1\rangle + b_2|x_2\rangle$

$|\phi_1\rangle = \frac{1}{\sqrt{13}}\left(2|x_1\rangle + 3|x_2\rangle\right)$; $|\phi_2\rangle = \frac{1}{\sqrt{13}}\left(3|x_1\rangle - 2|x_2\rangle\right)$



So $|\psi\rangle_f = \frac{4}{13}|x_1\rangle + \frac{9}{13}|x_2\rangle$

And the probability is $\frac{4^2+9^2}{13^2} = \frac{16+81}{169} = \boxed{\frac{97}{169}}$

QM F'03 #5

H-atom is in the ground state ($|100\rangle$) at $t<0$.

A time dependent electric field is applied

$$\mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \quad \text{for } t>0$$

A long time passes.

a) what is the fraction of atoms in the $|200\rangle$ state?

$$c_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle 100|H'|200\rangle e^{\frac{i\omega_0 t}{\hbar}}\,dt$$

$$\langle 100|H'|200\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{2}{a_0^{3/2}} e^{-r/a_0}\frac{1}{\sqrt{4\pi}}\Big(e\mathcal{E}_0 \overset{r\cos\theta}{z}\, e^{-t/\tau}\Big)\frac{1}{\sqrt{2}}\frac{1}{a_0^{3/2}}\Big(1-\frac{r}{2a_0}\Big)e^{-r/2a_0}\frac{1}{\sqrt{4\pi}}\,r^2 dr\sin\theta\, d\theta\, d\phi$$

$$= \frac{2e\mathcal{E}_0 e^{-t/\tau}}{4\pi a_0^3\sqrt{2}}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}\underbrace{\int_0^{\pi}\cos\theta\sin\theta\, d\theta}\int_0^\infty r^3\Big(1-\frac{r}{2a_0}\Big)e^{-\frac{3r}{2a_0}}dr = 0$$

$u=\sin\theta$

$du=\cos\theta\, d\theta$

$$\int_0^0 u\,du = 0$$

So $\langle 100|H'|200\rangle = 0$, hence there will be 0 atoms in the 2p state.

b) What is the fraction of atoms in the 2p state?

2p: $|211\rangle$, $|21\,{-1}\rangle$, $|210\rangle$

$$c_{1\to 2}(t) = \frac{-i}{\hbar}\int_0^\infty \langle 100|H'|21m\rangle e^{i\omega_0 t}\,dt$$

$$\langle 100|H'|211\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{2}{a_0^{3/2}}\frac{e^{-r/a_0}}{\sqrt{4\pi}}\left(e\mathcal{E}_0\, r\cos\theta\, e^{-t/\tau}\right)\frac{1}{\sqrt{24}}\frac{1}{a_0^{5/2}}\, r\, e^{-\frac{r}{2a_0}} -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{i\phi}\, r^2 dr\, \sin\theta\, d\theta\, d\phi$$

$$= \frac{-2e\mathcal{E}_0 e^{-t/\tau}}{a_0^4\sqrt{4\pi}\sqrt{24}}\left(\frac{3}{8\pi}\right)^{1/2}\underbrace{\int_0^{2\pi} e^{i\phi}d\phi}_{0}\int_0^{\pi}\cos\theta\,\sin^2\theta\, d\theta\int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}}dr = 0$$

$$\frac{1}{i}e^{i\phi}\Big|_0^{2\pi} = \frac{1}{i}\left[1-1\right] = 0$$

So $\langle 100|H'|211\rangle = 0$, similarly $\langle 100|H'|21{-1}\rangle$ as $Y_{1,-1} = (-1)Y_{1,1}^{*}$

$$\langle 100|H'|210\rangle = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{2e^{-r/a_0}}{a_0^{3/2}\sqrt{4\pi}}\left(e\mathcal{E}_0\, r\cos\theta\, e^{-t/\tau}\right)\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta \;\frac{1}{\sqrt{24}}\left(\frac{1}{a_0}\right)^{5/2} r\, e^{-r/2a_0}\; r^2 dr\,\sin\theta\, d\theta\, d\phi$$

$$= \frac{2e\mathcal{E}_0 e^{-t/\tau}}{4\pi a_0^4}\frac{1}{\sqrt{8}}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}\underbrace{\int_0^{\pi}\cos^2\theta\,\sin\theta\, d\theta}\;\underbrace{\int_0^{\infty} r^4 e^{-\frac{3r}{2a_0}}dr}$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$-\int_1^{-1}u^2du = +\int_{-1}^{1}u^2du = \frac{1}{3}\left[1+1\right] = 2/3$$

$$\int_0^{\infty}x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\;;\quad n=4;\ a=\frac{3}{2a_0}$$

$$= \frac{4!}{\left(\frac{3}{2a_0}\right)^5}$$

$$= \frac{2e\mathcal{E}_0 e^{-t/\tau}}{4\pi\sqrt{8}\, a_0^4}\, 2\pi\cdot\frac{2}{3}\cdot\frac{4!}{\left(\frac{3}{2a_0}\right)^5} = \frac{16e\mathcal{E}_0}{\sqrt{8}}e^{-t/\tau}a_0\frac{2^5}{3^5}$$

So

$$c_{1\to 2}(t) = \frac{+i}{\hbar}\frac{16e\mathcal{E}_0 a_0}{\sqrt{8}}\left(\frac{2}{3}\right)^5\int_0^{\infty}e^{-t/\tau}e^{i\omega_0 t}dt = \frac{+i}{\hbar}\frac{16e\mathcal{E}_0 a_0}{\sqrt{8}}\left(\frac{2}{3}\right)^5\int_0^{\infty}e^{-(1/\tau - i\omega_0)t}dt$$

$$= \frac{i}{\hbar}\frac{16e\mathcal{E}_0 a_0}{\sqrt{8}}\left(\frac{2}{3}\right)^5\left(\frac{-1}{\frac{1}{\tau}-i\omega_0}\right)e^{-(1/\tau - i\omega_0)t}\Big|_0^{\infty} = \frac{i}{\hbar}\frac{16e\mathcal{E}_0 a_0}{\sqrt{8}}\left(\frac{2}{3}\right)^5\frac{1}{(1/\tau - i\omega_0)}$$

The probability is given by: $|c_{1\to2}(t)|^2 = \dfrac{16^2e^2\mathcal{E}_0^2a_0^2}{8\hbar^2}\left(\dfrac{2}{3}\right)^{10}\underbrace{\dfrac{1}{(1/\tau - i\omega_0)^2}}_{\frac{1}{\frac{1}{\tau^2}+\omega_0^2}} = \dfrac{32e^2\mathcal{E}_0^2a_0^2}{\hbar^2}\left(\dfrac{2}{3}\right)^{10}\dfrac{1}{1/\tau^2+\omega_0^2}$

And hence if the total population of atoms is N the the fraction of atoms in the 2p state is:

$$\left(\frac{32e^2\mathcal{E}_0^2a_0^2}{\hbar^2}\left(\frac{2}{3}\right)^{10}\frac{1}{1/\tau^2+\omega_0^2}\right)N$$

Fall 2003 #5 (p 1 of 2)

A sample of hydrogen atoms in the ground state is placed between the plates of a parallel capacitor. A voltage pulse is applied to the capacitor at $t=0$ to produce a homogeneous electric field, $\mathcal{E}$, between the plates of:

$$\mathcal{E} = 0 \qquad t<0$$
$$\mathcal{E} = \mathcal{E}_0 e^{-t/\tau} \qquad t>0$$

where τ is a constant. A long time compared to τ passes ($t \gg \tau$)

(a) To first order, calculate the fraction of atoms in the $2p$ ($m=0$) state

once again, this is a time dependent perturbation problem. the general form of the transition probability is given by Zettili eq 10.41 (see also Spring 2003 #1)

$$P_{if}(t) = \left| -i \int_0^t \langle \psi_f | V'(t') | \psi_i \rangle e^{i\omega_{fi} t'} dt' \right|^2 \qquad (1)$$

where $V'(t')$ is given by

$$V'(t') = e\,\mathcal{E}_0 e^{-t'/\tau} z$$ ← time dependent stark effect

So, since $t \gg \tau$

$$P_{if}(t) = e^2 \mathcal{E}_0^2 \left| \int_0^\infty \langle \psi_f | z | \psi_i \rangle e^{(i\omega_{fi} - \frac{1}{\tau})t'} dt' \right|^2$$

where $\omega_{fi} = E_f - E_i = -\frac{\alpha^2 m}{2}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$

now, before we start, note that selection rules for

$$\langle n'l'm' | z | nlm \rangle \neq 0 \qquad (2)$$

tell us that $|\Delta l| = 1$ and $|\Delta m| = 0$ since z is odd and a rank one tensor.

Fall 2003 #5 (p 2 of 2)

So, for part a we want to find the 2p (m=0) state. So, we want

$$\underbrace{\langle 210|z|100\rangle}_{\text{selection rules tell us this will not vanish}} = \int R_{21}^* Y_1^{0*} z R_{10} Y_0^0 d^3r \quad , z = r\cos\theta$$

$$= \int \left[\frac{1}{2\sqrt{6}} \frac{r}{a^{5/2}} e^{-r/2a}\right]\left[\sqrt{\frac{3}{4\pi}}\cos\theta\right](r\cos\theta)\left[\frac{2}{a^{3/2}} e^{-r/a}\right]\left[\frac{1}{\sqrt{4\pi}}\right] d^3r$$

$$= \frac{a^{-4}}{4\pi\sqrt{2}} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\cos^2\theta \int_0^{\infty} r^4 e^{-3r/2a} dr$$

let $u = \cos\theta \Rightarrow du = -\sin\theta d\theta$

$$= \frac{a^{-4}}{4\pi\sqrt{2}} 2\pi \left(\int_{-1}^{1} du\, u^2\right)\left(\frac{4!}{\left(\frac{3}{2a}\right)^5}\right) = \frac{a 2^5 \cdot 2^3 \cdot 3}{2\sqrt{2}\, 3^5}\left[\frac{u^3}{3}\right]_{-1}^{1}$$

$$= \frac{a 2^8}{\sqrt{2}\, 3^5}$$

So, then the transition probability is

$$P = e^2 \mathcal{E}_0^2 \frac{a^2 2^{15}}{3^{10}} \left| \int_0^{\infty} e^{(i\omega_{fi} - \frac{1}{\tau})t'} dt' \right|^2$$

Thus,

$$\boxed{P = \frac{2^{15}}{3^{10}} \frac{e^2 \mathcal{E}_0^2 a^2}{\omega^2 + (\frac{1}{\tau})^2}} \quad \text{for } \omega = \frac{-\alpha^2 m}{2}\left(\frac{1}{4} - 1\right) = \frac{3\alpha^2 m}{8}$$

(b) to first order, what is the fraction of atoms in the 2s state ($|200\rangle$)

selection rules tell that $\langle 200|z|100\rangle = 0$ since $|\Delta \ell| \neq 1$.

Thus,

$$\boxed{P_{1s\to 2s} = 0}$$

Similar ~~to Fall 2003 #6~~. (p 1 of 2)

Here, I want you to obtain the thermodynamic properties of a gas of massless, relativistic, non-conserved particles (such as photons). Because the particles are masless and relativistic, the energy-versus-momentum relationship is $E = |\vec{p}|c$. The fact that they are not conserved means that you set the chemical potential equal to zero in the grand partition function.

a) Calculate the grand partition function of this gas. Assume a spin degeneracy factor of 3.

b) Find the energy of this gas as a function of temperature. The energy goes as a simple power law in the temperature. What is the power?

c) Find the pressure as a function of temperature.

$$\varepsilon_k = c|\vec{p}| = \hbar c|\vec{k}|$$

(a) The grand canonical partition function is

$$\Phi(T,V,z) = \ln Q = -\sum_k \ln(1 - z e^{-\beta \varepsilon_k}), \quad z = e^{\beta\mu}$$

(b) let's find the energy density $\nu(\varepsilon)$

The total number of states in the classical phase space is

$$\Sigma = \int \frac{d^3\vec{r}\, d^3\vec{p}}{h^3} = \frac{4\pi V}{h^3}\int_0^\infty p^2 dp = \frac{4\pi V}{h^3 c^3}\int_0^\infty \varepsilon^2 d\varepsilon$$

=> the one-particle density of states

$$\nu(\varepsilon) = \frac{4\pi V}{h^3 c^3}\varepsilon^2$$

$$\Rightarrow \Phi = -\frac{4\pi V}{(hc)^3}\int_0^\infty d\varepsilon\, \varepsilon^2 \ln(1 - e^{-\beta\varepsilon}) =$$

$$= \frac{4\pi V}{(hc)^3}\frac{1}{3}\beta\int_0^\infty d\varepsilon \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1}$$

we take into account that $\mu = 0$ since it costs no energy to insert massless particle into the system

~~####~~ Fall 2003 #6 (p 2 of 2)

For the internal energy:

$$U(T,V) = \frac{4\pi V}{(hc)^3}\int_0^\infty d\varepsilon \frac{\varepsilon^3}{\exp^{\beta\varepsilon}-1}$$

We make substitution $x = \beta\varepsilon$ and obtain

$$\Phi = \frac{PV}{kT} = \frac{1}{3}U\beta = \frac{4\pi V}{(hc)^3}\frac{2}{\beta^3}g_4(1) \qquad \beta = \frac{1}{k_B T} \qquad (*)$$

rem: $g_4(1) = \zeta(4) = \frac{\pi^4}{90}$

$$\Rightarrow U = \frac{4}{15}\frac{V\pi^5}{(hc)^3}k_B^4 T^4$$

$$\boxed{U \propto T^4}$$

(c) from $(*)$ $\boxed{P = \frac{1}{3}\frac{U}{V} = \frac{8\pi}{(hc)^3}(kT)^4\frac{\pi^4}{90}}$

Fall 2003 #7 (p 1 of 2)

7. *Statistical Mechanics*

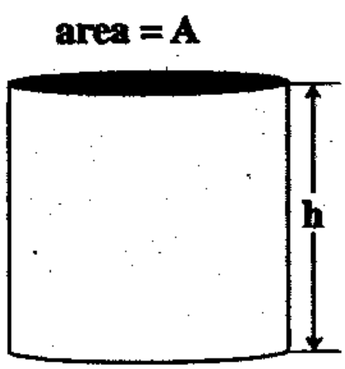
A gas of noninteracting particles fills a cylindrical container that has a cross-sectional area A and a height h. Each particle has a mass m, and is subject to the gravitational field at the surface of the earth. The circular bottom and top of the container are parallel to the surface of the Earth. There are N particles in the container, and the temperature of the container is T.

a) Find the partition function of the gas.

b) What is the pressure of the gas at the top of the container?

c) What is the pressure of the gas at the bottom of the container?

d) Finally, what is the difference between the pressure at the bottom of the container and the pressure at the top of the container? Interpret the answer that you get.



(a) The partition function is given by

$$Z = \frac{1}{N!}\frac{1}{h^{3N}}\left[\int_{-\infty}^{\infty} e^{-\beta p^2/2m}\,dp\right]^{3N} A^N \left[\int_{x_0}^{x_1 = x_0+h} e^{-\beta m g y}\,dy\right]^N$$

$$= \frac{1}{N!}\frac{A^N}{h^{3N}}(2\pi m k T)^{3/2}\left[\frac{kT}{mg}\left(e^{-\beta m g x_0} - e^{-\beta m g (x_0+h)}\right)\right]^N$$

(b) pressure is given by

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = kT\left(\frac{\partial \ln Z}{\partial V}\right) = kT\left(\frac{\partial \ln Z}{\partial A}\right)\left(\frac{\partial A}{\partial V}\right)$$

note: $V = Ah$

Fall 2003 #7 (p 2 of 2)

) where

$$\ln Z = \ln A^N + \ln\left[\frac{1}{N!}\frac{1}{h^{3N}}(2\pi m k T)^{3/2}\left\{e^{-\beta m g x_0} - e^{-\beta m g (x_0+h)}\right\}^N\right]$$

So,

$$\frac{\partial \ln Z}{\partial A} = \frac{\partial}{\partial A} N \ln A = \frac{N}{A}$$

$$\Rightarrow \quad p = \frac{kTN}{A}\frac{\partial A}{\partial V} = \frac{kTN}{A}\frac{\partial}{\partial V}\left(\frac{V}{h}\right) = \frac{kTN}{Ah}$$

$$\therefore \quad \boxed{P_{top} = \frac{kTN}{Ah}}$$

) (c) what is the pressure of the gas at the bottom of the container?

this is the same result as in part (b), but now we must evaluate p at $h = 0$.

Doing so yields

$$\boxed{P_{bottom} \to \infty}$$

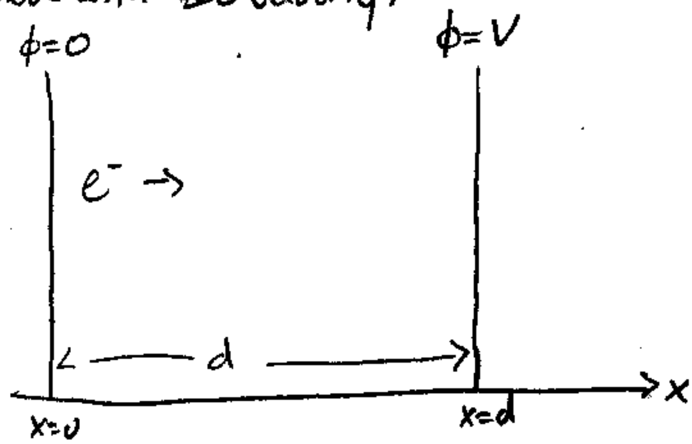
does this make sense?

(d) I am not sure how to interpret this result. It seems that if $h \to \infty$, then the pressure on the bottom should blow up. but not for a finite h. So ???

) Any ideas?

Fall 2003 #8 (p 1 of 3)

Consider a vacuum diode which is parallel plate capacitor (in vacuum) with plate area A and plate separation d. The cathode plate, which is at $\phi=0$, is heated as to thermionically emit electrons which then travel to the anode plate (at $\phi=V$). Assume a steady state bias V and diode current I. You may model the electrons in the diode as a cold fluid with density $n(x)$ and velocity $v(x)$. You may assume that the electrons are born from the cathode with zero velocity.



(this is just Fall 1998 #5)

(a) Find the 1-D potential distribution in the diode, $\phi(x)$ (hint: try a power law solution)

From Poisson's eq: $\left.\begin{array}{l}\nabla\cdot\vec{E}=4\pi\rho \\ \vec{E}=-\nabla\phi\end{array}\right\} \Rightarrow \nabla^2\phi=-4\pi\rho$ (1)

we also know that the current density $\vec{J}$ is given by (eq 5.26)

$$\vec{J}=\rho\vec{v}$$

in 1-D, this is

$$\rho(x)=\frac{J(x)}{v(x)}$$

and then eq (1) becomes

$$\boxed{\frac{d^2\phi(x)}{dx^2}=\frac{-4\pi J(x)}{v(x)}} \quad (2)$$

Now, we can get an expression for $v(x)$ by the relationship between work and change in kinetic energy. That is

$$W = \int_{x=0}^{x=d} \vec{F}_e \cdot d\vec{x} \quad , \quad \vec{F}_e = -e\left(\frac{-\partial V(x)}{\partial x}\right) = e\,\frac{\partial V(x)}{\partial x}$$

and

$$W = \frac{1}{2} m [v(x)]^2$$

so,

$$\frac{1}{2} m v^2 = e\,[V(x=d) - V(x=0)] = eV_0 - 0$$

$$\Rightarrow \boxed{v(x) = \sqrt{\frac{2e\phi(x)}{m}}} \qquad (3)$$

we are told to assume a power law solution, so, let $\phi(x) \sim x^y \Rightarrow \phi(x) = A x^y$
we can find the value of A from the boundary condition. That is,

$$\phi(x=d) = V = A d^y \Rightarrow A = V d^{-y}$$

so,

$$\phi(x) = V\left(\frac{x}{d}\right)^y \qquad (4)$$

substituting this result into eq (3) yields

$$v(x) = \sqrt{\frac{2eV}{m}\left(\frac{x}{d}\right)^y} \qquad (5)$$

substituting eq (4) & (5) into eq (2) yields

$$\frac{d^2}{dx^2}\left[V\left(\frac{x}{d}\right)^y\right] = \frac{-4\pi J(x)}{\sqrt{\frac{2eV}{m}\left(\frac{x}{d}\right)^y}}$$

$$\Rightarrow J(x) = -\frac{1}{4\pi}\sqrt{\frac{2e}{m}}\,\frac{V^{3/2}}{d^{3y/2}}\,y(y-1)\,x^{[(3y/2)-2]} \qquad (6)$$

Now, from the continuity equation $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$, we know that $J(x)$ must be constant wrt x. Otherwise, charge would be accumulating and not in motion. So, the power of x must vanish in our expression for $J(x)$ (eq 6).

That is,

$$\frac{3y}{2} - 2 = 0 \quad \Rightarrow y = \frac{4}{3}$$

Substituting this result into eq (4) yields

$$\boxed{\phi(x) = V\left(\frac{x}{d}\right)^{4/3}}$$

(b) Find the diode current as a function of the bias voltage V.

So, we have

$$I = JA$$, where A is the area the current is flowing through

and J is given by eq (6) with $y = \frac{4}{3}$. That is,

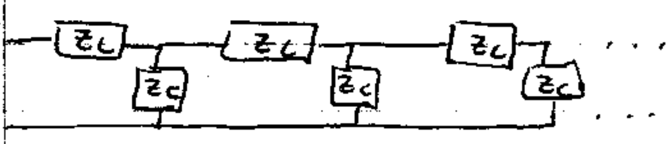
$$\boxed{J(x) = \frac{-V^{3/2}}{9\pi d^2}\sqrt{\frac{2e}{m}}}$$

(c) What unphysical result is caused by the assumption that electrons are born from the cathode with zero velocity?

The official solution says that this implies that $\rho = env \to \infty$, but i do not see how... insert tear here

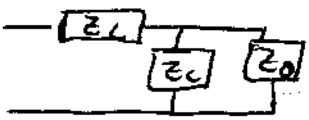
F'03 E.M. #9

a) Two wire transmission line with L and C per unit length. Show that $Z = \sqrt{L/C}$



Following Feynmanns argument:

Add an additional Z_L, Z_C block at the beginning and call everything else Z_0 :



$Z_{eq} = Z_L + \dfrac{Z_C Z_0}{Z_C + Z_0}$; now $Z_{eq} = Z_0$ as the line is infinitely long

$$Z_L + \frac{Z_C Z_0}{Z_C + Z_0} = Z_0 \Rightarrow Z_L Z_C + Z_L Z_0 + \cancel{Z_C Z_0} = \cancel{Z_C Z_0} + Z_0^2$$

so $\quad Z_0^2 - Z_L Z_0 - Z_L Z_C = 0$

similar to $\quad ax^2 + bx + c = 0$; $a = 1$; $b = -Z_L$; $c = -Z_L Z_C$

so

$$Z_0 = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{Z_L}{2} \pm \frac{1}{2}\sqrt{Z_L^2 + 4 Z_L Z_C}$$

now let's substitute in: $Z_L = i\omega L$; $Z_C = \frac{1}{i\omega C}$

$$Z_L^2 = -\omega L^2 ; \; Z_C Z_L = L/C$$

hence

$$Z_0 = \pm \frac{\sqrt{-\omega^2 L^2 + 4L/C}}{2} + \frac{i\omega L}{2}$$

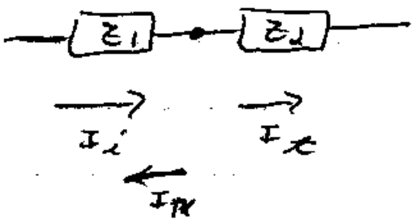
The real part of Z_0 is what matters:

$$\mathrm{Re}(Z_0) = \pm \frac{\sqrt{-\omega^2 L^2 + 4L/C}}{2} \approx \pm \sqrt{L/C} \quad \text{for} \quad \omega^2 < \frac{4}{LC}$$

(according to Feynmann)

So $\boxed{Z_0 = \sqrt{\frac{L}{C}}}$

b) What are the relative amplitudes of the reflected and transmitted waves $(V_r/V_i,\ V_t/V_i)$?



(1)

so $I_i - I_r = I_t$ also $V_i + V_r = V_t$

$$\frac{I_i}{Z_1} + \frac{I_r}{Z_1} = \frac{I_t}{Z_2} \qquad (2)$$

(2) $\Rightarrow$ $I_i + I_r = \frac{Z_1}{Z_2} I_t$ but from (1) $\cancel{I_i = I_t + I_r}$

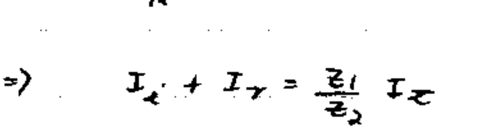
$\cancel{\Rightarrow 2I_r = I_t\left(\frac{Z_1}{Z_2} - 1\right) \Rightarrow I_r =}$ $\frac{Z_1 - Z_2}{Z_2}$ $\frac{Z_1+Z_2}{Z_2}$

$$I_i + I_r = \frac{Z_1}{Z_2}(I_i - I_r) \Rightarrow I_i\left(\frac{Z_1}{Z_2} - 1\right) = I_r\left(\frac{Z_1}{Z_2} + 1\right)$$

$$\boxed{I_r = \frac{Z_1 - Z_2}{Z_1 + Z_2} I_i}$$

now

$$I_t = I_i - I_r = I_i - \frac{Z_1 - Z_2}{Z_1 + Z_2} I_i = \frac{\cancel{Z_1} + Z_2 - \cancel{Z_1} + Z_2}{Z_1 + Z_2} I_i = \boxed{\frac{2Z_2}{Z_1 + Z_2} I_i}$$

c) 

For $Z_1 < Z_2$

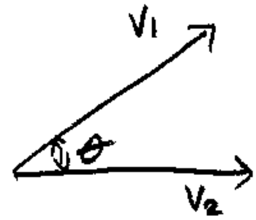
R is in parallel with Z_2

$$Z_1 = \frac{RZ_2}{R + Z_2} \Rightarrow Z_1 R + Z_1 Z_2 = RZ_2 \Rightarrow \boxed{R = \frac{Z_1 Z_2}{Z_2 - Z_1}}$$

For $Z_1 > Z_2$ R is in series with Z_2: $Z_1 = R + Z_2 \Rightarrow \boxed{R = Z_1 - Z_2}$

Fall 2003 # 10 (p 1 of 1)

Consider a wedge formed by two conducting half-planes, as depicted in the Figure. One plane is maintained at electrostatic potential V_1 while the other is at V_2. What is the electrostatic potential in the region between the two half-planes?



(see Spring 2005 #8 and Spring 2003 #9)

since ϕ is restricted (does not range to 2π), the general solution to the potential is given by

$$\Phi(r,\phi) = (a_0 + b_0 \ln r)(c_0 + d_0 \phi)$$

Now, apply the boundary conditions.

- $\Phi(r, \phi = 0) = V_2 = (a_0 + b_0 \ln r)\, c_0$

  the only way to satisfy this eq is for $b_0 = 0$ since $V_2 \neq V_2(r)$.

$$\Rightarrow \quad V_2 = a_0 c_0$$

- $\Phi(r, \phi = \theta) = V_1 = a_0 (c_0 + d_0 \theta) = V_2 + a_0 d_0 \theta$

$$\Rightarrow \quad a_0 d_0 = \frac{V_1 - V_2}{\theta}$$

Thus,

$$\Phi(r,\theta) = a_0 c_0 + b_0 d_0 \phi$$

$$\therefore \quad \boxed{\Phi(r,\theta) = V_2 + \frac{V_1 - V_2}{\theta}\phi}$$

Problem # 11 Fall 2003

$$k^2 c^2 = \omega^2 - 4\pi n e^2/m_e$$

$C = 3\times10^{10}$ $m_e = 9.11\times10^{-28}$ $e = 4.8\times10^{-10}$

a) <u>For Transmission</u>

$$\omega^2 - \frac{4\pi n e^2}{m_e} \geq 0$$

otherwise K is imaginary and the radiation is absorbed

$$\Rightarrow \frac{4\pi n e^2}{m_e} = \omega^2$$ is the frequency where transmission stops.

$$n = \frac{\omega^2 m_e}{4\pi e^2} = \frac{(2\pi\, 10^7)^2(9.11\times10^{-28})}{4\pi(4.8\times10^{-10})^2}$$

$$n \approx \frac{\pi}{25}\,\frac{10^{14}\times10^{-27}}{10^{-20}} = \frac{\pi}{25}\,10^7$$

So from the information given, one might conclude that Radiation with frequencies below 10 MHz can't be recieved from space because the earth is surrounded by a plasma of density $n \approx \frac{\pi}{25} 10^7$

I looked on the internet, and indeed this is the case.

b) $K = \frac{\omega}{v}$ $\quad n = 0.01\ cm^{-3}$ $\quad d = 1 \times 10^{22}\ cm$

$d = vt \quad t = \frac{d}{v} \quad \omega_2 = 10\ KHz \quad \omega_1 = 6\ KHz$

$$\frac{1}{v} = \frac{1}{c}\sqrt{1 - \frac{4\pi n e^2}{\omega^2 m_e}}$$

$$t = \frac{d}{v} = \frac{d}{c}\sqrt{1 - \frac{4\pi n e^2}{\omega^2 m_e}}$$

$$\boxed{t_2 - t_1 = \frac{d}{c}\left\{\sqrt{1 - \frac{4\pi n e^2}{\omega_2^2 m_e}} - \sqrt{1 - \frac{4\pi n e^2}{\omega_1^2 m_e}}\right\}}$$