FALL 2016

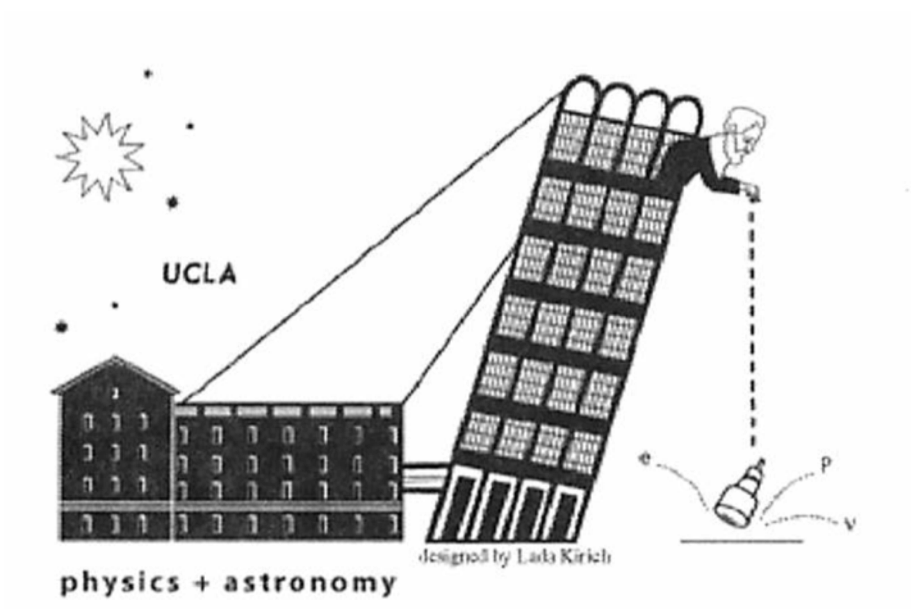
Physics Comprehensive Exam

September 12th , 2016 (Part 1) 9:00 – 1:00 pm

<u>Part 1: Classical Mechanics and Quantum Mechanics</u>

7 Total Problems/ 20 Points Each / Total 140 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the problem page.
- Use paper provided for additional pages. **<u>Use one side only.</u>**
- To ensure anonymous grading, use the Number Labels you are given on EACH of your response pages, including the question page. If you run out of labels, be sure to write the Number you have been assigned on each page.
- Return the question page as the first page of your answers.
- When submitting, please clip all pages together in question # order.
- If a part of any question seems ambiguous to you, state clearly your interpretations and answer the question accordingly.



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1. Classical Mechanics

Consider a pendulum made of a very heavy mass *m* suspended by a light wire of length *l* from a tall ceiling. The pendulum is allowed to swing freely for long periods of time, and moves in both the east-west and north-south directions.

a) (7 points) Choose a set of axes such that *x* is east and *y* is north and *z* is vertically upwards and assume small oscillations to calculate the two coupled differential equations for the motion of the pendulum in a frame rotating with angular velocity $\Omega \parallel z$, much smaller than the pendulum characteristic frequency so you can neglect all effects $\propto \Omega^2$.

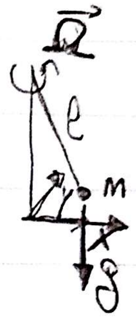
b) (7 points) If the pendulum starts oscillating in the *x* plane with zero initial velocity and initial offset x_0, use perturbation theory to obtain the first order correction to the trajectory.

c) (6 points) Find the full solution using the substitution $\eta = x + iy$ and then trying a solution for the resulting equation of motion of the form $\eta(t) = f(t)e^{-i\Omega t}$.

a) The pendulum is subject to gravity, wire Tension, Coriolis force and centrifugal force ($o(\Omega^2)$)

Projecting $m\vec{a} = m\vec{g} + T - 2m\,\vec{\Omega}\times\vec{v} - m\,\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$

along x and y we get



$$\begin{cases} m\ddot{x} = -mg\dfrac{x}{\ell} + 2m\Omega\dot{y} \\ m\ddot{y} = -mg\dfrac{y}{\ell} - 2m\Omega\dot{x} \end{cases}$$

setting $\omega_0^2 = g/\ell$

$$\ddot{x} = -\omega_0^2 x + 2\Omega\dot{y}$$

$$\ddot{y} = -\omega_0^2 y - 2\Omega\dot{x}$$

b) 0th order solution (i.e. $\Omega=0$) with initial conditions

$x(0) = x_0 \quad y(0)=0$

$\dot{x}(0)=0 \quad \dot{y}(0)=0$

$$x = x_0 \cos \omega_0 t$$

$$y=0$$

Plug in to find first order

$$\ddot{y} = -\omega_0^2 y + 2\Omega x_0 \omega_0 \sin \omega_0 t$$

Particular solution $y = -x_0 \Omega t \cos \omega_0 t$

$\Rightarrow \quad \eta = x + iy \qquad \eta = f e^{-i\Omega t} \quad \dot\eta = \dot f e^{-i\Omega t} - i\Omega f e^{-i\Omega t}$

$$\ddot\eta = -\omega_0^2 \eta - 2i\Omega\dot\eta$$

$$\ddot\eta = \ddot f e^{-i\Omega t} - 2i\Omega \dot f e^{-i\Omega t} - \Omega^2 f e^{-i\Omega t}$$

writing the equation

$$\ddot f e^{-i\Omega t} \cancel{- 2i\Omega \dot f e^{-i\Omega t}} - \Omega^2 f e^{-i\Omega t} = -\omega_0^2 f e^{-i\Omega t} \cancel{- 2i\Omega \dot f e^{-i\Omega t}} - 2\Omega^2 f$$

and f has to satisfy

$$\ddot f = \left(-\omega_0^2 - \Omega^2\right) f$$

Neglecting $O(\Omega^2)$ and applying initial conditions

$$f = x_0 \cos\omega_0 t$$

$\Rightarrow$ full solution

$$x = x_0 \cos\omega_0 t \cos\Omega t$$
$$y = -x_0 \cos\omega_0 t \sin\Omega t$$

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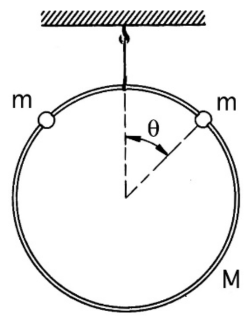
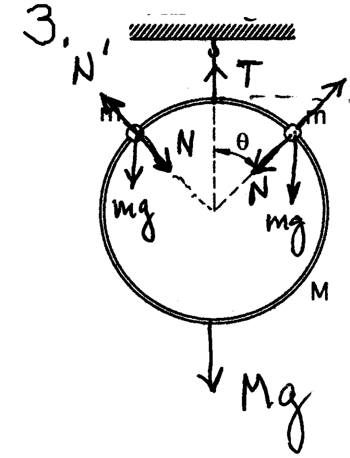
2. Classical Mechanics

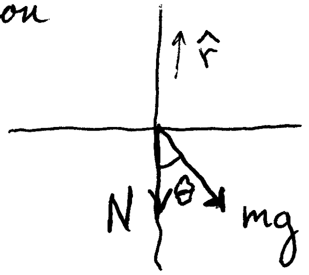
A ring of mass M = 0.1 kg hangs from a thread, and two beads of mass m = 0.2 kg slide on it without friction. The beads are released simultaneously from rest at the top of the ring and slide down opposite sides. The ring is initially motionless, but when the beads pass a critical angle θ_c the ring is observed to start moving upwards. Find the value of θ_c.



3.



forces on one m - take "up" as radial direction



($\vec{N}$ switches direction with θ, assume radial inward here)

$$N + mg\cos\theta = m a_c = m\frac{v^2}{R}$$

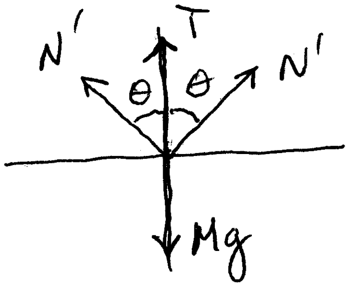
Energy conservation: after moving to θ the mass m has dropped $h = R - R\cos\theta$

$$E_i = 0 = E_f = \frac{1}{2}mv^2 - mgR(1-\cos\theta)$$

$$\Rightarrow v^2 = 2gR(1-\cos\theta)$$

and so $N = 2mg(1-\cos\theta) - mg\cos\theta = mg(2-3\cos\theta)$

Forces on the ring:



3rd law $\vec{N}' = -\vec{N}$

for the ring to move up T must go to zero, and if $2N'\cos\theta > Mg$ it will move up. The value of θ_c where this occurs is given by

$$2(mg(2-3\cos\theta_c))\cos\theta_c = Mg$$

$$\cos\theta_c = \frac{4 \pm \sqrt{16 - 24\frac{M}{m}}}{12} \Rightarrow \boxed{\theta_c = 60^\circ}, ~~80.4^\circ~~$$

net force up positive here

net force negative here

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3. Quantum Mechanics

An apparatus is constructed that emits pairs of photons whose polarizations are quantum-mechanically correlated because each pair is in the state

$$|\psi\rangle = (|H_\mathrm{A}, H_\mathrm{B}\rangle - |V_\mathrm{A}, V_\mathrm{B}\rangle)/\sqrt{2}$$

where H (Horizontal) and V (Vertical) correspond to orthogonal linear polarizations for the photons. Each photon from the pair is collected in a separate (polarization-maintaining) optical fiber and the output of the first fiber is sent to Alice while the output of the second is sent to Bob (corresponding to the A and B subscripts above).

Alice and Bob each have a fancy single-photon polarization detection system that will report a result of $+1$ if the measurement of the polarization finds H and -1 if it finds V. Expressed in the H, V basis, each detector implements the measurement given by the operator

$$M_p = |H_p\rangle\langle H_p| - |V_p\rangle\langle V_p| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_p$$

where $p \in \{\mathrm{A,B}\}$.

a) (6 points) The H and V directions of each detector may be said to point along the x and y axes of some coordinate systems that are fixed with respect to each detector. If Alice or Bob decides to rotate their detector about the $+z$ axis through some angle θ, the resulting measurement may be called $M_p(\theta)$. Find expressions for the following four measurement operators in the H, V basis: $M_\mathrm{A}(0)$, $M_\mathrm{A}\left(\frac{\pi}{4}\right)$, $M_\mathrm{B}\left(\frac{\pi}{8}\right)$, and $M_\mathrm{B}\left(\frac{3\pi}{8}\right)$.

b) (6 points) Find the expectation value of the following operator in the state $|\psi\rangle$:

$$\widehat{W} \equiv 2 \cdot \mathbb{I} - M_\mathrm{A}(0) M_\mathrm{B}\left(\frac{\pi}{8}\right) + M_\mathrm{A}(0) M_\mathrm{B}\left(\frac{3\pi}{8}\right) + M_\mathrm{A}\left(\frac{\pi}{4}\right) M_\mathrm{B}\left(\frac{\pi}{8}\right) + M_\mathrm{A}\left(\frac{\pi}{4}\right) M_\mathrm{B}\left(\frac{3\pi}{8}\right)$$

where $\mathbb{I}$ is the identity matrix.

c) (8 points) The operator $\widehat{W}$ is known as an *entanglement witness*; if $\langle\widehat{W}\rangle \leq 0$, the state is incompatible with a local hidden variable theory. Now consider the effect of group velocity birefringence in the optical fibers, which would lead to a polarization-dependent time-delay. Consider the case where the photons emitted by the source (i.e. at some position in space before they are collected by the fibers) are in Gaussian temporal wavepackets of width τ given by

$$\psi_\mathrm{temporal}(t) = (2\pi\,\tau^2)^{-\frac{1}{4}}\; \mathrm{e}^{-\frac{t^2}{4\,\tau^2}}.$$

Find the relative delay between H and V necessary to give $\langle\widehat{W}\rangle > 0$ for photons created in the initial state $|\psi\rangle$. You should assume that the fibers (and therefore the birefringence-induced delays) are identical for Alice and Bob. The following identity may be useful:

$$\int_{-\infty}^{\infty} \mathrm{d}t\, (2\,\pi\,\tau^2)^{-\frac{1}{2}} \exp\left(-\frac{t^2}{4\,\tau^2}\right) \exp\left(-\frac{(t+T)^2}{4\,\tau^2}\right) = \exp\left(-\frac{T^2}{8\,\tau^2}\right)$$

(a) Since M_p was represented in 2x2 matrix form, we can follow that same convention and recognize the top and bottom entries of a column vector as H and V, respectively, which we can treat as the x and y components. This allows us to write $R(\theta)$ as the standard 2D rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Giving us

$$M(\theta) = R(\theta)M(0)R(-\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$$M_{\mathrm{A}}(0) = |H_{\mathrm{A}}\rangle\langle H_{\mathrm{A}}| - |V_{\mathrm{A}}\rangle\langle V_{\mathrm{A}}| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathrm{A}}$$

$$M_{\mathrm{A}}\left(\frac{\pi}{4}\right) = |H_{\mathrm{A}}\rangle\langle V_{\mathrm{A}}| + |V_{\mathrm{A}}\rangle\langle H_{\mathrm{A}}| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{\mathrm{A}}$$

$$M_{\mathrm{B}}\left(\frac{\pi}{8}\right) = \frac{(|H_{\mathrm{B}}\rangle\langle H_{\mathrm{B}}| + |H_{\mathrm{B}}\rangle\langle V_{\mathrm{B}}| + |V_{\mathrm{B}}\rangle\langle H_{\mathrm{B}}| - |V_{\mathrm{B}}\rangle\langle V_{\mathrm{B}}|)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_{\mathrm{B}}$$

$$M_{\mathrm{B}}\left(\frac{3\pi}{8}\right) = \frac{(-|H_{\mathrm{B}}\rangle\langle H_{\mathrm{B}}| + |H_{\mathrm{B}}\rangle\langle V_{\mathrm{B}}| + |V_{\mathrm{B}}\rangle\langle H_{\mathrm{B}}| + |V_{\mathrm{B}}\rangle\langle V_{\mathrm{B}}|)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}_{\mathrm{B}}$$

(b) Using these forms of these operators, we find

$$\langle \widehat{W} \rangle = 2 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\big(1 - \sqrt{2}\big).$$

(c) The state vector can now be written something like

$$|\psi'\rangle = (|H_{\mathrm{A}}, H_{\mathrm{B}}, 0\rangle - |V_{\mathrm{A}}, V_{\mathrm{B}}, T\rangle)/\sqrt{2}$$

where the last entry denotes the delay of the temporal wavepacket. The first three terms in $\langle \widehat{W} \rangle$ are unchanged, but the last two involve inner products between temporal wavepackets that are delayed by different amounts, such as $\langle 0|T\rangle$. The expectation value of the witness is

$$\langle \psi' | \widehat{W} | \psi' \rangle = 2 - \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} \exp\left(-\frac{T^2}{8\,\tau^2}\right)$$

Which leads to the condition for this expectation value to be positive,

$$T > 2\tau\sqrt{2\ln\left(\frac{1}{\sqrt{2}-1}\right)}.$$

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4. Quantum Mechanics

We consider a one-dimensional system with Hamiltonian $H = \frac{p^2}{2m} + V$, where the potential V gives an effective description of the physical system, and V is non-local. The matrix elements of V in the position eigen-basis $|x\rangle$, for real x, is given in terms of a real-valued square integrable function $u(x)$ which decreases to zero exponentially as $|x| \to \infty$,

$$\langle x|V|x'\rangle = \frac{\hbar^2}{2m} u(x)u(x')$$

a) (4 points) Derive the integro-differential equation obeyed by a wave function $\psi_k(x)$ of energy E given in terms of the wave number k by $E = \hbar^2 k^2/(2m)$.

b) (6 points) Establish the solution to the equation obtained in point 1 above for the scattering of an incoming plane wave of wave-number k and unit amplitude in terms of a suitable Green function $G(x, x'; k)$. (The resulting solution is usually referred to as the Lippmann-Schwinger equation.)

c) (6 points) Compute the reflection and transmissions coefficients, respectively r_k, t_k as a function of the Fourier transform of $u(x)$.

d) (4 points) Verify that probability is conserved during the process so that ${r_k}^2 + {t_k}^2 = 1$

1. The Schrödinger equation for this non-local potential is an integro-differential equation,

$$-\frac{d^2}{dx^2}\psi_k(x) + u(x)\int_{-\infty}^{\infty} dx' u(x')\psi_k(x') = k^2\psi_k(x)$$

2. We define the Green function G by,

$$\left(\frac{d^2}{dx^2} + k^2\right) G(x,x';k) = \delta(x-x')$$

It will be convenient to choose a solution G,

$$G(x,x';k) = \frac{e^{ik|x-x'|}}{2ik}$$

which is symmetric $G(x',x;k) = G(x,x';k)$ and has only the branch proportional to e^{ikx} for $x > x'$. The imaginary part of G is a solution to the homogeneous equation chosen uniquely

to enforce these conditions. For given energy $E = \hbar^2 k^2/(2m)$, two linearly independent solutions are given by $\psi_{\pm k}$ where,

$$\psi_k(x) = e^{+ikx} + \varphi_k \int_{-\infty}^{\infty} dx' G(x,x';k)u(x') \tag{0.1}$$

and φ_k is defined by,

$$\varphi_k = \int_{-\infty}^{\infty} dx\, u(x)\psi_k(x)$$

We obtain φ_k by integrating equation (0.1) against $u(x)$, and we obtain,

$$\varphi_k = \frac{v_k}{1-K} \qquad\qquad v_k = \int_{-\infty}^{\infty} dx\, u(x)\, e^{+ikx}$$

and where,

$$K = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' u(x)G(x,x';k)u(x') \tag{0.2}$$

3. The reflection and transmission coefficients, respectively denoted r_k and t_k, are defined by the following asymptotic behavior,

$$\begin{array}{ll} x \to -\infty & \qquad \phi_k(x) \approx e^{+ikx} + r_k\, e^{-ikx} \\ x \to +\infty & \qquad \phi_k(x) \approx t_k\, e^{+ikx} \end{array}$$

The solution obtained in (0.1) has already been nicely tailored to this asymptotic form, as may be seen by using the explicit expression for the Green function,

$$\psi_k(x) = e^{+ikx} + \frac{\varphi_k}{2ik}\int_{-\infty}^{x} dx'\, e^{ik(x-x')}u(x') + + \frac{\varphi_k}{2ik}\int_{x}^{\infty} dx'\, e^{ik(x'-x)}u(x')$$

Since $u(x)$ rapidly decays as $|x| \to \infty$, we readily read off r_k, t_k,

$$r_k = \frac{v_k\varphi_k}{2ik} \qquad\qquad t_k = 1 + \frac{v_k^*\varphi_k}{2ik}$$

4. To verify unitarity for this self-adjoint, though non-local Hamiltonian, we decompose K of (0.2) into real and imaginary parts:

$$K = K_r + \frac{|v_k|^2}{2ik}$$

with K_r real, in terms of which the coefficients become,

$$r_k = \frac{-|v_k|^2}{|v_k|^2 - 2ik(1 - K_r)} \qquad\qquad t_k = \frac{-2ik(1 - K_r)}{|v_k|^2 - 2ik(1 - K_r)}$$

which manifestly satisfy $|r_k|^2 + |t_k|^2 = 1$.

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5. Quantum Mechanics

Confinement of a quark anti-quark pair with masses much larger than the typical confinement scale may be described by a non-relativistic Hamiltonian with an attractive potential which is linear in the distance between the quarks, and given by,

$$H = \frac{\boldsymbol{p}^2}{2\mu} + \frac{\hbar^2 a^3}{2\mu}|\boldsymbol{r}|$$

where μ is the reduced mass, and $a > 0$ is a constant related to the confinement scale.

a) (12 points) Estimate the ground state energy by using the variational method and a family of trial wave functions depending on one parameter $\lambda > 0$,

$$\psi_\lambda(\boldsymbol{r}) = \begin{cases} (\lambda^2 - \boldsymbol{r}^2) & |\boldsymbol{r}| < \lambda \\ 0 & |\boldsymbol{r}| > \lambda \end{cases}$$

b) (8 points) How can the variational method be used to estimate the energy of the first excited state with zero orbital angular momentum as well? Please give a careful explanation, but there is no need to perform any calculations.

1. We begin by computing the norm of the trial wave function,

$$\langle\Psi_\lambda|\Psi_\lambda\rangle = \int_{r\leq\lambda} d^3r\,(\lambda^2 - r^2)^2 = \frac{32}{105}\pi\lambda^7 \tag{0.3}$$

and the expectation value of the Hamiltonian in the unnormalized trial wave function,

$$\langle\Psi_\lambda|H|\Psi_\lambda\rangle = \frac{\hbar^2}{2\mu}\int_{r\leq\lambda} d^3r\left(|2\mathbf{r}|^2 + r(\lambda^2 - r^2)^2\right) = \pi\frac{\hbar^2}{\mu}\left(\frac{8}{5}\lambda^5 + \frac{1}{12}a^3\lambda^8\right) \tag{0.4}$$

Hence the physical expectation value of the Hamiltonian gives the trial energy,

$$E_\lambda = \frac{\langle\Psi_\lambda|H|\Psi_\lambda\rangle}{\langle\Psi_\lambda|\Psi_\lambda\rangle} = \frac{21\hbar^2}{4\mu}\left(\frac{1}{\lambda^2} + \frac{5a^3\lambda}{96}\right) \tag{0.5}$$

Minimizing the energy as a function of $\lambda > 0$, we find the minimum at,

$$\lambda_0 = \frac{\kappa}{a} \qquad\qquad \kappa = 4\left(\frac{3}{5}\right)^{\frac{1}{3}} \tag{0.6}$$

and the corresponding energy,

$$E_{\lambda_0} = \frac{63}{4\kappa^2}\frac{\hbar^2 a^2}{\mu} \tag{0.7}$$

2. The variational approximation to computing the energy of the ground state and first excited state in the s-wave channel is to use a family of a pair of mutually orthogonal normalized trial wave functions functions $\psi_\lambda^{(1)}$ and $\psi_\lambda^{(2)}$ where λ may stand for a single trial parameter, or an array of trial parameters. Thus, we should have,

$$\langle\psi_\lambda^{(i)}|\psi_\lambda^{(j)}\rangle = \delta_{ij} \tag{0.8}$$

for all values of λ. We then evaluate the reduced Hamiltonian projected on this subspace,

$$H_\lambda^{ij} = \langle\psi_\lambda^{(i)}|H|\psi_\lambda^{(j)}\rangle \tag{0.9}$$

and diagonalize H_λ^{ij} to give two eigenvalues $E_\lambda^{(0)} < E_\lambda^{(1)}$. The ground state energy is then determined by minimizing $E_\lambda^{(0)}$ as a function of λ. If this minimum gives a unique value for $\lambda = \lambda_0$ then the first excited state energy is $E_{\lambda_0}^{(1)}$. If the value of λ is not uniquely fixed at the minimum, then one can further minimize the value of $E_\lambda^{(1)}$ on this subspace.

Generally, the more parameters are available for variation, the more accurate the approximation can become. In the present case the trial wave functions, in the s-wave state, may be taken to be orthogonal polynomials of r.

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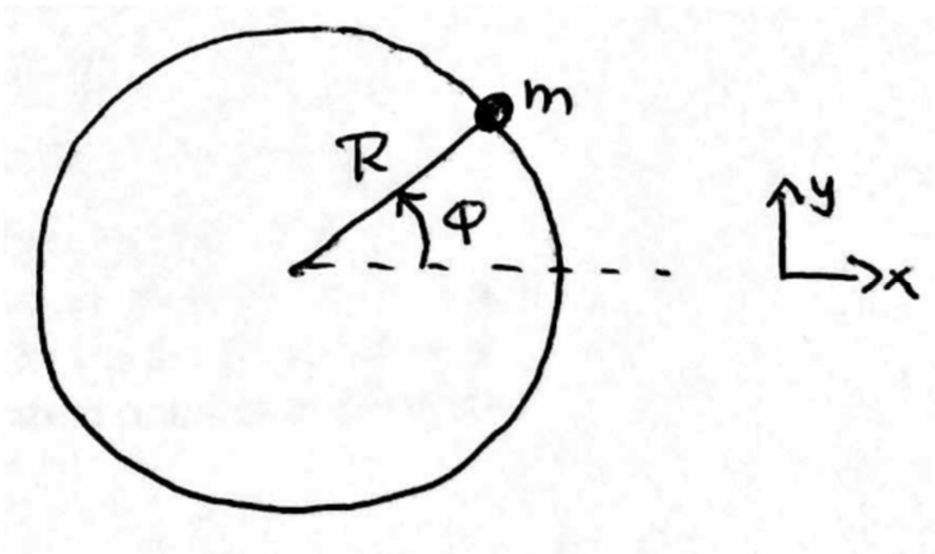
6. Quantum Mechanics

A particle of mass m is constrained to move on a ring of radius R lying in the xy plane. The system is in its ground state. A time-dependent potential is applied to the particle of the form:

$$V(y,t) = \begin{cases} 0 & t < 0 \\ yV_0 e^{-t/\tau} & t > 0 \end{cases}$$

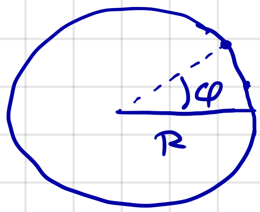


(20 points) At long times $t/\tau \gg 1$ the system is observed. Find the probability of finding it in each of the excited states of the unperturbed Hamiltonian.

You may assume that V_0 is sufficiently small that only first-order perturbation theory is necessary.

1. Quantum Mechanics Solution

moment of inertia $I = mR^2$ and the unperturbed Hamiltonian is:



$$H_0 = \frac{1}{2I} L_z^2 \implies \text{Unperturbed energy eigenstates are}$$

$$E_m = \langle m|H|m\rangle = \frac{m^2\hbar^2}{2I} \qquad \langle\varphi|m\rangle = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad m \in \mathbb{Z}.$$

Adding the perturbation

$$H_1(t) = V_0 e^{-t/\tau} R\sin\varphi$$ for $t>0$. And 0 otherwise.

Transition amplitude to state m_f: from initial state m_i

$$A^{(1)}_{fi} = \frac{-i}{\hbar}\int_0^\infty \langle m_f| V_0 R \sin\varphi\, e^{-t/\tau} |m_i\rangle e^{i\bar{\omega}t}$$

where $\bar{\omega} = \frac{E_f - E_i}{\hbar} = \frac{(m_f^2 - m_i^2)\hbar}{2I}$

Now $$\langle m_f|\sin\varphi|m_i\rangle = \int_0^{2\pi} \frac{d\varphi}{4\pi i}\left\{e^{i\varphi[m_f - m_i + 1]} - e^{+i\varphi[m_f - m_i - 1]}\right\}$$

$$= \frac{1}{2i}\left\{\delta_{m_f, m_i - 1} - \delta_{m_f, m_i+1}\right\}$$

And the time integral is:

$$\int_0^\infty dt\, e^{t(-\frac{1}{\tau} + i\bar{\omega})} = \frac{-1}{-\frac{1}{\tau} + i\bar{\omega}} = \frac{\tau}{1 - i\bar{\omega}\tau}$$

Now we put the pieces together:

$$A^{(1)}_{1,0} = \frac{-i}{\hbar}\left(-\frac{1}{2i}\right) V_0 R \frac{\tau}{1 - i\bar{\omega}\tau}; \quad \bar{\omega} = \hbar/2I$$

$$A^{(1)}_{-1,0} = \frac{-i}{\hbar}\left(\frac{1}{2i}\right) V_0 R \frac{\tau}{1 + i\bar{\omega}\tau} \quad \text{since here } \bar{\omega} \to -\bar{\omega}$$

Both transition probabilities are:

$$P_{\pm 1,0} = \frac{1}{4\hbar^2}(V_0 R)^2 \frac{\tau^2}{1 + \bar{\omega}^2\tau^2}; \quad \bar{\omega} = \frac{\hbar}{2mR^2}$$ and all others are zero at

first order. Transitions to the $m = \pm 2$ state have a probability of $O(V_0^2 R^2)$ and so on.

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7. Quantum Mechanics

Consider a particle of charge *e*, mass m_0 , constrained to move on the surface of a sphere of radius *R* (we do not consider spin in this problem). There is a uniform magnetic field **B**.

a) (10 points) Write the Hamiltonian in terms of the momentum and angular momentum operators, neglecting terms second order in the field.

b) (10 points) Find the energy levels of the system.

(Hint: work in the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$)

4. a) $\hat{H} = \frac{1}{2m_0}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m_0}|\vec{p}|^2 - \frac{2e}{2m_0 c}\vec{p}\cdot\vec{A}$

$+ \mathcal{O}(A^2)$

$\vec{p}\cdot\vec{A} = \frac{1}{2}\vec{p}\cdot(\vec{B}\times\vec{r}) = \frac{1}{2}\vec{B}\cdot(\vec{r}\times\vec{p}) = \frac{1}{2}\vec{B}\cdot\vec{L}$

$\Rightarrow \quad \hat{H} = \frac{|\vec{p}|^2}{2m_0} - \frac{e}{2m_0 c}\vec{B}\cdot\vec{L}$

b) $|\vec{p}|^2 = -\nabla^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}|\vec{L}|^2$

but for our problem $r = \text{const.} = R$, so

$|\vec{p}|^2 = \frac{1}{R^2}|\vec{L}|^2$

and $$\hat{H} = \frac{1}{2m_0R^2}|\hat{\vec{L}}|^2 - \frac{eB}{2m_0c}\hat{L}_z$$

(choosing the z axis along $\vec{B}$)

Energy levels: $\hat{H}\psi = E\psi$

ψ are the $Y_{\ell m}$ (simult. eigenfcts. of $|\hat{\vec{L}}|^2$, $\hat{L}_z$)

$$\rightarrow \quad E = \frac{\ell(\ell+1)}{2m_0R^2} - \frac{eB}{2m_0c}m$$

where $\ell = 0, 1, 2, \dots$ and $m = \ell, \ell-1, \dots$
$\dots, -\ell$.

FALL 2016

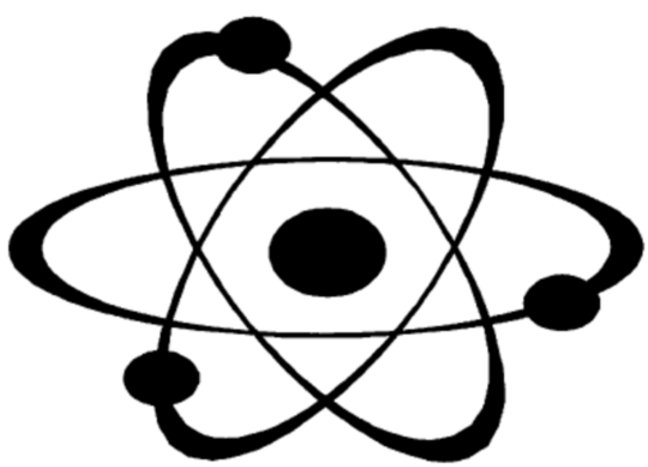
Physics Comprehensive Exam

September 13th, 2016 (Part 2) 9:00 – 1:00 pm

Part 2: Statistical Mechanics and Electromagnetism

7 Total Problems/ 20 Points Each / Total 140 Points

- Closed book exam.
- Calculators not allowed.
- Begin your solution on the problem page.
- Use paper provided for additional pages. **Use one side only.**
- To ensure anonymous grading, use the Number Labels you are given on EACH of your response pages, including the question page. If you run out of labels, be sure to write the Number you have been assigned on each page.
- Return the question page as the first page of your answers.
- When submitting, please clip all pages together in question # order.
- If a part of any question seems ambiguous to you, state clearly your interpretations and answer the question accordingly.



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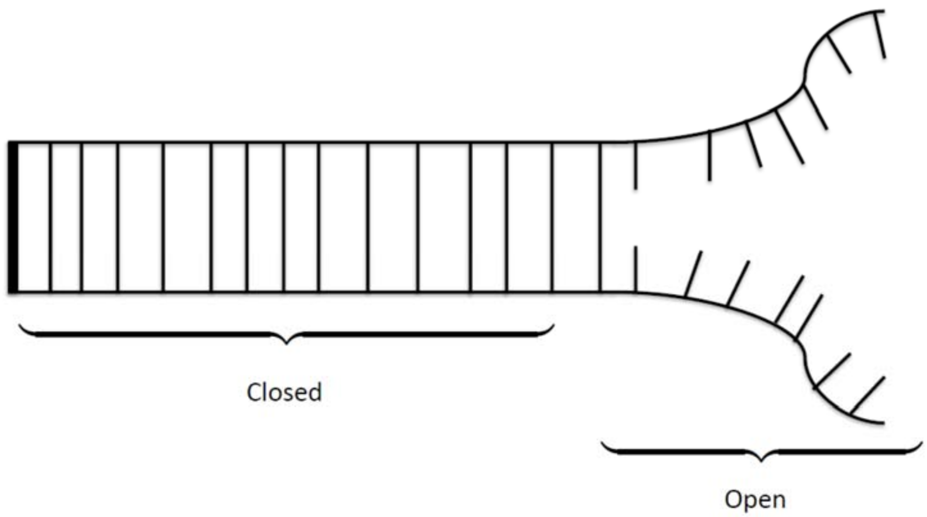
1. Statistical Mechanics

A molecular zipper. Consider the following simple model for the melting of DNA. Two unbreakable molecular strands are coupled by *N* links. See the figure below. Each link can be in one of $G+1$ states. Of these states, G are unbound (open) and have energy ϵ, and one is bound (closed) and has energy 0. You must assume that links unbind sequentially. In other words, the only states of the system with finite energy are those in which all links to the right of a given link are unbound. Let $n \leq N-1$ be the number of unbound links.



a) (5 points) Find the partition function of the system

b) (10 points) Find the mean fraction of unbound links $\chi = \frac{\langle n \rangle}{N}$ at temperature T in the thermodynamic limit $N \to \infty$. Show that there is a critical temperature where this mean fraction changes in the thermodynamic limit.

c) (5 points) Compute the fraction of completely unzipped ($n = N-1$) zippers in a noninteracting solution of these molecules as function of temperature T.

Statistical Mechanics Solution

a) Partition sum

$$Z = 1 + g e^{-\varepsilon/T} + \left(g e^{-\varepsilon/T}\right)^2 + \cdots + \left(g e^{-\varepsilon/T}\right)^{N-1}$$

$$Z = \sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}; \quad \text{where } x = g e^{-\varepsilon/T}$$

b) $$\langle n \rangle = x \frac{d}{dx} \log Z = x \frac{d}{dx}\left[\log(1-x^N) - \log(1-x)\right]$$

$$= x \left[\frac{-N x^{N-1}}{1-x^N} + \frac{1}{1-x}\right]$$

$$= \frac{x}{(1-x^N)(1-x)}\left[1 - x^N - N x^{N-1}(1-x)\right] = \frac{x}{(1-x^N)(1-x)}\left[1 - x^N - N x^{N-1} + N x^N\right]$$

$$\langle n \rangle = \frac{x}{(1-x^N)(1-x)}\left[(N-1)x^N + 1 - N x^{N-1}\right]$$

Now $f = \frac{\langle n \rangle}{N} = \frac{x}{(1-x^N)(1-x)}\left[x^N - x^{N-1}\right] + \mathcal{O}(1/N)$ if $x > 1$

↑ open fraction

$$= \frac{x\, x^{N-1}}{1-x^N} \frac{x-1}{1-x} = \frac{x^N}{x^N - 1} \xrightarrow[N\to\infty]{} 1$$

If $x < 1$ then we get the thermodynamic limit:

$$\langle n \rangle \longrightarrow \frac{x}{(1-x^N)(1-x)}\left[1 - x^N + N x^{N-1}(x-1)\right]$$

$$f = \frac{\langle n \rangle}{N} \longrightarrow \frac{x}{1-x}\left(\frac{1}{N}\right) - \frac{x^N}{1-x^N} \xrightarrow[N\to\infty]{} 0$$

So different limits depending on the value of

$$g e^{-\varepsilon/T} \gtrless 1$$

$\Rightarrow$ critical temp $T_c = \dfrac{\varepsilon}{\log(g)}$

c) $\mathcal{F}$ = fraction of open chains:

$$\mathcal{F} = \frac{x^{N-1}}{Z} = \frac{x^{N-1}(1-x)}{1-x^N}$$

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2. Statistical Mechanics

You have 3 concentric spheres with radii $R_1 < R_2 < R_3$. The sphere at R_1 is maintained at temperature T_1 and the sphere at R_3 is maintained at temperature T_3. Assume that the spheres are black and that the only heat transport occurs via photons.

a) (4 points) Draw a diagram (or two) and label it with the variables you will use to solve the problem.

b) (13 points) Find the steady-state temperature T_2 of the sphere at R_2.

c) (3 points) Evaluate the interesting limiting cases of your result from (b).

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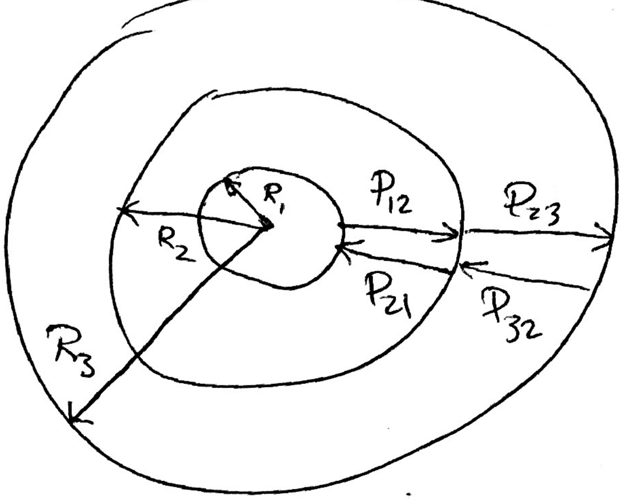


In the steady-state the power emitted by the sphere @ R_2 must equal the power absorbed:

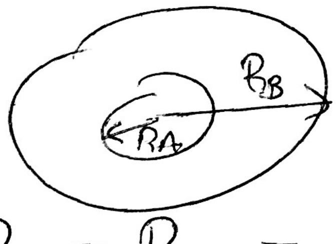
$$P_{12} + P_{32} = P_{21} + P_{23}$$

↑ incident ↑ emitted

Expressions for P_{12} and P_{23} follow directly from the Stefan-Boltzmann Law

$$P = \varepsilon A \sigma T^4 = A \sigma T^4 \text{ for a blackbody,}$$

which has emissivity $\varepsilon = 1$. The other powers are not so obvious, since some power radiated from a larger sphere will miss the smaller. If neighboring spheres



happen to have $T_A = T_B$, then detailed balance says

$$P_{AB} = P_{BA} = 4\pi R_A^2 \sigma T_A^4,$$ identifying the geometric factor. Thus

$$A_1 \sigma T_1^4 + A_2 \sigma T_3^4 = A_1 \sigma T_2^4 + A_2 \sigma T_2^4$$

$$R_1^2 T_1^4 + R_2^2 T_3^4 = R_1^2 T_2^4 + R_2^2 T_2^4$$

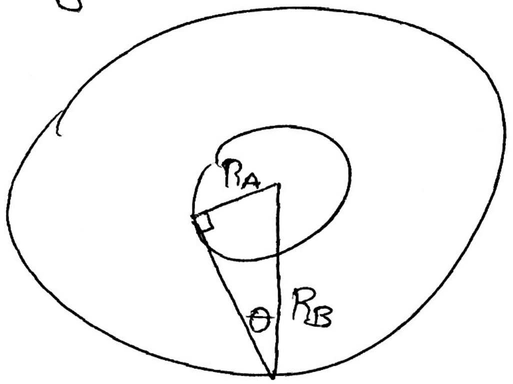
$$\boxed{T_2 = \left(\frac{R_1^2 T_1^4 + R_2^2 T_3^4}{R_1^2 + R_2^2} \right)^{\frac{1}{4}}},$$ independent of R_3.

limits $R_1 \to R_2$ $\quad T_2 \to \left(\frac{T_1^4 + T_3^4}{2} \right)^{\frac{1}{4}}$ sensible

$R_1 \to 0$ or $R_2 \to \infty$ $\quad T_2 \to T_3$, the background temperature

SM1

If the detailed balance argument does not occur to you, you must calculate the geometric factor explicitly.



A blackbody is a Lambertian emitter.

Only radiation at angles $\theta \le \arcsin\left(\frac{R_A}{R_B}\right)$ is incident on the inner sphere.

$$\int \cos\theta \, d\Omega = \int \cos\theta \, d(\cos\theta) \, d\phi = 2\pi \frac{\cos^2\theta}{2}$$

$$= \pi \left(1 - \sin^2\theta\right)\Big|_{\arcsin\left(\frac{R_A}{R_B}\right)}^{0} = \pi\left[1 - \left(1 - \left(\frac{R_A}{R_B}\right)^2\right)\right]$$

$$= \pi \frac{R_A^2}{R_B^2}$$

The usual result has $\theta = \frac{\pi}{2}$ (i.e. $R_A = R_B$), giving π for the integral.

Thus $P_{BA} = \sigma A_B \cdot \frac{R_A^2}{R_B^2} \cdot T_B^4$

$= \sigma A_A \cdot T_B^4$ recovering the result we found with detailed balance.

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3. Statistical Mechanics

a) (10 points) Find the temperature dependence of the heat capacity at constant volume (C_V) for an ideal Bose gas below the BE condensation temperature.

b) (10 points) Give an expression for the condensation temperature T_0 in terms of *N*, *V*. Definite integrals which are just numerical factors may be left indicated as such.

Solution a) The single particle energy is $\varepsilon = \frac{p^2}{2m}$ and the occupation number of the single particle states

$$\bar{n} = \frac{1}{e^{(\varepsilon-\mu)/T} - 1}$$

For $T \le T_0$, $\mu = 0$ and while there is a macroscopic number $N(\varepsilon = 0)$ of particles in the ground state, only particles with $\varepsilon > 0$ contribute to the energy so we may write:

$$E = \int \frac{d^3p\, d^3x}{h^3} \frac{p^2/2m}{e^{p^2/2mT} - 1} \qquad (T \le T_0)$$

$$= \frac{V}{h^3} \frac{4\pi}{2m} A_4 (2mT)^{5/2} \quad \text{where} \quad A_4 = \int_0^\infty du \frac{u^4}{e^{u^2} - 1}$$

2

$$\Rightarrow \quad C_v = \left.\frac{\partial E}{\partial T}\right|_V \propto T^{3/2}$$

b) At $T = T_0$, $\mu = 0$ and we may write:

$$N = \int \frac{d^3x\, d^3p}{h^3} \frac{1}{e^{p^2/2mT_0} - 1} = \frac{(2mT_0)^{3/2}}{h^3} V 4\pi A_2$$

where $A_2 = \int_0^\infty du \frac{u^2}{e^{u^2} - 1}$

$$\Rightarrow \quad T_0 = \left(\frac{2\pi^2}{A_2}\right)^{2/3} \frac{\hbar^2}{2m} \left(\frac{N}{V}\right)^{2/3}$$

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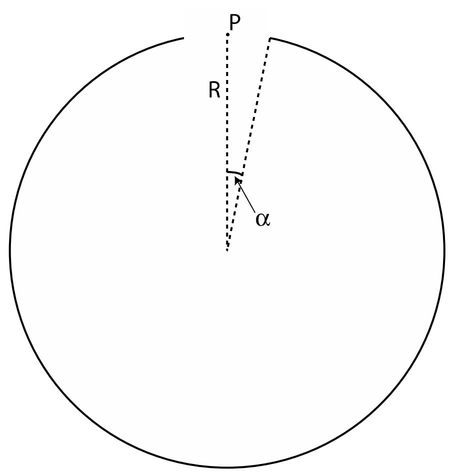
4. Electromagnetism

A thin spherical shell of radius R is uniformly charged so that the charge per unit area on the surface is σ. You take a sword and chop off the very top of the sphere, so that there is a hole at the apex with polar opening angle α, as shown below.



a) (10 points) If the angle α is small (so that the diameter of the opening is much smaller than the radius of the sphere), what is the electric field at the center of the sphere (magnitude and direction)?

b) (10 points) What is the electric field at point P in the diagram (in the opening, at the location where the apex of the sphere used to be before I sliced off the top)?

SOLUTION: We solve this by superposition. We superimpose the electric field generated by two different charge distributions: (1) E_1 due to a complete spherical shell with uniform charge density σ and (2) E_2 due to a uniformly negatively charged spherical cap (charge density σ) with radius of curvature R and with extent in the polar angle α. Adding these two together gives me the charge distribution shown in the figure.

The first of these two objects produces no electric field at the center of the sphere (in fact, it produces no electric field for any $r < R$. So we just need to deal with the negative spherical cap.

The problem statement lets us consider the leading order contribution from the cap (we are told that the cap is very small compared to R). The leading term in the electric field from the cap will be the monopole term; so we can treat the field like that from a point charge located at the apex of the sphere. The total charge of the cap is the surface area times $-\sigma$. The surface area of the cap is:

$$A = R^2 \int_0^{2\pi} d\phi \int_0^{\alpha} \sin\theta d\theta = \sigma 2\pi R^2 (1 - \cos\alpha) \approx \pi\alpha^2 R^2$$

The field at the center of the sphere is then:

$$\mathbf{E} \approx \frac{\sigma A}{4\pi\epsilon_o R^2}\hat{z}$$

The field points up, toward the opening.

SOLUTION: The field at point P is again the superposition of the fields from the two charge distributions. For whole spherical shell, the electric field just above the sphere at point P is:

$$\mathbf{E}_{1,\text{above}} = \hat{z}\frac{\sigma}{\epsilon_o}$$

You can get the above from knowing the field of the whole sphere is the same as if a point charge of total charge $\sigma 4\pi R^2$ was sitting at the origin (center of the sphere). You can also get it from the jump condition for E at the surface.

$$\hat{z} \cdot (\mathbf{E}_{1,\text{above}} - \mathbf{E}_{1,\text{below}}) = \frac{\sigma}{\epsilon_o}$$

And you use $E_{1,\text{below}} = 0$; the field below is made zero by contributions from elsewhere on the sphere.

The field at point P due to the cap can be obtained the same way. If the cap is small, so that we can treat it as a small disk/plane, then the field just above and just below the plane is:

$$E_{2,\text{above}} = -\frac{\sigma}{2\epsilon_o}\hat{z}$$

and

$$E_{2,\text{below}} = \frac{\sigma}{2\epsilon_o}\hat{z}$$

If the cap were large, this answer would not be correct – the cap would have too much curvature and the field would not be the same as an infinite plane (e.g. take the limit of the cap being the whole sphere, and you get zero electric field under the surface at point P).

We add these two together to find the same field above and below point P (we had better as there is no charge there to introduce a discontinuity!):

$$\mathbf{E}_P = \frac{\sigma}{2\epsilon_o}\hat{z}$$

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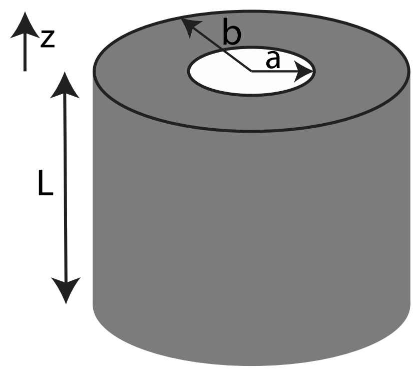
5. Electromagnetism

A permanent magnet is created from a hollow cylindrical piece of magnetic material with inner radius *a*, outer radius *b*, and length *L*. It is given magnetization

$$\boldsymbol{M} = \frac{M_o \rho}{a} \hat{\boldsymbol{\phi}}$$

where ρ is the distance from the axis of the hollow cylinder and $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector.



a) (10 points) Where are the bound currents in this magnet and what are their values?

b) (10 points) What is the magnetic field produced by this magnet ?

Useful formulas:

$$\boldsymbol{\nabla}\psi = \boldsymbol{e_1}\frac{\partial\psi}{\partial\rho} + \boldsymbol{e_2}\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \boldsymbol{e_3}\frac{\partial\psi}{\partial z}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\varphi} + \frac{\partial A_3}{\partial z}$$

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{e_1}\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\varphi} - \frac{\partial A_2}{\partial z}\right) + \boldsymbol{e_2}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \boldsymbol{e_3}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\varphi}\right)$$

SOLUTION: The bound currents are given by:

$$\mathbf{j}_B = \nabla \times \mathbf{M}$$

in the volume and

$$\mathbf{K}_B = \mathbf{M} \times \hat{n}$$

at the surfaces. Looking at the expression for the curl in cylindrical coordinates (see equation sheet), we see that $\nabla \times \mathbf{M}$ is zero in the bulk of the magnet, so $\mathbf{j}_B$ is zero. We have four surfaces to deal with: (1) at the inner cylindrical surface, $\rho = a$, we have $\hat{n} = -\hat{\rho}$ (always the normal vector that points OUTWARD from the material), (2) at the outer cylindrical surface, $\rho = b$, we have $\hat{n} = \hat{\rho}$, (3) at the bottom surface, $z = 0$, we have $\hat{n} = -\hat{z}$, and (4) at the top surface, $z = L$, we have $\hat{n} = \hat{z}$. Looking at $\mathbf{M} \times \hat{n}$ at these four surfaces we have:

$$\mathbf{K}_B = \begin{cases} M_o \hat{z} & \rho = a \\ -M_o \frac{a}{b} & \rho = b \\ -M_o \frac{a}{\rho} \hat{\rho} & z = 0 \\ M_o \frac{a}{\rho} \hat{\rho} & z = L \end{cases}$$

I didn't ask you to, but you can show that the total current flowing through each surface is the same (as we should expect).

SOLUTION: Based on the currents above, this is a toroidal solenoid. You can take this distribution of currents and solve for $\mathbf{B}$ in the usual way (Ampere's law will work). But the easiest way to do this problem is to note that $\mathbf{H} = 0$ everywhere. This is because the free current is zero everywhere and $\nabla \cdot \mathbf{M} = 0$ (can see this from the cylindrical divergence formula on the equation sheet or by the fact that lines of $\mathbf{M}$ don't start or end anywhere). So if $\mathbf{H} = 0$ then:

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) = \mu_o \mathbf{M} = \mu_o M_o \frac{a}{\rho} \hat{\phi}$$

The field is zero outside the magnet ($\mathbf{M}$ is zero there). You'll get the same answer applying Ampere's law (and symmetry arguments) to the currents given above.

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6. Electromagnetism

A plane wave with intensity $\langle S \rangle$ J/m^2·s is incident on a totally reflecting, plane surface at an angle θ, where θ is measured relative to the plane normal.

a) (8 points) Find the radiation pressure normal to the surface.

b) (8 points) Find the total radiation force produced by this plane wave when incident on a perfectly reflecting sphere of radius R.

c) (4 points) Find the total radiation force produced by this plane wave when incident on a perfectly absorbing sphere of radius R, and compare with the result of (b).

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a) At normal incidence the relationship between pressure and intensity is $p = \frac{\langle S \rangle}{c}$.

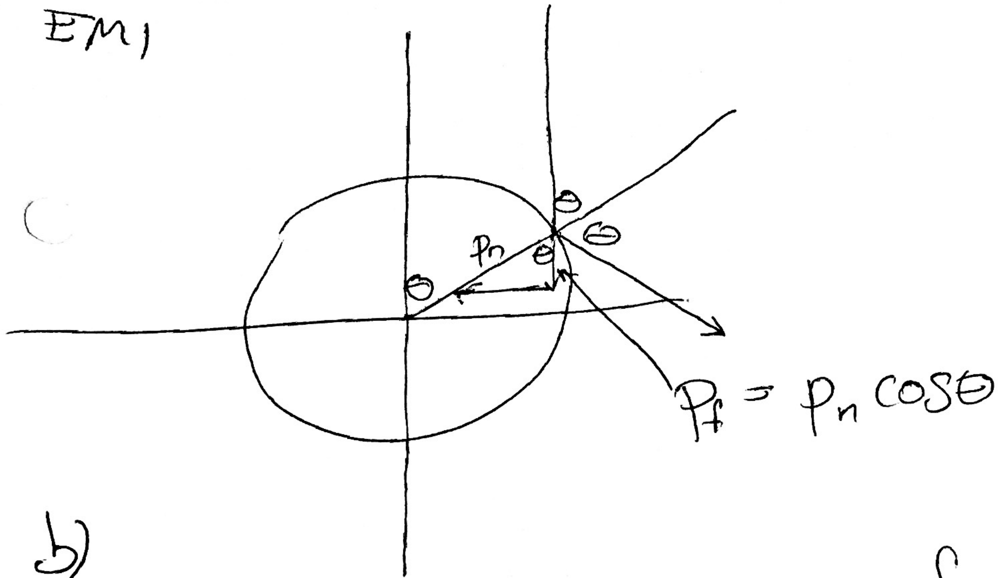
(This relation can be recalled directly from the energy-momentum relation for light, $E = pc$).

At non-normal incidence we have two cosines factors

$S_\perp = S\cos\theta$ $\qquad$ $A_\perp \cos\theta = A$

and a factor of 2 for the reaction from the reflected radiation.

Thus, the normal pressure $P_n = \frac{2 \cdot S_\perp}{c} \cdot \frac{A}{A_\perp} = \frac{2\langle S \rangle}{c}\cos^2\theta$



b)

The total forward force $F = \int p_f \, da$

↑ forward pressure ↑ area

The forces tending to compress the sphere cancel by symmetry.

$$F_{ref} = \int p_n \cos\theta \cdot R^2 d\Omega$$

$$= \int \frac{2\langle S\rangle}{c} \cos^2\theta \cos\theta \; R^2 \, d(\cos\theta) \, d\phi$$

$$= \frac{2\langle S\rangle}{c} R^2 \cdot 2\pi \int \cos^3\theta \, d(\cos\theta)$$

$$= \frac{\pi R^2 \langle S\rangle}{c}$$

c) An absorbing sphere presents a cross-section πR^2 to the incident plane wave, absorbing all of its momentum (and without the factor of 2).

$$F_{abs} = p \cdot A = \frac{\langle S\rangle}{c} \cdot \pi R^2$$

Thus the reflecting and absorbing cases give the same result.

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7. Electromagnetism

Find the total energy radiated in the head-on collision of a non-relativistic particle of charge q, velocity v_0 against a fixed target of charge Q ($qQ > 0$). Write your result in terms of q, Q, v_0.

1.

The instantaneous power radiated is $P = \frac{2}{3}\frac{q^2}{c^3}\dot{v}^2$

the eq. of motion: $m\dot{v} = \frac{Qq}{r}$ $(Qq > 0)$

with an integral: $\frac{1}{2}mv^2 + \frac{Qq}{r} = E$ (1)

where $E = \frac{1}{2}mv_0^2$ is the energy of the particle.

The energy radiated is

$$W = 2\int_{-\infty}^{t_0} P\,dt = 2\,\frac{2}{3}\frac{q^2}{c^3}\frac{Q^2q^2}{m^2}\int_{-\infty}^{t_0}\frac{dt}{r^4}$$

where t_0 is the time of closest approach.

To calculate the integral: $r = r(t) \Rightarrow dr = -v\,dt$ (see drawing)

$v = \sqrt{\frac{2}{m}}\left(E - \frac{Qq}{r}\right)^{1/2}$ from (1)

so
$$\int_{-\infty}^{t_0}\frac{dt}{r^4} = -\int_{\infty}^{r_0}\frac{1}{v}\frac{dr}{r^4} = \sqrt{\frac{m}{2}}\int_{r_0}^{\infty}\frac{dr}{r^4\left(E - \frac{Qq}{r}\right)^{1/2}}$$

where r_0 is the distance of closest approach

i.e. $E = \frac{Qq}{r_0}$.

To evaluate the integral: $E - \frac{Qq}{r} = y \Rightarrow r = \frac{Qq}{E-y}$

$\Rightarrow dy = \frac{Qq}{r^2} dr$ and the last integral is

$$\frac{1}{Qq} \int_{E-\frac{Qq}{r_0}}^{E} \frac{dy\,(E-y)^2}{(Qq)^2\, y^{1/2}} = \frac{1}{(Qq)^3} \int_0^E dy \left[\frac{E^2}{y^{1/2}} - 2E y^{1/2} + y^{3/2} \right].$$

$$= \frac{1}{(Qq)^3} \left[2E^2 y^{1/2} - 2E \frac{2}{3} y^{3/2} + \frac{2}{5} y^{5/2} \right]_0^E$$

$$= \frac{1}{(Qq)^3} \left(2 - \frac{4}{3} + \frac{2}{5} \right) E^{5/2} = \frac{16}{15} \frac{E^{5/2}}{(Qq)^3}$$

so $W = \frac{4}{3} \frac{16}{15} \frac{q^4 Q^2}{c^3 m^2} \left(\frac{m}{2} \right)^{1/2} \frac{E^{5/2}}{(Qq)^3}$, $E = \frac{1}{2} m V_0^2$

$$\Rightarrow W = \frac{8}{45} \frac{m q V_0^5}{c^3 Q} .$$