Q8 -- anisotropic medium

Tuesday, July 21, 2020 12:11 PM

kdbarajas @ live.com

8. *Electricity and Magnetism* (Fall 2005)

An anisotropic medium has a tensor conductivity given by

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}$$

where $\sigma_\parallel$ and $\sigma_\perp$ are real and independent of frequency. The symbol $\perp$ refers to the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ direction and the symbol $\parallel$ to the $\hat{\mathbf{z}}$ direction in a Cartesian coordinate system.

(a) Find the dispersion relation $k = k(\omega)$ for an electromagnetic wave with O-mode (ordinary mode) polarization with the k vector along $\hat{\mathbf{x}}$.

(b) Write an expression for the damping decrement $k_I = \text{Im } k$ in the limit of high frequency.

(c) If the amplitude of the electric field is E_0 at $x = 0$, find the time-avaraged power per unit volume delivered to this medium at the location $x > 0$. (No need to write down k_I explicitly.)

(a) Find dispersion relation $k(\omega)$

- Ordinary mode $\leftrightarrow$ linearly polarized
- $\vec{k} = k\hat{x}$ (ordinary mode, $\vec{E} \cdot \hat{k} = 0$ $\Rightarrow \vec{E} = \hat{z} E_0 e^{i(\underbrace{\vec{k}\cdot\vec{r}}_{kx} - \omega t)}$
- We are only told that the medium is anisotropic in the conductivity tensor, so we can assume that μ and ε are constant

First starting w/ ME in medium

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 = 0$$

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon \vec{E}) = \varepsilon \vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \mu \vec{H})$$

$$= -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{J}_f = \overleftrightarrow{\sigma} \cdot \vec{E} \qquad = -\mu \left(\frac{\partial \vec{J}_f}{\partial t} + \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

$$\frac{\partial \vec{J}_f}{\partial t} = \overleftrightarrow{\sigma} \cdot \frac{\partial \vec{E}}{\partial t} \qquad = -\mu \left(\overleftrightarrow{\sigma} + \varepsilon \frac{\partial}{\partial t}\right) \frac{\partial \vec{E}}{\partial t} = -\mu \left(\overleftrightarrow{\sigma} + \varepsilon(-i\omega)\right)(-i\omega \vec{E})$$

$$= -\mu \left(-i\omega \overleftrightarrow{\sigma} - \omega^2 \varepsilon \mathbb{1}\right) \vec{E}$$

Finding $\vec{\nabla} \times \vec{\nabla} \times \vec{E}$:

(1) $\vec{\nabla} \times \vec{E} = \varepsilon_{ijk} (\nabla_i E_j)_k$

$j = z$ (only comp)

$= \varepsilon_{xzy} (\nabla_x E_z)_y = (-1)(ik) E_z(x)\, \hat{y}$

$$= \epsilon_{xzy} (\nabla_x E_z)_y = (-1)(ik) E_z(x)\, \hat{y} \quad \leftarrow j=z \text{ (only comp)}$$

$$\nabla \times (\nabla \times E) = \epsilon_{ijk} [\nabla_i (-ik E_z(x))_j]_k \quad \leftarrow j=y$$

$$= \epsilon_{xyz} (-ik) [(ik) E_z(x)\, \hat{z}]$$

$$= k^2 \vec{E}$$

(2) or using BAC-CAB:

$$= \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}}) - (\vec{\nabla}^2)\vec{E}$$

$$= -(ik)(ik)\vec{E} = k^2 \vec{E} \quad \text{(much easier)}$$

$$= -\mu(-i\omega \overleftrightarrow{\sigma} - \omega^2 \epsilon \mathbb{1}) E$$

$$\Rightarrow k^2 \vec{E} = [i\mu\omega \overleftrightarrow{\sigma} + \omega^2 \mu\epsilon \mathbb{1}] \vec{E}$$

$$(k^2 - \omega^2 \mu\epsilon)\vec{E} = i\mu\omega \overleftrightarrow{\sigma} \cdot \vec{E}$$

for O-mode polarization: $(k^2 - \omega^2\mu\epsilon) E_z = i\mu\omega\, \sigma_\perp E_z$

$$\boxed{k^2(\omega) = \mu\epsilon\omega^2 \left(1 + \frac{i\sigma_\perp}{\epsilon\omega}\right)}$$

(b) $$\lim_{\frac{\omega\epsilon}{\sigma_\perp} \gg 1} k(\omega) = \sqrt{\mu\epsilon}\,\omega \left(1 + \frac{1}{2}\frac{i\sigma_\perp}{\epsilon\omega}\right) \quad n = c\sqrt{\mu\epsilon}$$

$$\boxed{k_z = \mathrm{Im}(k) = \left(\frac{n}{c}\right)\frac{\sigma_\perp}{2\epsilon} \quad @ \text{ high freq.}}$$

(c) Find time-averaged power/volume @ $x>0$

Want power density delivered to volume for $x>0$

$$-\vec{\nabla}\cdot\vec{S} = \frac{\partial}{\partial t}(u_{em} + u_{mech})$$

$$-\vec{\nabla}\cdot\vec{S} - \frac{\partial u_{em}}{\partial t} = \bar{P} = \vec{E}\cdot\vec{J}$$

rate of energy density transport

u_{em}: EM field energy density;

power density delivered to medium $\left(\vec{E}\cdot\vec{J} = \frac{\partial u_{mech}}{\partial t}\right)$

$-\frac{\partial u_{em}}{\partial t}$: absorbed by medium from EM fields (positive would be taken to be stored in fields)

- $\vec{S} = \vec{E} \times \vec{H}$ (power per unit area; energy density flow)
- Time-averaging:

$$\langle \bar{P}(x>0)\rangle_t = \langle \vec{E}\cdot\vec{J}(x>0)\rangle_t = -\vec{\nabla}\cdot\langle\vec{S}\rangle_t - \left\langle \frac{\partial u_{em}}{\partial t}\right\rangle_t$$

power density $\bar{P} = P/V$

Method 1: Calc LHS $\langle \vec{E}\cdot\vec{J}(x>0)\rangle_t$

power density $p = \cdot/v$

Method 1: Calc LHS $\langle \vec{E}\cdot\vec{J}(x>0)\rangle_t$

$$\langle \vec{E}\cdot\vec{J}\rangle_t = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, \vec{E}\cdot\vec{J} = \frac{1}{2}\,\mathrm{Re}\left[\vec{E}_0\cdot\vec{J}_0^*\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\left[E_0 e^{ikx}\cdot(\overleftrightarrow{\sigma}\cdot\vec{E}_0)^*\right]$$

$$\hat{E}_0 = \hat{E}_0^* = \hat{z}$$

$$= \frac{1}{2}E_0^2\,\mathrm{Re}\left[e^{-k_I x}\cancel{e^{in\omega/c}}\,\sigma_1\, e^{-k_I x}\cancel{e^{-in\omega/c}}(\cancel{\hat{E}_0\cdot\hat{E}_0^*})\right]$$

$$= \frac{\sigma_1}{2}E_0^2 e^{-2k_I x} \Rightarrow \boxed{\langle \bar{P}\rangle_t = \left(\frac{c k_I}{n}\right)\varepsilon E_0^2 e^{-2k_I x}}$$

Method 2: Calc RHS

Using time-average theorem for complex vector fields

$$\langle \vec{S}\rangle_t = \frac{1}{2}\,\mathrm{Re}\left[\vec{E}_0\times\vec{H}_0^*\right] \qquad \vec{B} = \mu\vec{H} = \sqrt{\mu\varepsilon}\,\hat{k}\times\vec{E}$$

$$\vec{H} = \sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\vec{E}\times\left\{\sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}\right\}^*\right]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,\mathrm{Re}\left[\hat{k}(\vec{E}\cdot\vec{E}^*) - \underbrace{(\vec{E}\cdot\hat{k})}_{0}\vec{E}^*\right]$$

$$= \hat{x}\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,\mathrm{Re}\left[E_0^2 e^{ikx}e^{-ik^*x}\right] = \hat{x}\frac{1}{2}\sqrt{\varepsilon/\mu}\,E_0^2 e^{-2k_I x} \Rightarrow \vec{\nabla}\cdot\langle S\rangle_t = -k_I\sqrt{\varepsilon/\mu}\,E_0^2 e^{-2k_I x}$$

$$k = n\frac{\omega}{c} + ik_I$$

$$\left\langle \frac{\partial u_{em}}{\partial t}\right\rangle_t = \frac{1}{T}\int_0^T\left(\frac{\partial u_{em}}{\partial t}\right)dt = \left(\frac{\omega}{2\pi}\right)u_{em}\Big|_0^{2\pi/\omega} \quad w/\ \omega = 2\pi/T$$

$$u_{em} = \frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2}\mu|\vec{H}|^2$$

$$= \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x} + \frac{1}{2}\mu\left|\sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}\right|^2$$

$$= \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x} + \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x}|\hat{k}\times\hat{z}|^2$$

$$u_{em} = \varepsilon E_0^2 e^{-2k_I x}$$

$$\left\langle \frac{\partial u_{em}}{\partial t}\right\rangle_t = \frac{\omega}{2\pi}\varepsilon E_0^2 e^{-2k_I x}\Big|_0^{2\pi/\omega} = 0$$

$$\langle \bar{P}(x>0)\rangle = -\vec{\nabla}\cdot\langle\vec{S}\rangle_t = k_I\sqrt{\frac{\varepsilon}{\mu}}E_0^2 e^{-2k_I x}$$

$$\boxed{\langle \bar{P}(x>0)\rangle = (c k_I/n)\,\varepsilon E_0^2 e^{-2k_I x}}$$