1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a \cdot \hat{\mathbf{n}}_a \, \mathbf{s}_b \cdot \hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

2. *Quantum Mechanics* (Fall 2004)

The van der Waals interaction between two neutral atoms is due to dipole-dipole interactions. Consider the following simplified 1-D model. Each atom consists of a fixed nucleus of charge $+e$ and electron of charge $-e$, bound by a harmonic spring. Two such oscillators are a distance R ($\gg$ size of the atom) apart. The Hamiltonian of the system consists of the two harmonic oscillator terms plus a dipole-dipole perturbation.

(a) Write the perturbation part of the Hamiltonian.

(b) Calculate the correction to the energy of the unperturbed ground state. This is the van der Waals interaction potential. (*Hint*: it should come out $\propto 1/R^6$.)

3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\left\langle \mathbf{r}_+, \mathbf{r}_- \middle| \mathbf{r}'_+, \mathbf{r}'_- \right\rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\, \delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium.* For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\,|\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

[1]Write your answer in terms of m, e^2 or α, $\hbar$, c, the Bohr radius, etc. You may use units in which $\hbar = c = 1$.

4. *Quantum Mechanics* (Fall 2004)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L}+\mathbf{s}$. Conventionally, the states are labeled $|n, l, j, m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

(a) If the electron is in the state $|n, l, j, m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $\mathbf{J}_\pm = J_x \pm iJ_y$. What are

(i) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_+ \middle| 3, 1, \frac{3}{2}, -\frac{1}{2} \right\rangle = ?$

(ii) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_+ \middle| 3, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(iii) $\left\langle 2, 1, \frac{3}{2}, \frac{3}{2} \middle| p_z \middle| 2, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(iv) $\left\langle 2, 1, \frac{1}{2}, -\frac{1}{2} \middle| \mathbf{L}^2 \middle| 2, 1, \frac{1}{2}, -\frac{1}{2} \right\rangle = ?$

(v) $\left\langle 3, 2, \frac{3}{2}, -\frac{1}{2} \middle| \mathbf{J}^2 \middle| 3, 2, \frac{3}{2}, -\frac{1}{2} \right\rangle = ?$

(vi) $\left\langle 3, 1, \frac{3}{2}, \frac{3}{2} \middle| J_z \middle| 3, 1, \frac{3}{2}, \frac{1}{2} \right\rangle = ?$

(d) What is $\left\langle 1, 0, \frac{1}{2}, \frac{1}{2} \middle| p_i\, p_j \middle| 1, 0, \frac{1}{2}, \frac{1}{2} \right\rangle = ?$

(e) For given n, l, j, and m, what are the conditions on n', l', j', and m' so that

$$\left\langle n', l', j', m' \middle| \mathbf{s}\cdot\mathbf{r} \middle| n, l, j, m \right\rangle \neq 0?$$

5. *Quantum Mechanics* (Fall 2004)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, ...$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1| e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $\big|\psi(0)\big\rangle = |\phi\rangle$. What is $\big|\psi(t)\big\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

6. *Statistical Mechanics and Thermodynamics* (Fall 2004)

If the specific heat of a gas of non-interacting fermions in d dimensions varies with temperature as $C \sim T^{\alpha}$ for $k_B T \ll E_F$, then what is α? What is α for a system of non-interacting bosons?

7. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Some organic molecules have a triplet excited state at energy $k_B\Delta$ above a singlet ground state.

(a) Find an expression for the magnetic moment in a field B in terms of Δ, B, the temperature T, the Bohr magneton μ_B, and the gyromagnetic ratio g.

(b) Show that the susceptibility for $T \gg \Delta$ is given by $N(g\mu_B)^2/2k_BT$, where N is the total number of molecules in the system.

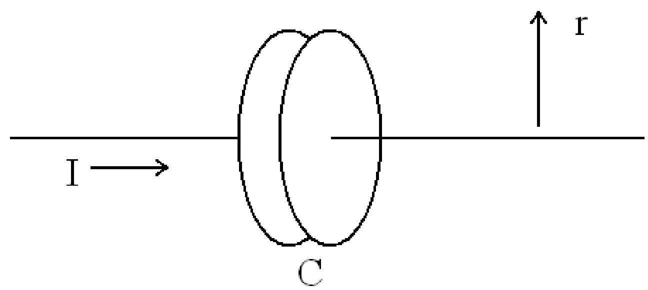
(c) With the help of a diagram of energy levels versus field and a rough sketch of entropy versus field, explain how this system might be cooled by adiabatic magnetization (*not demagnetization*).

8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?

9. *Electricity and Magnetism* (Fall 2004)

A wire carrying current I is connected to a circular capacitor of capacitance C, as depicted in the figure. What is the magnetic field outside the wire, far from the capacitor (as a function of the distance r from the wire)? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor. What is this magnetic field?



10. *Electricity and Magnetism* (Fall 2004)

The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

11. *Electricity and Magnetism* (Fall 2004)

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $\mathbf{r}(t)$. You do not need to find dimensionless proportionality constants (i.e., only find the dependence on q, $\mathbf{r}(t)$, and universal constants).

12. *Electricity and Magnetism* (Fall 2004)

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy E_{min} of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2}rM^2 + uM^4 - hM$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^\delta$. What is δ?

14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature T. What is the average energy per photon in units of kT? You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$

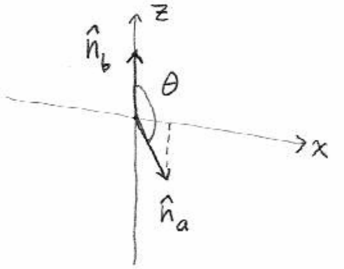
1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a \cdot \hat{\mathbf{n}}_a\, \mathbf{s}_b \cdot \hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

Pick axes:



Note:

$$\sigma_z|\pm\rangle = \pm|\pm\rangle$$
$$\sigma_x|\pm\rangle = |\mp\rangle$$
$$\sigma_y|\pm\rangle = \pm i|\mp\rangle$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$\langle\psi|\vec{s}_a\cdot\hat{n}_a\,\vec{s}_b\cdot\hat{n}_b|\psi\rangle = \frac{\hbar^2}{4}\langle\psi|\vec{\sigma}_a\cdot\hat{n}_a\,\vec{\sigma}_b\cdot\hat{n}_b|\psi\rangle$$

$$= \frac{\hbar^2}{4}\langle\psi|\left(\cos\theta\,\sigma_{az} + \sin\theta\,\sigma_{ax}\right)\sigma_{bz}\frac{1}{\sqrt{2}}\left[|+-\rangle - |-+\rangle\right]$$

$$= \frac{\hbar^2}{4}\frac{1}{2}\left[\langle+-| - \langle-+|\right]\Big[\left(+\cos\theta|+-\rangle + \sin\theta\,\cancel{|--\rangle}\right)(-1)$$
$$-\left(-\cos\theta|-+\rangle + \sin\theta\,\cancel{|++\rangle}\right)(+1)\Big]$$

$$= \frac{\hbar^2}{4}\frac{1}{2}\left[-\cos\theta\langle+-|+-\rangle - \cos\theta\langle-+|-+\rangle\right]$$

$$= -\frac{\hbar^2}{4}\cos\theta$$

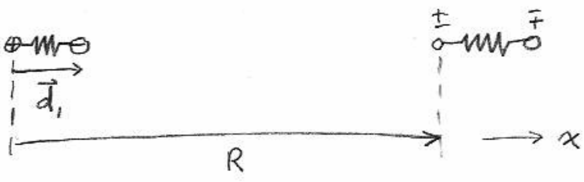
2. *Quantum Mechanics* (Fall 2004)

The van der Waals interaction between two neutral atoms is due to dipole-dipole interactions. Consider the following simplified 1-D model. Each atom consists of a fixed nucleus of charge $+e$ and electron of charge $-e$, bound by a harmonic spring. Two such oscillators are a distance R ($\gg$ size of the atom) apart. The Hamiltonian of the system consists of the two harmonic oscillator terms plus a dipole-dipole perturbation.

(a) Write the perturbation part of the Hamiltonian.

(b) Calculate the correction to the energy of the unperturbed ground state. This is the van der Waals interaction potential. (Hint: it should come out $\propto 1/R^6$.)

a) $H = H_1 + H_2 + H'$



$K = \frac{1}{4\pi\varepsilon_0}$

$H' = -\vec{p}_2 \cdot \vec{E}_1$

$$= -\vec{p}_2 \cdot \left[K \frac{3(\vec{p}_1 \cdot \hat{R})\hat{R} - \vec{p}_1}{R^3}\right] = -K\, \frac{3\, p_{1x} p_{2x} - p_{1x} p_{2x}}{R^3} = -2K \frac{d_{1x} d_{2x} e^2}{R^3}$$

$$= \mp 2K \frac{e^2}{R^3} d_1 d_2 \qquad \begin{array}{l} - \Rightarrow \text{aligned} \\ + \Rightarrow \text{antialigned} \end{array}$$

b) States $|n_1 n_2\rangle$ $\quad n_1, n_2 \in \mathbb{N} = \{0, 1, 2, \dots\}$

$$d_i = d_0 (a_i^\dagger + a_i) \qquad d_0 = \sqrt{\frac{\hbar}{2m\omega}} \qquad H' = \mp 2K \frac{e^2}{R^3} d_0^2 (a_1^\dagger + a_1)(a_2^\dagger + a_2)$$

$\Delta E_{00}^{(1)} = \langle 00 | H' | 00 \rangle = 0$ since the $a^\dagger$'s and a's only connect states with $\Delta n_i = \pm 1$.

$$\Delta E_{00}^{(2)} = -\sum_{m \neq 0} \sum_{k \neq 0} \frac{|\langle mk | H' | 00 \rangle|^2}{E_{mk}^0 - E_{00}^0} = -\left(\frac{2Ke^2 d_0^2}{R^3}\right)^2 \sum_m \sum_k \frac{|\langle mk | (a_1^\dagger + a_1)(a_2^\dagger + a_2) | 00 \rangle|^2}{\hbar\omega(m + k - 1)}$$

$$= -\left(\frac{2Ke^2\hbar}{2m\omega R^3}\right)^2 \frac{1}{\hbar\omega} \left[\frac{|\langle 11 | 11 \rangle|^2}{(1 + 1 - 1)}\right] = -\frac{K^2 e^4 \hbar}{m^2 \omega^3 R^6}$$

(No need for degenerate perturbation theory in this case.)

3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\langle \mathbf{r}_+, \mathbf{r}_- | \mathbf{r}'_+, \mathbf{r}'_- \rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\, \delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\, |\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

a) At least one particle farther than $r = b$

$\Rightarrow$ Not (both particles closer than $r = b$) $\quad V_b \equiv$ sphere, radius b, at origin

$$P = 1 - \int_{V_b}\int_{V_b} |\Psi(\vec{r}_+, \vec{r}_-)|^2 \, dv_1\, dv_2 = 1 - \int\int_{\text{all angles}} \int_0^b\int_0^b |\psi(\vec{r}_+, \vec{r}_-)|^2 r_+^2 dr_+ \, r_-^2 dr_- \, d\Omega_+ d\Omega_-$$

b) $$H = \frac{\vec{p}_+^{\,2}}{2m} + \frac{\vec{p}_-^{\,2}}{2m} - K\frac{e^2}{|\vec{r}_+ - \vec{r}_-|}$$

c) $$H = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^{\,2}}{2\mu} - K\frac{e^2}{r} \qquad M = 2m, \quad \mu = \tfrac{1}{2}m$$

d) $$E_n = -\frac{\mu K^2 e^4}{2\hbar^2 n^2} = -\frac{mK^2e^4}{4\hbar^2}\frac{1}{n^2} \approx -\frac{1}{2}(13.6\text{ eV})\frac{1}{n^2} \qquad E_1 \approx (6.8\text{ eV})$$

e) Since $\hat{C}^2 = \hat{1}$, $\quad \hat{C}\hat{H}\hat{C} = \hat{H} \iff [\hat{C}, \hat{H}] = \hat{\emptyset}$

$$\hat{C}\hat{\vec{r}}\hat{C}|\vec{r}_+, \vec{r}_-\rangle = \hat{C}\hat{\vec{r}}|\vec{r}_-, \vec{r}_+\rangle = \hat{C}(\hat{\vec{r}}_+ - \hat{\vec{r}}_-)|\vec{r}_-, \vec{r}_+\rangle = \hat{C}(\vec{r}_- - \vec{r}_+)|\vec{r}_-, \vec{r}_+\rangle$$

$$= (\vec{r}_- - \vec{r}_+)\hat{C}|\vec{r}_-, \vec{r}_+\rangle = -(\vec{r}_+ - \vec{r}_-)|\vec{r}_+, \vec{r}_-\rangle = -\vec{r}\,|\vec{r}_+, \vec{r}_-\rangle = -\hat{\vec{r}}\,|\vec{r}_+, \vec{r}_-\rangle$$

$\hat{C}\hat{\vec{r}}\hat{C} = -\hat{\vec{r}}$ for a complete basis $\Rightarrow$ $\hat{C}\hat{\vec{r}}\hat{C} = -\hat{\vec{r}}$ (for all state kets)

[1] Write your answer in terms of m, e^2 or α, $\hbar$, c, the Bohr radius, etc. You may use units in which $\hbar = c = 1$.

3. Quantum Mechanics (Fall 2004)

e) (continued)

$$\hat{C}\hat{\vec{p}}\hat{C}\,|\vec{r}_+,\vec{r}_-\rangle = \hat{C}\,(\hat{\vec{p}}_+ - \hat{\vec{p}}_-)|\vec{r}_-,\vec{r}_+\rangle = \hat{C}\,(-i\hbar)(\vec{\nabla}_- - \vec{\nabla}_+)|\vec{r}_-,\vec{r}_+\rangle$$

$$= (-i\hbar)(\vec{\nabla}_- - \vec{\nabla}_+)\hat{C}\,|\vec{r}_-,\vec{r}_+\rangle = -(-i\hbar)(\vec{\nabla}_+ - \vec{\nabla}_-)|\vec{r}_+,\vec{r}_-\rangle$$

$$= -(\hat{\vec{p}}_+ - \hat{\vec{p}}_-)|\vec{r}_+ - \vec{r}_-\rangle = -\hat{\vec{p}}\,|\vec{r}_+,\vec{r}_-\rangle \qquad \Rightarrow \quad \hat{C}\hat{\vec{p}}\hat{C} = -\hat{\vec{p}}$$

$$\hat{C}\hat{\vec{P}}\hat{C} = \hat{C}\,\tfrac{1}{2}(\hat{\vec{p}}_+ + \hat{\vec{p}}_-)\hat{C} = \tfrac{1}{2}(\hat{\vec{p}}_- + \hat{\vec{p}}_+) = \hat{\vec{P}} \qquad \Rightarrow \quad \hat{C}\hat{\vec{P}}\hat{C} = \hat{\vec{P}}$$

$$\Rightarrow \quad \hat{C}\hat{H}\hat{C} = \hat{C}\left[\frac{\hat{\vec{P}}^2}{2M} + \frac{\hat{\vec{p}}^2}{2\mu} - \frac{Ke^2}{|\hat{\vec{r}}|}\right]\hat{C} = \left[\frac{(\hat{\vec{P}})^2}{2M} + \frac{(-\hat{\vec{p}})^2}{2\mu} - \frac{Ke^2}{|-\hat{\vec{r}}|}\right]$$

$$= \left[\frac{\hat{\vec{P}}^2}{2M} + \frac{\hat{\vec{p}}^2}{2\mu} - \frac{Ke^2}{|\hat{\vec{r}}|}\right] = \hat{H}$$

$$\Rightarrow \quad [\hat{C},\hat{H}] = \hat{\emptyset} \qquad \checkmark$$

Since the lowest energy eigenstate is spherically symmetric with respect to $\vec{r}$ and $\hat{C}$ acts as the spatial parity operator with respect to $\vec{r}$, the lowest energy eigenstate is even in $\vec{r}$ and is an eigenstate of $\hat{C}$ with eigenvalue $+1$.

5. *Quantum Mechanics* (Fall 2004)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, \ldots$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0\,|\psi_0\rangle + c_1\,|\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|\,e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} = \frac{1}{\sqrt{2}}\,x_0 \qquad x_0 = \sqrt{\frac{\hbar}{2m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

a) $\langle\phi|H|\phi\rangle = |c_0|^2\langle\psi_0|H|\psi_0\rangle + |c_1|^2\langle\psi_1|H|\psi_1\rangle = \hbar\omega\left[|c_0|^2\tfrac{1}{2} + |c_1|^2\left(1+\tfrac{1}{2}\right)\right] = \hbar\omega$

$\Rightarrow \quad |c_0|^2 + 3|c_1|^2 = 2 \quad$ and $\quad \langle\phi|\phi\rangle = |c_0|^2 + |c_1|^2 = 1$

$$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \Rightarrow \quad |c_0| = |c_1| = \frac{1}{\sqrt{2}}$$

b) Let $c_0 = \frac{1}{\sqrt{2}} \qquad c_1 = \frac{1}{\sqrt{2}}e^{i\theta}$

$\langle\phi|x|\phi\rangle = x_0\langle\phi|(a^\dagger + a)|\phi\rangle = x_0\left(c_0^*\langle\psi_0| + c_1^*\langle\psi_1|\right)\left(c_0|\psi_1\rangle + c_1\cancel{\sqrt{2}|\psi_2\rangle} + c_1|\psi_0\rangle\right)$

$= x_0\left(c_0^* c_1 + c_1^* c_0\right) = x_0\, c_0\left(c_1 + c_1^*\right) = x_0\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(e^{i\theta} + e^{-i\theta}\right)$

$= x_0\cos\theta = \frac{1}{\sqrt{2}}x_0 \quad \Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta_1 = \pm\frac{\pi}{4}$

c) $|\psi(t)\rangle = e^{-iHt/\hbar}|\phi\rangle = c_0 e^{-i\omega t/2}|\psi_0\rangle + c_1 e^{-i3\omega t/2}|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|\psi_0\rangle + e^{-i(3\omega_0 t \mp \frac{\pi}{4})}|\psi_1\rangle\right)$

where $\omega_0 = \frac{\omega}{2}$

$\langle\psi(t)|x|\psi(t)\rangle = x_0\langle\psi(t)|(a^\dagger + a)|\psi(t)\rangle = x_0\frac{1}{2}\left(e^{i\omega_0 t} \quad e^{i(3\omega_0 t \mp \frac{\pi}{4})} \quad 0\right)\begin{pmatrix} e^{-i(3\omega_0 t \mp \frac{\pi}{4})} \\ e^{-i\omega_0 t} \\ \sqrt{2}\,e^{-i(3\omega_0 t \mp \frac{\pi}{4})} \end{pmatrix}$

$= \frac{1}{2}x_0\left[e^{-i(2\omega_0 t \mp \frac{\pi}{4})} + e^{i(2\omega_0 t \mp \frac{\pi}{4})}\right]$

$= x_0\cos\left[\omega t \mp \frac{\pi}{4}\right] = \sqrt{\frac{\hbar}{2m\omega}}\cos\left(\omega t \mp \frac{\pi}{4}\right) \qquad$ frequency ω

2. *Quantum Mechanics* (Spring 2006)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, ...$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

a. For the simple harmonic oscillator, $H|\psi_n\rangle = (n+\frac{1}{2})\hbar\omega$

$$\hbar\omega = \langle\phi|H|\phi\rangle = \left(\langle\psi_0|c_0^* + \langle\psi_1|c_1^*\right)H\left(c_0|\psi_0\rangle + c_1|\psi_1\rangle\right)$$

$$= |c_0|^2\langle\psi_0|H|\psi_0\rangle + |c_1|^2\langle\psi_1|H|\psi_1\rangle \quad \text{by orthogonality}$$

$$= |c_0|^2\left(\tfrac{1}{2}\hbar\omega\right) + |c_1|^2\left(\tfrac{3}{2}\hbar\omega\right)$$

$$\Rightarrow 1 = \tfrac{1}{2}|c_0|^2 + \tfrac{3}{2}|c_1|^2$$

Normalization of $|\phi\rangle$ implies $\langle\phi|\phi\rangle = 1 \Rightarrow |c_0|^2 + |c_1|^2 = 1$

$$\Rightarrow 1 = \tfrac{1}{2}|c_0|^2 + \tfrac{3}{2}(1 - |c_0|^2) = \tfrac{3}{2} - |c_0|^2$$

$$\Rightarrow |c_0|^2 = \tfrac{1}{2} \quad \text{and} \quad |c_1|^2 = \tfrac{1}{2}$$

Therefore $|c_0| = \frac{1}{\sqrt{2}}$ and $|c_1| = \frac{1}{\sqrt{2}}$

b. Recall $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$

$$\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} = \langle\phi|x|\phi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\psi_0|c_0^* c_1 a|\psi_1\rangle + \langle\psi_1|c_1^* c_0 a^\dagger|\psi_0\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[c_0^* c_1 + c_1^* c_0\right]$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2m\omega}}\left[c_1 + c_1^*\right] \quad \text{Since } c_0 \text{ real and pos} \Rightarrow c_0 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 1 = c_1 + c_1^* = \frac{1}{\sqrt{2}}e^{i\theta_1} + \frac{1}{\sqrt{2}}e^{-i\theta_1} = \frac{1}{\sqrt{2}}2\cos(\theta_1)$$

$$\Rightarrow \cos(\theta_1) = \frac{\sqrt{2}}{2} \Rightarrow \theta_1 = \pi/4$$

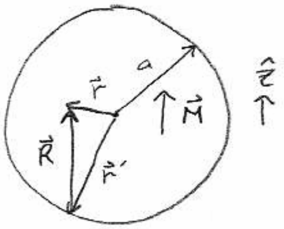
c. $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t/2}|\psi_0\rangle + \frac{1}{\sqrt{2}}e^{-3i\omega t/2 + i\pi/4}|\psi_1\rangle$

$\langle\psi(t)|x|\psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\frac{1}{2}e^{-i\omega t + i\pi/4} + \frac{1}{2}e^{i\omega t - i\pi/4}\right] = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \frac{\pi}{4})$

The angular frequency of oscillation is ω.

8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?



$$\vec{B} = \mu_0(\vec{H} + \vec{M})$$

$$\vec{\nabla}\times\vec{H} - \cancel{\partial_t \vec{D}}^{\,\to \vec{0}} = \vec{J}^f = \vec{0} \quad \Rightarrow \quad \vec{H} = -\vec{\nabla}\Phi_m$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} = \rho_m$$

$$\Rightarrow \quad \nabla^2 \Phi_m = -\rho_m = \vec{\nabla}\cdot\vec{M} \qquad \text{and} \qquad \Sigma_m = -\Delta M\cdot\hat{r} = M\cos\theta$$

"surface magnetic charge"

$$\Rightarrow \quad \Phi_m = \frac{1}{4\pi}\int_{R^3} \frac{\rho_m\, dv'}{R} \qquad \text{where } \rho_m = \rho_{m,\text{interior}} + \Sigma_m\,\delta(r-a)$$

$$\text{and } \rho_{m,\text{interior}} = -\vec{\nabla}\cdot\vec{M}_{\text{interior}} = 0$$

since $\vec{M}$ is constant in the interior

$$= \frac{1}{4\pi}\oint_S \frac{\Sigma_m\, a^2\, d\Omega'}{R} = \frac{Ma^2}{4\pi}\oint_S \frac{\cos\theta'\, d\Omega'}{R} \qquad \text{where } Y_{10}(\theta',\phi') = C\cos\theta'$$

$$= \frac{Ma^2}{4\pi}\oint_S d\Omega' \left[\frac{1}{C}Y_{10}(\theta',\phi')\right] \sum_{lm} \frac{4\pi}{2l+1}\frac{r_<^{\,l}}{r_>^{\,l+1}} Y_{lm}(\theta,\phi)\, Y^*_{lm}(\theta',\phi')$$

$$= \frac{Ma^2}{4\pi}\frac{1}{C}\sum_{lm}\frac{4\pi}{2l+1}\frac{r^l}{a^{l+1}} Y_{lm}(\theta,\phi)\,\delta_{1l}\,\delta_{0m}$$

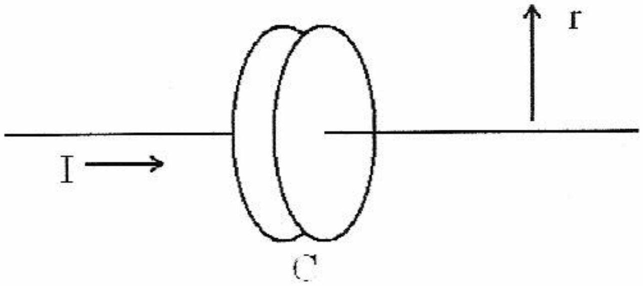
$$= \frac{M\cancel{a^2}}{\cancel{4\pi}}\,\frac{\cancel{4\pi}}{2+1}\,\frac{r^1}{\cancel{a^{1+1}}}\left(\frac{1}{C}Y_{10}(\theta,\phi)\right) = \frac{1}{3}Mr\cos\theta = \frac{1}{3}Mz$$

$$\vec{H} = -\vec{\nabla}\Phi_m = -\partial_z \Phi_m(z)\,\hat{z} = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$$

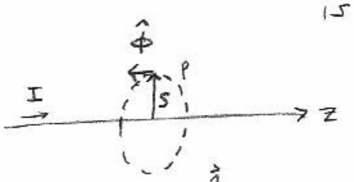
$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0\left(-\frac{1}{3}+1\right)\vec{M} = \frac{2}{3}\mu_0\vec{M}$$

9. *Electricity and Magnetism* (Fall 2004)

A wire carrying current I is connected to a circular capacitor of capacitance C, as depicted in the figure. What is the magnetic field outside the wire, far from the capacitor (as a function of the distance r from the wire)? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor. What is this magnetic field?



Far from wire: (the capacitor is negligible)



$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\cancel{\partial_t \vec{E}} = \mu_0 \vec{J}$$

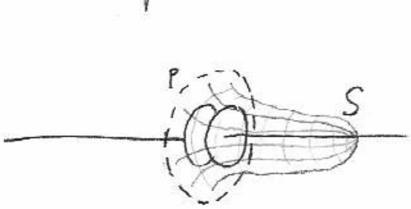
$$\Rightarrow \oint_P \vec{B}\cdot d\vec{l} = \int_S \mu_0 \vec{J}\cdot d\vec{a} = \mu_0 I = B_\phi 2\pi s \quad \text{by cylindrical symmetry}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \quad \text{or} \quad \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

Field outside capacitor:

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\partial_t\vec{E}$$ $\Rightarrow$ A changing electric field can also be seen as a source for the magnetic field. There is a changing electric field in the capacitor.

To solve for the field outside the capacitor, one may solve for the changing electric field in the capacitor or note that the surface of integration S may be manipulated to avoid the fields in the capacitor:



$$\oint_P \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \frac{1}{c^2}\partial_t \cancel{\int_S \vec{E}\cdot d\vec{a}} = \mu_0 I$$

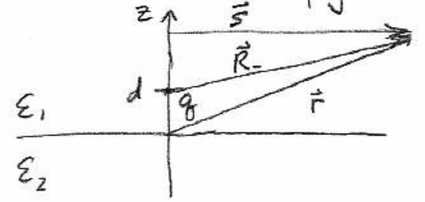
$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

(These issues are simple when one assumes no fringing effects.)

10. *Electricity and Magnetism* (Fall 2004)

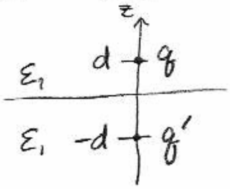
The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

cf. Jackson pg 254 (I cannot yet explain why this works...)

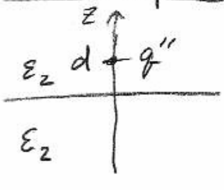


We use a method analogous to the method of images, using two different physical setups when solving for the potential in the...

...Upper space



...Lower space



$$\Phi(\vec{r}) = \Phi(s,z) = \begin{cases} \Phi_1(s,z) = \frac{1}{4\pi\varepsilon_1}\left[\frac{q}{R_-} + \frac{q'}{R_+}\right] \\ \Phi_2(s,z) = \frac{1}{4\pi\varepsilon_2}\frac{q''}{R_-} \end{cases}$$

where $R_\pm = \sqrt{s^2 + (z \pm d)^2}$

B.C.s: $(\Delta D)^\perp = \Sigma^f = 0 \qquad (\Delta \vec{E})^{\parallel} = \vec{0}$

$$\Rightarrow \quad \varepsilon_1 \vec{\nabla}\Phi_1 \cdot \hat{z}\Big|_{z=0} = \varepsilon_2 \vec{\nabla}\Phi_2 \cdot \hat{z}\Big|_{z=0} \quad \Rightarrow \quad \varepsilon_1 \partial_z \Phi_1\Big|_{z=0} = \varepsilon_2 \partial_z \Phi\Big|_{z=0}$$

$$\Rightarrow \varepsilon_1 \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_1}\left[q\,\partial_z\!\left(\frac{1}{R_-}\right) + q'\,\partial_z\!\left(\frac{1}{R_+}\right)\right]\Bigg|_{z=0} = \varepsilon_2 \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_2} q''\,\partial_z\!\left(\frac{1}{R_-}\right)\Bigg|_{z=0}$$

$$\text{and} \quad \partial_z\left(\frac{1}{R_\pm}\right)\Bigg|_{z=0} = \frac{1}{2}\left[s^2 + (z \pm d)^2\right]^{-1/2} 2(z \pm d)\Big|_{z=0} = \pm d\left[s^2 + d^2\right]^{-1/2}$$

$$\text{so} \quad \partial_z\left(\frac{1}{R_+}\right)\Bigg|_{z=0} = -\,\partial_z\left(\frac{1}{R_-}\right)\Bigg|_{z=0}$$

$$\Rightarrow \quad q - q' = q'' \quad ①$$

$$\text{and} \quad (\vec{\nabla}\Phi_1)^{s\phi}\Big|_{z=0} = (\vec{\nabla}\Phi_2)^{s\phi}\Big|_{z=0} \Rightarrow (\text{no } \phi\text{-dependence}) \quad \partial_s \Phi_1\Big|_{z=0} = \partial_s \Phi_2\Big|_{z=0}$$

$$\Rightarrow \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_1}\left[q\,\partial_s\!\left(\frac{1}{R_-}\right) + q'\,\partial_s\!\left(\frac{1}{R_+}\right)\right]\Bigg|_{z=0} = \frac{\cancel{1}}{\cancel{4\pi}\varepsilon_2} q''\,\partial_s\!\left(\frac{1}{R_-}\right)\Bigg|_{z=0}$$

$$\text{and} \quad \partial_s\left(\frac{1}{R_\pm}\right)\Bigg|_{z=0} = \frac{1}{2}\left[s^2 + (z \pm d)^2\right]^{-1/2} 2s \qquad \text{so} \quad \partial_s\left(\frac{1}{R_+}\right)\Bigg|_{z=0} = \partial_s\left(\frac{1}{R_-}\right)\Bigg|_{z=0}$$

$$\Rightarrow \quad q + q' = \frac{\varepsilon_1}{\varepsilon_2} q'' \quad ②$$

$$\text{Thus} \quad ① + ② \Rightarrow 2q = (1 + \varepsilon_1/\varepsilon_2)q'' \Rightarrow q'' = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} q \qquad \bar{\varepsilon} \equiv \frac{\varepsilon_1 + \varepsilon_2}{2}$$

$$② - \frac{\varepsilon_1}{\varepsilon_2}① \Rightarrow \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right)q + \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\right)q' = 0 \qquad \Rightarrow q' = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} q$$

$$\Rightarrow \Phi_1(s,z) = \frac{1}{4\pi\varepsilon_1}\frac{q}{\sqrt{s^2 + (z-d)^2}} + \frac{1}{4\pi\varepsilon_1}\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\frac{q}{\sqrt{s^2 + (z+d)^2}} \qquad \Phi_2(s,z) = \frac{1}{4\pi\bar{\varepsilon}}\frac{q}{\sqrt{s^2 + (z-d)^2}}$$

11. *Electricity and Magnetism* (Fall 2004)

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $\mathbf{r}(t)$. You do not need to find dimensionless proportionality constants (i.e., only find the dependence on q, $\mathbf{r}(t)$, and universal constants).

Let $a \equiv |\ddot{\vec{r}}|$

Assume the radiation electric field is proportional to a and to $\frac{1}{R}$ where R is the distance from the particle to the observation point:

$E_a = A \frac{1}{4\pi\varepsilon_0} \frac{q}{R} a$ where A is a constant of unknown dimensions

$$[\vec{E}_a] = \left[A \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{R^2}\right) R a\right] = [\vec{E}]\,[A R a] \Rightarrow [A] = \left[\frac{1}{Ra}\right] = \frac{s^2}{m^2} = \left[\frac{1}{c^2}\right]$$

$$\Rightarrow \quad E_a \propto \frac{1}{\varepsilon_0} \frac{q}{c^2} \frac{a}{R}$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2} \mathrm{Re}\left[\vec{S}_a \cdot R^2 \hat{n}\right] \qquad \vec{S}_a = \frac{1}{\mu_0} \vec{E}_a \times \vec{B}_a = \frac{1}{\mu_0 c} |\vec{E}_a|^2 \hat{k}$$

since $E_a = cB_a$ and $\hat{E}_a \times \hat{B}_a = \hat{k}$, the radiation propagation direction

Far away, $\hat{k} = \hat{n}$, and Ω is dimensionless

$$\Rightarrow \quad P \propto |\vec{S}_a| R^2 = \frac{1}{\mu_0 c} E_a^2 R^2 = \frac{1}{\mu_0 c} \frac{q^2 a^2}{\varepsilon_0^2 c^4 R^2} R^2$$

$$= \frac{c^2}{c} \frac{q^2 a^2}{\varepsilon_0 c^4} \qquad \text{since } \frac{1}{\mu_0 \varepsilon_0} = c^2$$

$$= \frac{1}{\varepsilon_0} \frac{q^2}{c^3} a^2$$

(The Larmor formula proportionality. ✓)

12. *Electricity and Magnetism* (Fall 2004)

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy E_{min} of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

a) $\xrightarrow{e^-} \; \xleftarrow{e^+}$

$\gamma \cdot$ ✗

$X \cdot$ ✓

In the center-of-mass frame the total momentum is zero, so the resultant particle must have zero momentum. Any massless particles, including photons, always have nonzero instantaneous momentum, so a single photon cannot be the product. (The total momentum of multiple massless particles could add to zero, though.) A massive particle such as X can have zero momentum and so can be a product of this annihilation.

b)

(Lab frame)	Before	After	
Collider	$\xrightarrow[E_c]{e^+} \quad \xleftarrow[E_c]{e^-}$	X	E_c is the minimum energy to produce one X particle (so it is not excited)
Fixed	$\xrightarrow[E_f \ \vec{p}]{e^+} \quad e^-$	$\xrightarrow[\vec{p}]{X}$	E_f is also the minimum (X not excited)

$$E_c + E_c = Mc^2 \quad \Rightarrow \quad E_c = \tfrac{1}{2}Mc^2$$

$$E_f + mc^2 = \sqrt{p^2c^2 + M^2c^4}$$

$$E_f^2 = p^2c^2 + m^2c^4 \quad \Rightarrow \quad p^2c^2 = E_f^2 - m^2c^4$$

$$(E_f + mc^2)^2 = p^2c^2 + M^2c^4 = (E_f^2 - m^2c^4) + M^2c^4$$

$$\Rightarrow \quad \cancel{E_f^2} + 2E_f mc^2 + m^2c^4 = E_f^2 - m^2c^4 + M^2c^4$$

$$\Rightarrow \quad E_f = \frac{1}{2mc^2}\left[M^2c^4 - 2m^2c^4\right] = \frac{1}{2}\left(\frac{M^2}{m} - 2m\right)c^2 = \frac{1}{2}\left(\frac{M}{m} - 2\frac{m}{M}\right)Mc^2$$

$$\frac{E_f}{E_c} = \frac{M}{m} - 2\frac{m}{M} = \frac{M}{m}\left(1 - 2\left(\frac{m}{M}\right)^2\right) \approx \frac{M}{m} \gg 1 \quad \text{since } M \gg m \ \left(\text{and } \frac{m}{M} \ll 1\right)$$

$$\therefore \ E_f \gg E_c \ ✓$$

13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2}rM^2 + uM^4 - hM$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^\delta$. What is δ?

(units $\Rightarrow$ specific free energy) $f(T) = F(T, M_{min})$

(h = ext. B-field, not H-field)

a) $h = 0$ $\quad F(T,M) = \frac{1}{2}rM^2 + uM^4 \qquad r = a(T-T_c)$

$F' = rM + 4uM^3 = M(r + 4uM^2) = 0 \quad \Rightarrow \quad M = 0, \pm\sqrt{-\frac{r}{4u}} \equiv \pm A$

$F'' = r + 12uM^2$

$F''(0) = r \qquad F''(\pm A) = r + 12u\left(-\frac{r}{4u}\right) = -2r$

minimize $F \Rightarrow F' = 0,\ F'' > 0$

T	a	u	r	$M=0$	$M=\pm A$	F''	M_{min}	Conclusion
$T > T_c$	$\boxed{a > 0}$	$\boxed{u > 0}$	+	min	complex	+	0	correct *
		$u < 0$	+	min	max.s	$\pm$	$0, \pm\infty$	(probably not)
	$a < 0$	$u > 0$	−	max	min.s	$\pm$	$\pm A$	
		$u < 0$	−	max	complex	−	$\pm\infty$	(not physical)
$T < T_c$	$\boxed{a > 0}$	$\boxed{u > 0}$	−	max	min.s	$\pm$	$\pm A$	
		$u < 0$	−	max	complex	−	$\pm\infty$	X
	$a < 0$	$u > 0$	+	min	complex	+	0	
		$u < 0$	+	min	max.s	$\pm$	$0, \pm\infty$	X

$*\ (T > T_c \Rightarrow M = 0)$

Assuming $a > 0$ and $u > 0$ (which makes the most sense above)

$M = 0$ minimizes $F(T,M)$ for $T > T_c$ and

$M = \pm A = \pm\sqrt{-\frac{r}{4u}} = \pm\frac{1}{2}\sqrt{\frac{a}{u}|T - T_c|}$ minimizes F for $T < T_c$.

13. Statistical Mechanics and Thermodynamics (Fall 2004)

b) $h=0$

$$C = \frac{dQ}{dT} = T\frac{ds}{dT}$$

$$h=0 \Rightarrow \quad du = T\,ds \qquad f = u - Ts \qquad df = -s\,dT \Rightarrow -s = \frac{df}{dT}$$

$$u = u(s) \qquad\qquad f = f(T)$$

$$c = T\frac{ds}{dT} = T\frac{d}{dT}\left(-\frac{df}{dT}\right) = -T\frac{d^2f}{dT^2}$$

$$f(T) = F(T, M_{min}) = \begin{cases} 0 & \text{if } T > T_c \\ \frac{1}{2}a(T-T_c)\frac{1}{4}\frac{a}{u}|T-T_c| + u\frac{1}{16}\frac{a^2}{u^2}|T-T_c|^2 \end{cases}$$

$$= -\frac{1}{8}\frac{a^2}{u}|T-T_c|^2 + \frac{1}{16}\frac{a^2}{u}|T-T_c|^2$$

$$= -\frac{1}{16}\frac{a^2}{u}|T-T_c|^2 \quad \text{if } T < T_c$$

$$\Rightarrow c = -T\frac{d^2f}{dT^2} = -T\frac{d}{dT}\left[-\frac{1}{8}\frac{a^2}{u}|T-T_c|\right] = -T\left[-\frac{1}{8}\frac{a^2}{u}\right] = \frac{1}{8}\frac{a^2}{u}T$$

$$= \frac{1}{8}\frac{a^2}{u}(T-T_c) + \frac{1}{8}\frac{a^2}{u}T_c = -\frac{1}{8}\frac{a^2}{u}|T-T_c| + \frac{1}{8}\frac{a^2}{u}T_c$$

for $T < T_c$ $\quad$ ($c = 0$ for $T \to T_c$ from above)

As $T \to T_c$ from below $|T-T_c| \ll 1$, so the term $\frac{1}{8}\frac{a^2}{u}T_c$ dominates.

$$\Rightarrow \quad c \sim |T-T_c|^0 \quad \Rightarrow \quad \alpha = 0$$

c) $T = T_c \Rightarrow r = 0$

$$F(T,M) = uM^4 - hM$$

$$F' = 4uM^3 - h = 0 \quad \Rightarrow \quad M = \left(\frac{h}{4u}\right)^{1/3}$$

$$F'' = 12uM^2 > 0$$

$$\therefore \quad M \sim h^{1/3}$$

$$\delta = \frac{1}{3}$$

1. *Quantum Mechanics* (Fall 2004)

Two spin-half particles are in a state with total spin zero. Let $\hat{\mathbf{n}}_a$ and $\hat{\mathbf{n}}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{\mathbf{n}}_a$ and the spin of the second along $\hat{\mathbf{n}}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\mathbf{s}_a\cdot\hat{\mathbf{n}}_a\,\mathbf{s}_b\cdot\hat{\mathbf{n}}_b|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

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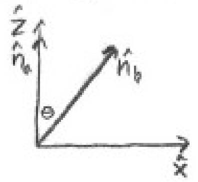
Let the z-axis lie in the direction of $\hat{n}_a$ and the x-axis in the direction of $\hat{n}_b$. Then



$$\vec{S}_a\cdot\hat{n}_a = S_{az} \quad\text{and}\quad \vec{S}_b\cdot\hat{n}_b = S_{bx}n_{bx} + S_{bz}\cos(\theta)$$

where θ is the angle between $\hat{n}_a$ and $\hat{n}_b$.

The spin-singlet state is $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right)$

$$\langle\Psi|\vec{S}_a\cdot\hat{n}_a\vec{S}_b\cdot\hat{n}_b|\Psi\rangle = \langle\Psi|S_{az}\left(S_{bx}n_{bx}+S_{bz}\cos(\theta)\right)|\Psi\rangle$$
$$= \langle\Psi|S_{az}S_{bx}|\Psi\rangle n_{bx} + \langle\Psi|S_{az}S_{bz}|\Psi\rangle\cos(\theta)$$

$$\langle\Psi|S_{az}S_{bx}|\Psi\rangle = \frac{1}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}S_{bx}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]$$
$$= \frac{1}{2}\frac{\hbar}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| + \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{bx}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]$$
$$= \frac{\hbar}{4}\left[(0\;1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - (0\;1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (1\;0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - (1\;0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$
$$= \frac{\hbar}{4}\left[0 - 1 + 1 - 0\right] = 0$$

$$\langle\Psi|S_{az}S_{bz}|\Psi\rangle = \frac{1}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}S_{bz}\left[|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]$$
$$= \frac{1}{2}\frac{\hbar}{2}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]S_{az}\left[-|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]$$
$$= \frac{\hbar^2}{8}\left[\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}| - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|\right]\left[-|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle + |-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right]$$
$$= \frac{\hbar^2}{8}\left[-\langle\tfrac{\hbar}{2},-\tfrac{\hbar}{2}|\tfrac{\hbar}{2},-\tfrac{\hbar}{2}\rangle - \langle-\tfrac{\hbar}{2},\tfrac{\hbar}{2}|-\tfrac{\hbar}{2},\tfrac{\hbar}{2}\rangle\right] = -\frac{\hbar^2}{4}$$

Therefore $\langle\Psi|\vec{S}_a\cdot\hat{n}_a\vec{S}_b\cdot\hat{n}_b|\Psi\rangle = -\frac{\hbar^2}{4}\cos(\theta)$

3. *Quantum Mechanics* (Fall 2004)

A positron has the same mass m as the electron, but the opposite charge. Consider a set of states containing one electron and one positron. A complete set of these states can be labeled $|\mathbf{r}_+, \mathbf{r}_-\rangle$, where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the positions of the positron and electron, respectively. Normalize these states so that

$$\langle \mathbf{r}_+, \mathbf{r}_- | \mathbf{r}'_+, \mathbf{r}'_- \rangle = \delta_3(\mathbf{r}'_+ - \mathbf{r}_+)\, \delta_3(\mathbf{r}'_- - \mathbf{r}_-)$$

Then if the system is in any state $|\psi\rangle$, the wave function is

$$\psi(\mathbf{r}_+, \mathbf{r}_-) = \langle \mathbf{r}_+, \mathbf{r}_- | \psi \rangle$$

In this problem ignore spin.

(a) In terms of $\psi(\mathbf{r}_+, \mathbf{r}_-)$, what is the probability that at least one of the two particles is farther than a distance b from the origin?

(b) Write down the Hamiltonian for this electron-positron system, including the electrostatic (Coulomb) interactions between the two particles.

(c) Let $\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_-$ and $\mathbf{R} = \frac{1}{2}(\mathbf{r}_+ + \mathbf{r}_-)$. Write the Hamiltonian in terms of the new coordinates and their canonically conjugate momenta $\mathbf{p}$ and $\mathbf{P}$.

(d) The bound electron-positron system is called *positronium*. For states with zero total momentum, write a formula for the possible negative values of the energy[1]. What is the approximate numerical value, in electron volts, of the ground state energy?

(e) Define the *charge conjugation* operator C on this system by

$$C\,|\mathbf{r}_+, \mathbf{r}_-\rangle = |\mathbf{r}_-, \mathbf{r}_+\rangle$$

Show that C commutes with the Hamiltonian. What is the eigenvalue of C on the state of lowest energy?

a. $P = 1 - \int_0^{2\pi}\int_0^{\pi}\int_0^{b}\left[\int_0^{2\pi}\int_0^{\pi}\int_0^{b} |\psi(\vec{r}_+, \vec{r}_-)|^2\, r_+^2 \sin(\theta_+)\,dr_+\,d\theta_+\,d\phi_+\right] r_-^2 \sin(\theta_-)\,dr_-\,d\theta_-\,d\phi_-$

b. $H = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}_+ - \vec{r}_-|}$

c. $T = \frac{p_+^2}{2m} + \frac{p_-^2}{2m} = \frac{(m\dot{\vec{r}}_+)^2}{2m} + \frac{(m\dot{\vec{r}}_-)^2}{2m} = \frac{1}{2}m\dot{\vec{r}}_+^{\,2} + \frac{1}{2}m\dot{\vec{r}}_-^{\,2} = \frac{1}{2}m\left(2\dot{\vec{R}}^2 + \frac{1}{2}\dot{\vec{r}}^2\right)$

$V = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}_+ - \vec{r}_-|} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}|}$

$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial}{\partial \dot{\vec{r}}}(T - V) = \frac{1}{2}m\dot{\vec{r}}$ $\qquad \vec{P} = \frac{\partial L}{\partial \dot{\vec{R}}} = 2m\dot{\vec{R}}$

$H = T + V = m\left(\frac{P}{2m}\right)^2 + \frac{1}{4}m\left(\frac{2p}{m}\right)^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}|} = \frac{P^2}{4m} + \frac{p^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}|}$

d. Total Momentum zero $\Rightarrow P = 0 \Rightarrow H = \frac{p^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{|\vec{r}|}$

Which is Hydrogen atom with $m \to \frac{m}{2}$

$\Rightarrow E_1 = -\frac{mk^2e^4}{4n^2\hbar^2}\Big|_{n=1} = -\frac{mk^2e^4}{4\hbar^2} = \frac{1}{2}(-13.6\,\text{eV}) = -6.8\,\text{eV}$

e. $C\vec{r}\,|\vec{r}_+{}', \vec{r}_-{}'\rangle = C(\vec{r}_+{}' - \vec{r}_-{}')\,|\vec{r}_+{}', \vec{r}_-{}'\rangle = (\vec{r}_+{}' - \vec{r}_-{}')\,|\vec{r}_-{}', \vec{r}_+{}'\rangle$

$\vec{r}C\,|\vec{r}_+, \vec{r}_-{}'\rangle = \vec{r}\,|\vec{r}_-{}', \vec{r}_+{}'\rangle = (\vec{r}_-{}' - \vec{r}_+{}')\,|\vec{r}_-{}', \vec{r}_+{}'\rangle \Rightarrow C\vec{r} = -\vec{r}C$

True for all states since its true for an entire basis.

Similary $C\vec{R} = \vec{R}C$, so $C\vec{p} = C\left(\frac{2\dot{\vec{r}}}{m}\right) = -\left(\frac{2\dot{\vec{r}}}{m}\right)C = -\vec{p}C$, $C\vec{P} = C(2m\dot{\vec{R}}) = (2m\dot{\vec{R}})C = \vec{P}C$

So $CH = \left[\frac{P^2}{4m} + \frac{(-p)^2}{m} - \frac{1}{4\pi\epsilon_0}\frac{q^2}{|-\vec{r}|}\right]C = HC \Rightarrow [H, C] = 0 \Rightarrow C$ eigenstates are H eigenstates

Like Hydrogen, lowest state is spherically symmetric $\Rightarrow C|\psi(\vec{r})\rangle = |\psi(-\vec{r})\rangle = |\psi(\vec{r})\rangle \Rightarrow \lambda = 1$

4. *Quantum Mechanics* (Fall 2004)

Let H be the Hamiltonian for the hydrogen atom, including spin. $\hbar\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\hbar\mathbf{s}$ are the orbital and spin angular momentum, respectively, and $\mathbf{J} = \mathbf{L} + \mathbf{s}$. Conventionally, the states are labeled $|n, l, j, m\rangle$ and they are eigenstates of H, $\mathbf{L}^2$, $\mathbf{J}^2$, and J_z.

(a) If the electron is in the state $|n, l, j, m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

(b) What are the restrictions on the possible values of n, l, j, and m?

(c) Let $\mathbf{J}_\pm = J_x \pm iJ_y$. What are   Recall $J_\pm |n \ell j m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|n \ell j\, m\pm 1\rangle$

(i) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_+|3, 1, \frac{3}{2}, -\frac{1}{2}\rangle = ?$ $\sqrt{\frac{15}{4} + \frac{1}{4}}\,\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{1}{2}\rangle = 0$ by orthogonality

(ii) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_+|3, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ $\sqrt{\frac{15}{4} - \frac{3}{4}}\,\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{3}{2}\rangle = \sqrt{\frac{12}{4}} = \sqrt{3}$

(iii) $\langle 2, 1, \frac{3}{2}, \frac{3}{2}|p_z|2, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ See below

(iv) $\langle 2, 1, \frac{1}{2}, -\frac{1}{2}|\mathbf{L}^2|2, 1, \frac{1}{2}, -\frac{1}{2}\rangle = ?$ $1(1+1) = 2$

(v) $\langle 3, 2, \frac{3}{2}, -\frac{1}{2}|\mathbf{J}^2|3, 2, \frac{3}{2}, -\frac{1}{2}\rangle = ?$ $\frac{3}{2}(\frac{3}{2}+1) = \frac{15}{4}$

(vi) $\langle 3, 1, \frac{3}{2}, \frac{3}{2}|J_z|3, 1, \frac{3}{2}, \frac{1}{2}\rangle = ?$ $\frac{1}{2}\langle 3,1,\frac{3}{2},\frac{3}{2}|3,1,\frac{3}{2},\frac{1}{2}\rangle = 0$ by orthogonality

(d) What is $\langle 1, 0, \frac{1}{2}, \frac{1}{2}|p_i p_j|1, 0, \frac{1}{2}, \frac{1}{2}\rangle = ?$

(e) For given n, l, j, and m, what are the conditions on n', l', j', and m' so that

$$\langle n', l', j', m'|\mathbf{s} \cdot \mathbf{r}|n, l, j, m\rangle \neq 0?$$

a. $H|n\ell jm\rangle = (-\frac{\alpha^2}{2n^2}mc^2)|n\ell jm\rangle$ $L^2|n\ell jm\rangle = \ell(\ell+1)|n\ell jm\rangle$

$J^2|n\ell jm\rangle = j(j+1)|n\ell jm\rangle$ $J_z|n\ell jm\rangle = m|n\ell jm\rangle$

b. $n \in \{1, 2, 3, \dots\}$ $\ell \in \{0, 1, 2, \dots, n-1\}$ $j \in \{\ell - \frac{1}{2}, \ell + \frac{1}{2}\}$

$m \in \{-j, \dots, 0, \dots, j\}$

c. See above, (iii) $[L_z, p_z] = [xp_y - yp_x, p_z] = 0$

$\Rightarrow \langle 2,1,\frac{3}{2},\frac{3}{2}|L_z p_z - p_z L_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

$\Rightarrow (\frac{3}{2} - \frac{1}{2})\langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

$\Rightarrow \langle 2,1,\frac{3}{2},\frac{3}{2}|p_z|2,1,\frac{3}{2},\frac{1}{2}\rangle = 0$

d. If $i \neq j$, then $p_i p_j$ is a cartesian componet of a rank two tensor, which is always some linear combination of rank two spherical tensors, so by the Wigner-Eckart theorem it is zero because $|j-2| \leq j' \leq j+2$ is not satisfied by $j = j' = \frac{1}{2}$.

If $i = j$ $\langle p_i^2\rangle = \langle p_x^2\rangle = \langle p_y^2\rangle = \langle p_z^2\rangle$ by symmetry

$= \frac{1}{3}(\langle p_x^2\rangle + \langle p_y^2\rangle + \langle p_z^2\rangle) = \frac{1}{3}\langle p^2\rangle = \frac{1}{3}2m\langle\frac{p^2}{2m}\rangle = \frac{2m}{3}\langle T\rangle$

$= -\frac{2m}{3}\langle E\rangle$ by the virial theorem

Therefore $\langle 1,0,\frac{1}{2},\frac{1}{2}|p_i p_j|1,0,\frac{1}{2},\frac{1}{2}\rangle = -\frac{2m}{3}E_1\delta_{ij}$ where $E_1 = -\frac{mk^2e^4}{2\hbar^2}$

e. $\vec{s}\cdot\vec{r}$ is a scalar, so it is the 0^{th} spherical component of a rank 0 tensor

$\Rightarrow \ell' = \ell$ and $j' = j$ and $m' = m$, but n' and n can be anything, according to the Wigner-Eckart theorem selection rules.

7. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Some organic molecules have a triplet excited state at energy $k_B\Delta$ above a singlet ground state.

(a) Find an expression for the magnetic moment in a field B in terms of Δ, B, the temperature T, the Bohr magneton μ_B, and the gyromagnetic ratio g.

(b) Show that the susceptibility for $T \gg \Delta$ is given by $N(g\mu_B)^2/2k_BT$, where N is the total number of molecules in the system.

(c) With the help of a diagram of energy levels versus field and a rough sketch of entropy versus field, explain how this system might be cooled by adiabatic magnetization (*not demagnetization*).

a. $\vec{\mu} = g\mu_B \vec{S}$ where $\hbar\vec{S}$ is the spin angular momentum

The singlet state has $|\vec{S}|=0$ and the triplet state has $|\vec{S}|=1$.

But we want to find $\langle \mu_z \rangle$ where $\hat{z}$ is the direction of the applied magnetic field. So $\mu_z = g\mu_B S_z$

$$(\mu_S)_z = 0 \qquad (\mu_T^1)_z = g\mu_B \qquad (\mu_T^0)_z = 0 \qquad (\mu_T^{-1})_z = -g\mu_B$$

The energy of a magnetic moment in a field is $U = -\vec{\mu}\cdot\vec{B} = -\mu_z B$

$$\epsilon_S = 0 \qquad \epsilon_T^1 = K\Delta - g\mu_B B \qquad \epsilon_T^0 = K\Delta \qquad \epsilon_T^{-1} = K\Delta + g\mu_B B$$

$$\langle \mu_z \rangle = \frac{(\mu_S)_z e^{-\beta\epsilon_S} + (\mu_T^1)_z e^{-\beta\epsilon_T^1} + (\mu_T^0) e^{-\beta\epsilon_T^0} + (\mu_T^{-1}) e^{-\beta\epsilon_T^{-1}}}{e^{-\beta\epsilon_S} + e^{-\beta\epsilon_T^1} + e^{-\beta\epsilon_T^0} + e^{-\beta\epsilon_T^{-1}}}$$

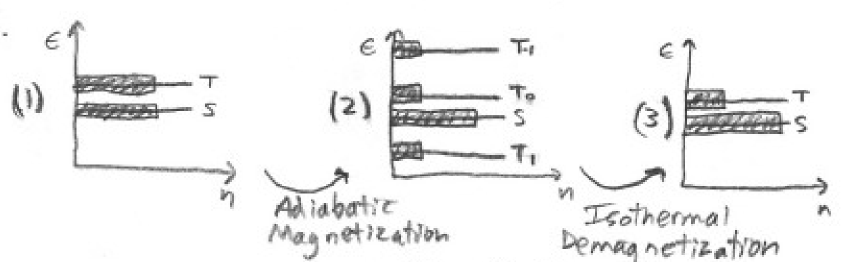
$$= g\mu_B \frac{e^{-\beta(K\Delta - g\mu_B B)} - e^{-\beta(K\Delta + g\mu_B B)}}{1 + e^{-\beta(K\Delta - g\mu_B B)} + e^{-\Delta/T} + e^{-\beta(K\Delta + g\mu_B B)}}$$

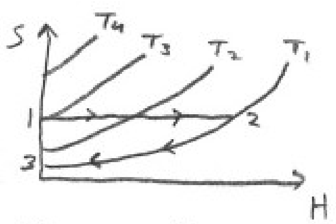
$$M = N\langle \mu_z \rangle = Ng\mu_B \frac{e^{\beta g\mu_B B} - e^{-\beta g\mu_B B}}{e^{\Delta/T} + e^{\beta g\mu_B B} + 1 + e^{-\beta g\mu_B B}}$$

b. $T \gg \Delta \Rightarrow$ $M \cong Ng\mu_B \left(\dfrac{1 + \beta g\mu_B B - 1 + \beta g\mu_B B}{1 + 1 + 1 + 1} \right) = \dfrac{N(g\mu_B)^2}{2KT} B$

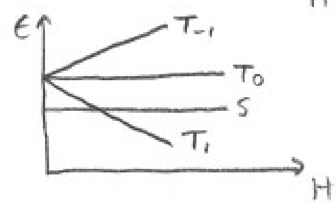
$$M = \chi B \Rightarrow \chi = \frac{N(g\mu_B)^2}{2KT}$$

c.





All the particles in the triplet state are split among the three levels when the field is applied adiabatically and settle down preferentially into the singlet state when the field is removed isothermally.



8. *Electricity and Magnetism* (Fall 2004)

Consider a sphere of radius a with uniform magnetization $\mathbf{M}$, pointing in the z-direction. What are the magnetic induction $\mathbf{B}$ and magnetic field $\mathbf{H}$ inside the sphere?

See Jackson Page 198.

$\vec{\nabla}\times\vec{H} = \rho_f = 0 \Rightarrow \vec{H}$ is curl free $\Rightarrow \vec{H} = -\vec{\nabla}\Phi_M$ for some scalar field Φ_M

$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{\nabla}\Phi_M = -\nabla^2\Phi_M$ and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot(\frac{1}{\mu_0}\vec{B}-\vec{M}) = -\vec{\nabla}\cdot\vec{M} \Rightarrow \nabla^2\Phi_M = \vec{\nabla}\cdot\vec{M}$

We know the solution to poisson's equation $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$ is $\Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho}{|\vec{x}-\vec{x}'|}d^3x'$ where V is any volume that encloses all $\vec{x}$ such that $\rho(\vec{x}) \neq 0$.

Therefore $\Phi_M = -\frac{1}{4\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$ and lets take V as all space.

Let V_0 be the interior of the sphere and let V_0' be the complement of V_0 (V_0' is a closed set containg the boundary of the sphere).

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0}\cancel{\frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}}d^3x' - \frac{1}{4\pi}\int_{V_0'}\frac{\vec{\nabla}'\cdot\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x'$$

The first term is zero because $\vec{M}$ is constant in the interior.

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0'}\left[\vec{\nabla}'\cdot\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) - \vec{M}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\right]d^3x'$$

using the product rule $\vec{\nabla}\cdot(f\vec{A}) = f(\vec{\nabla}\cdot\vec{A}) + \vec{A}\cdot\vec{\nabla}f$. The second term has only an infinitesimal contribution to the result because $\vec{M}$ is finite and is not nonzero over any finite subspace of V_0'.

$$\Phi_M = -\frac{1}{4\pi}\int_{V_0'}\vec{\nabla}'\cdot\left(\frac{\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)d^3x' = -\frac{1}{4\pi}\int_S \frac{\vec{M}(\vec{x}')\cdot(-\hat{n}')}{|\vec{x}-\vec{x}'|}da'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{M\cos(\theta)}{|\vec{x}-\vec{x}'|}a^2\sin(\theta)d\theta d\phi$$

$$= \frac{Ma^2}{4\pi}\int\frac{\cos(\theta)}{|\vec{x}-\vec{x}'|}d\Omega'$$

Now we use the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)$

$$\Phi_M = \frac{Ma^2}{4\pi}\int 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)\cos(\theta')d\Omega'$$

$$= Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\int Y_{\ell m}^*(\theta',\phi')\left(\sqrt{\tfrac{4\pi}{3}}Y_{10}(\theta',\phi')\right)d\Omega'$$

since $Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$. Now $\int Y_{\ell' m'}^*(\theta,\phi)Y_{\ell m}(\theta,\phi)d\Omega = \delta_{\ell\ell'}\delta_{mm'}$

$$\Phi_M = Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}}\delta_{\ell 1}\delta_{m0}$$

$$= Ma^2\left(\frac{1}{3}\frac{r_<}{r_>^2}\right)Y_{10}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}} = \frac{1}{3}Ma^2\frac{r_<}{r_>^2}\cos(\theta)$$

And $r_>, r_<$ are the greater and lesser between r and a, so inside $r<a$:

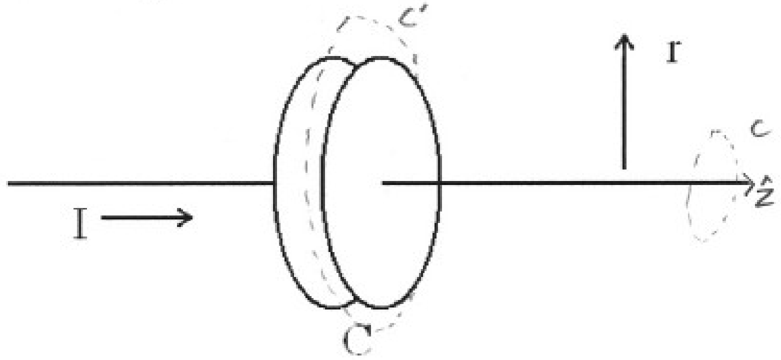
$$\Phi_m = \frac{1}{3}Ma^2\frac{r}{a^2}\cos(\theta) = \frac{1}{3}Mr\cos(\theta) = \frac{1}{3}Mz$$

Therefore $\vec{H} = -\vec{\nabla}\Phi_M = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$

and $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M}$

9. *Electricity and Magnetism* (Fall 2004)

A wire carrying current I is connected to a circular capacitor of capacitance C, as depicted in the figure. What is the magnetic field outside the wire, far from the capacitor (as a function of the distance r from the wire)? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor. What is this magnetic field?



Far from the capacitor it looks like a regular current carrying wire, so using Ampere's Law, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\Rightarrow \int_S(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\int_S\vec{J}\cdot d\vec{a} + \mu_0\epsilon_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$$

$$\Rightarrow \int_c \vec{B}\cdot d\vec{l} = \mu_0 I + \cancel{\mu_0\epsilon_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}} \quad \text{since } \vec{E}=0$$

$$\Rightarrow 2\pi r B = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

Outside the capacitor, we can get the field on C' by integrating over a surface that balloons out around the plates to intersect the wire and we get the same answer. If we choose to use the minimal surface spanning C', then

$$\int_c \vec{B}\cdot d\vec{l} = \cancel{\mu_0 I} + \mu_0\epsilon_0\int_S\frac{\partial\vec{E}}{\partial t}\cdot d\vec{a} \quad \text{since } I=0$$

$$2\pi r B = \mu_0\cancel{\epsilon_0}\int_S\frac{\partial}{\partial t}\left(\frac{\sigma}{\cancel{\epsilon_0}}\hat{z}\right)\cdot d\vec{a}$$

$$2\pi r B = \mu_0\frac{\partial}{\partial t}\left(\frac{Q}{A}\right)\int_S da = \mu_0 I$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r} \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

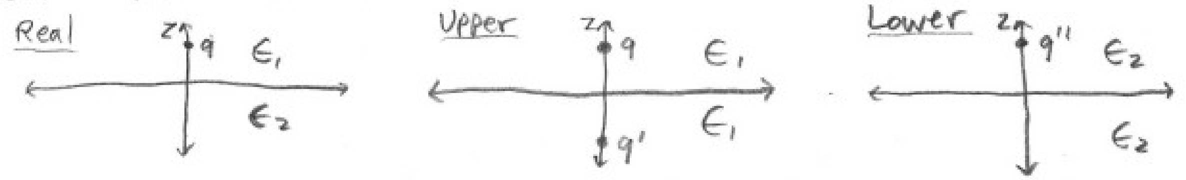
The field has the same expression outside the capacitor because the changing electric field creates a displacement current.

10. *Electricity and Magnetism* (Fall 2004)

The upper half-space is filled with a material of permittivity ϵ_1, while the lower half space is filled with a different material with permittivity ϵ_2. Their interface is located at the $z = 0$ plane. A point charge q is located at $\mathbf{r}_q = d\hat{\mathbf{z}}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

See Jackson Page 154

For this problem you have to know the trick that you can satisfy the Laplace/Poisson equation and the boundary conditions for the upper half space by replacing the ϵ_2 medium with an ϵ_1 medium and a point charge in the image location, and for the lower half space by replacing the ϵ_1 medium with an ϵ_2 medium and a point charge on top of q.



Note: This is not a direct consequence of the method of images.

So $$\Phi = \begin{cases} \Phi_1 \equiv \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) & z>0 \\ \Phi_2 \equiv \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{R_1}\right) & z<0 \end{cases}$$ where $R_1 = \sqrt{r^2+(z-d)^2}$, $R_2 = \sqrt{r^2+(z+d)^2}$

Subject to the boundary conditions $\vec{E}_{\parallel}^2 - \vec{E}_{\parallel}^1 = 0$, $\vec{D}_{\perp}^2 - \vec{D}_{\perp}^1 = \sigma_f = 0$

which we can express as $$\lim_{z\to 0^+}\begin{Bmatrix} E_x \\ E_y \\ \epsilon_1 E_z \end{Bmatrix} = \lim_{z\to 0^-}\begin{Bmatrix} E_x \\ E_y \\ \epsilon_2 E_z \end{Bmatrix}$$

To facilitate the calculation we can observe that
$$\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\Big|_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\Big|_{z=0} \equiv A$$
$$\frac{\partial}{\partial r}\left(\frac{1}{R_1}\right)\Big|_{z=0} = \frac{\partial}{\partial r}\left(\frac{1}{R_2}\right)\Big|_{z=0} \equiv B$$

Now $$\vec{E}_{\parallel}^1 = \vec{E}_{\parallel}^2 \Rightarrow -\frac{\partial \Phi_1}{\partial r}\Big|_{z=0} = -\frac{\partial \Phi_2}{\partial r}\Big|_{z=0} \Rightarrow \frac{-1}{4\pi\epsilon_1}(qB + q'B) = \frac{-1}{4\pi\epsilon_2}q''B$$
$$\Rightarrow \frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2}$$

and $$\vec{D}_{\perp}^1 = \vec{D}_{\perp}^2 \Rightarrow -\epsilon_1\frac{\partial \Phi_1}{\partial z}\Big|_{z=0} = -\epsilon_2\frac{\partial \Phi_2}{\partial z}\Big|_{z=0}$$
$$\Rightarrow \frac{-1}{4\pi}(qA - q'A) = -\frac{1}{4\pi}q''A \Rightarrow q - q' = q''$$

Combining, $$\frac{q+q'}{\epsilon_1} = \frac{q-q'}{\epsilon_2} \Rightarrow q'\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}\right) = q\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right)$$
$$\Rightarrow q'(\epsilon_2 + \epsilon_1) = q(\epsilon_1 - \epsilon_2) \Rightarrow q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}q$$

and $$q'' = q - q' = \left(1 - \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right)q = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}q$$

Therefore $$\Phi = \begin{cases} \frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right)\frac{q'}{\sqrt{r^2+(z+d)^2}}\right) & z>0 \\ \frac{1}{4\pi(\epsilon_1 + \epsilon_2)}\frac{2q}{\sqrt{r^2+(z-d)^2}} & z<0 \end{cases}$$

11. *Electricity and Magnetism* (Fall 2004)

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $\mathbf{r}(t)$. You do not need to find dimensionless proportionality constants (i.e., only find the dependence on q, $\mathbf{r}(t)$, and universal constants).

We will use the general principle that the radiation field is an acceleration field that goes like $\frac{1}{r}$

$\Rightarrow \vec{E}_a \propto \frac{b}{4\pi\epsilon_0} \frac{e}{r} a$ where $a = |\ddot{\vec{r}}(t)|$ is the acceleration

and b is a constant of unknown dimension.

$$[\vec{E}] = \left[\frac{1}{4\pi\epsilon_0}\frac{e}{r^2}\right] = [\vec{E}_a] \Rightarrow \left[\frac{1}{4\pi\epsilon_0}\frac{e}{r^2}\right] = \left[\frac{b}{4\pi\epsilon_0}\frac{e}{r}a\right]$$

$$\Rightarrow [b] = \left[\frac{1}{ra}\right] = \frac{s^2}{m^2} \Rightarrow b \propto \frac{1}{c^2}$$

$$\Rightarrow \vec{E}_a \propto \frac{ea}{\epsilon_0 c^2 r}$$

$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0 c}|\vec{E}_a|^2\hat{k}$ since $E = cB$ and $\hat{E}\times\hat{B} = \hat{k}$ and $\vec{E}\perp\vec{B}$

$\Rightarrow \vec{S} \propto \frac{1}{\mu_0 c}\frac{e^2a^2}{\epsilon_0^2 c^4 r^2}\hat{k} = \frac{\mu_0^2 \cancel{\epsilon_0^2}}{\mu_0 c}\frac{e^2a^2}{\cancel{\epsilon_0^2} r^2}\hat{k} = \frac{\mu_0}{c}\frac{e^2a^2}{r^2}\hat{k}$ since $c^2 = \frac{1}{\mu_0\epsilon_0}$

$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}[r^2\vec{S}\cdot\hat{n}] \propto \frac{1}{2}\mathrm{Re}\left[\frac{\mu_0}{c}e^2a^2\cos(\theta)\right]$ by taking $\hat{z} = \hat{k}$

$$\Rightarrow \frac{dP}{d\Omega} \propto \frac{\mu_0}{c}e^2a^2\cos(\theta)$$

$$P = \int \frac{dP}{d\Omega}d\Omega \propto \frac{\mu_0}{c}e^2a^2\int_0^{\pi}\cos(\theta)\sin(\theta)d\theta\int_0^{2\pi}d\phi \propto \frac{\mu_0}{c}e^2a^2$$

12. *Electricity and Magnetism* (Fall 2004)

(a) Show that the annihilation of an electron and a positron can produce a single massive particle (say, X), but not a single photon.

(b) A positron beam of energy E can be made to annihilate against electrons by hitting electrons at rest in a fixed-target machine or by hitting electrons moving in the opposite direction with the same energy E in an electron-positron collider (colliding-beam accelerator). Show that the minimum energy $E_{\rm min}$ of a positron beam needed to produce neutral particles X of mass $M \gg m_e$ (where m_e is the electron rest mass) is much greater in a fixed-target machine than in a collider.

a. If an electron and a positron were to annihilate into a single photon, it would be impossible to conserve momentum in all frames: in the center of mass frame where the total momentum is zero, but the photon created must have nonzero momentum. A massive particle can still be created because it can be stationary in the center of mass frame.

b. We want to find the total energy in the center of mass frame $E_{tot}^{cm} = 2E_1^{cm}$ so that we don't have to take into account leftover kinetic energy required by momentum conservation. Then E_{min} is the value of E_1^{lab} in the lab frame when $E_{tot}^{cm} = Mc^2 \Leftrightarrow E_1^{cm} = \frac{1}{2}Mc^2$

For a collider, we are already in the CM frame, so $E_1^{lab} = E_1^{cm}$

$$E_{min}^{col} = E_1^{lab}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = E_1^{cm}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = \frac{1}{2}Mc^2$$

For a fixed target, $E_1^{lab} \neq E_1^{cm}$. If you start from the center of mass where each particle has velocity u_{cm}, say, then the velocity of the projectile in the lab frame, u_{lab}, is

$$u_{lab} = \frac{2u_{cm}}{1+u_{cm}^2/c^2} \qquad \left(\text{from} \quad u' = \frac{u+v}{1+uv/c^2}\right)$$

since you are adding the velocities of the particles relativistically.

$$\text{So } E_1^{lab} = \gamma_{lab} mc^2 = \frac{mc^2}{\sqrt{1-\beta_{lab}^2}} = \frac{mc^2}{\sqrt{1-\frac{4\beta_{cm}^2}{(1+\beta_{cm}^2)^2}}} = \frac{mc^2}{\sqrt{\frac{1-2\beta_{cm}^2+\beta_{cm}^4}{(1+\beta_{cm}^2)^2}}}$$

$$= mc^2\sqrt{\frac{(1+\beta_{cm}^2)^2}{(1-\beta_{cm}^2)^2}} = mc^2\gamma_{cm}^2(1+\beta_{cm}^2) = mc^2\gamma_{cm}^2\left(1+1-\frac{1}{\gamma_{cm}^2}\right)$$

$$= 2mc^2\gamma_{cm}^2 - mc^2 = \frac{2(E_1^{cm})^2}{mc^2} - mc^2$$

$$\text{Therefore } E_{min}^{fix} = E_1^{lab}\Big|_{E_1^{cm}=\frac{1}{2}Mc^2} = \frac{2}{mc^2}\left(\frac{1}{2}Mc^2\right)^2 - mc^2 = \frac{1}{2}\frac{M^2}{m}c^2 - mc^2$$

$$\Rightarrow E_{min}^{fix} \cong \frac{1}{2}\frac{M^2}{m}c^2 \gg \frac{1}{2}Mc^2 = E_{min}^{col}$$

since $M \gg m$

13. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider the Landau-Ginzburg free energy functional for a magnet with magnetization M:

$$F(M) = \frac{1}{2} r M^2 + u M^4 - hM$$

M takes values $M \in [-\infty, \infty]$. (The rotational symmetry of the magnet is broken by the crystal so that M is a scalar, not a vector.) $r = a(T - T_c)$, u is only weakly dependent on T, and h is the magnetic field. We will make the mean-field approximation that M is equal to the value which minimizes $F(M)$, and $F(M)$ is given by its minimum value.

(a) For $T > T_c$ and $h = 0$, what value of M minimizes F? For $T < T_c$ and $h = 0$, what value of M minimizes F?

(b) For $h = 0$, the specific heat takes the asymptotic form $C \sim |T - T_c|^{-\alpha}$ as $T \to T_c$. What is α?

(c) At $T = T_c$, $M \sim h^\delta$. What is δ?

a. Mean-field approximation $\Rightarrow \frac{\partial F}{\partial M} = rM + 4uM^3 - h = 0$

And $h = 0 \Rightarrow 4uM^3 = -rM \Rightarrow M = 0$ or $M = \sqrt{-\frac{r}{4u}}$

If $T > T_c$, then $r = a(T - T_c) > 0$, so all terms in F are positive, so $M = 0$ minimizes F.

If $T < T_c$, then $r = a(T - T_c) < 0$, so $M = \sqrt{\frac{-r}{4u}}$ is a real solution that makes $F(M) = \frac{1}{2} r \left(-\frac{r}{4u}\right) + u \left(\frac{r^2}{16u^2}\right) = -\frac{r^2}{16u}$ which is less than zero, so $M = \sqrt{-\frac{r}{4u}}$ minimizes F.

b. The free energy functional is a type of Gibbs free energy so we use $\left(\frac{\partial G}{\partial T}\right)_p = -S$ and $C_v = T\left(\frac{\partial S}{\partial T}\right)_v$

We assume that $T \to T_c$ from below T_c because the Mean field approximation used here gives a trivial result otherwise

$$G = F(M) = \frac{1}{2} a(T - T_c) \frac{-a(T-T_c)}{4u} + u \frac{a^2(T-T_c)^2}{16u^2} = -\frac{a^2(T-T_c)^2}{16u^2}$$

$$\Rightarrow C_v = T\left(\frac{\partial S}{\partial T}\right)_v = T\frac{\partial^2 G}{\partial T^2} = T\frac{\partial}{\partial T}\left(-\frac{a^2(T-T_c)}{8u^2}\right) = -\frac{a^2}{8u^2} T$$

$$= -\frac{a^2}{8u^2}\left[(T - T_c) + T_c\right] \cong -\frac{a^2}{8u^2} T_c \quad \text{since } |T - T_c| \ll T_c$$

in the asymptotic limit, so $C_v \sim |T - T_c|^0 \Rightarrow \alpha = 0$

c. $F(M)\big|_{T=T_c} = uM^4 - hM$

$$\frac{\partial F}{\partial M}\bigg|_{T=T_c} = 4uM^3 - h = 0 \Rightarrow h = 4uM^3 \Rightarrow M = \left(\frac{h}{4u}\right)^{1/3} \Rightarrow \delta = \frac{1}{3}$$

14. *Statistical Mechanics and Thermodynamics* (Fall 2004)

Consider black body radiation at temperature T. What is the average energy per photon in units of kT?

You may find the following formulae useful:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \approx 6.5; \qquad \int_0^\infty \frac{x^2 dx}{e^x - 1} \approx 2.4$$

$$\epsilon = pc = \hbar k c = \hbar c \sqrt{\left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2 + \left(\frac{n_z \pi}{L}\right)^2}$$

$$= \frac{\hbar c \pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\hbar c \pi}{L} n$$

$$\Rightarrow n = \frac{L}{\hbar c \pi} \epsilon \quad \text{and} \quad dn = \frac{L}{\hbar c \pi} d\epsilon$$

$$\rho(\epsilon) d\epsilon = \frac{1}{8} 4\pi n^2 dn = \frac{1}{2}\pi \left(\frac{L}{\hbar c \pi}\right)^3 \epsilon^2 d\epsilon = \frac{V}{2\pi^2} \frac{\epsilon^2}{(\hbar c)^3} d\epsilon$$

$$\langle \epsilon \rangle = \frac{\int_0^\infty \epsilon f(\epsilon) \rho(\epsilon) d\epsilon}{\int_0^\infty f(\epsilon) \rho(\epsilon) d\epsilon} \quad \text{where } f(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1}$$

$$\int_0^\infty \epsilon f(\epsilon) \rho(\epsilon) d\epsilon = \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \int_0^\infty \frac{\epsilon^3}{e^{\beta\epsilon} - 1} d\epsilon$$

Let $x = \beta\epsilon \Rightarrow d\epsilon = \frac{1}{\beta} dx$

$$= \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \frac{1}{\beta^4} \int_0^\infty \frac{x^3}{e^x - 1} dx$$

$$\cong \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} (kT)^4 (6.5)$$

$$\int_0^\infty f(\epsilon) \rho(\epsilon) d\epsilon = \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \int_0^\infty \frac{\epsilon^2}{e^{\beta\epsilon} - 1} d\epsilon$$

Let $x = \beta\epsilon \Rightarrow d\epsilon = \frac{1}{\beta} dx$

$$= \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} \frac{1}{\beta^3} \int_0^\infty \frac{x^2}{e^x - 1} dx$$

$$\cong \frac{V}{2\pi^2} \frac{1}{(\hbar c)^3} (kT)^3 (2.4)$$

$$\Rightarrow \langle \epsilon \rangle \cong \frac{6.5}{2.4} (kT) \cong 2.7 kT$$

Fall 2004 #1

$\langle \psi | (\vec{S}_a \cdot \hat{n}_a)(\vec{S}_b \cdot \hat{n}_b) | \psi \rangle$

$$= \sum_{ij} (n_a)_i (n_b)_j \langle \psi | (S_a)_i (S_b)_j | \psi \rangle$$

$$\langle \psi | \tfrac{1}{3} \delta_{ij} \sum_K (S_a)_K (S_b)_K | \psi \rangle + \ldots$$

abers eq. 5.40

(This is expanding $(S_a)_i (S_b)_j$ into traceless matrices)

Only need first term since $(S_a)_i (S_b)_j$ is a scalar

$$\rightarrow \langle \psi | (S_a \cdot \hat{n}_a)(S_b \cdot \hat{n}_b) | \psi \rangle = \tfrac{1}{3} \hat{n}_a \cdot \hat{n}_b \langle \psi | S_a \cdot S_b | \psi \rangle$$

But $S^2 = 0 \quad S = S_a + S_b$

$$S_a \cdot S_b = \tfrac{1}{2}(S^2 - S_a^2 - S_b^2) = -\tfrac{3}{4}$$

$$\langle \psi | (S_a \cdot \hat{n}_a)(S_b \cdot \hat{n}_b) | \psi \rangle = -\tfrac{1}{3} \cos\theta \, \tfrac{3}{4} = -\tfrac{1}{4} \cos\theta$$

Fall 2004 #1 (p1of2)

Two spin-half particles are in a state with total spin zero. Let $\hat{n}_a$ and $\hat{n}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the product of the spin of the first particle along $\hat{n}_a$ and the spin of the second along $\hat{n}_b$. That is, if $\vec{S}_a$ and $\vec{S}_b$ are the two spin operators, calculate

$$\langle \Psi | \vec{S}_a \cdot \hat{n}_a \; \vec{S}_b \cdot \hat{n}_b | \Psi \rangle$$

(See Abers 4.11)

From the definition of the dot product

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i$$

So, we can write

$$\gamma \equiv \langle \Psi | \vec{S}_a \cdot \hat{n}_a \vec{S}_b \cdot \hat{n}_b | \Psi \rangle = \langle \Psi | \sum_{ij} (S_a)_i (n_a)_i (S_b)_j (n_b)_j | \Psi \rangle$$

since $(n_a)_i$ and $(n_b)_j$ are just numbers, we get

$$\gamma = \sum_{ij} (n_a)_i (n_b)_j \langle \Psi | (S_a)_i (S_b)_j | \Psi \rangle \qquad (1)$$

Now, $(S_a)_i (S_b)_j$ has the form of a 2^{nd} rank tensor

$$T_{ij} = (S_a)_i (S_b)_j$$

So, from Abers eq 5.40, we can write the tensor as

$$T_{ij} = \underbrace{\left[\tfrac{1}{3} \delta_{ij} \sum_k T_{kk}\right]}_{\text{trace, spin zero}} + \underbrace{\left[\tfrac{1}{2}(T_{ij} - T_{ji})\right]}_{\text{anti-symmetric part}} + \underbrace{\left[\tfrac{1}{2}(T_{ij} + T_{ji}) - \tfrac{1}{3}\delta_{ij} \sum_k T_{kk}\right]}_{\text{traceless, symmetric part}}$$

Since the spin of our system is equal to zero, only the 1st term on the RHS survives. So,

$$T_{ij} = \tfrac{1}{3} \delta_{ij} \sum_k T_{kk} = \tfrac{1}{3} \sum_k (S_a)_k (S_b)_k = \tfrac{1}{3} \vec{S}_a \cdot \vec{S}_b$$

Fall 2004 #1 (p 2 of 2)

note:

$$\underbrace{\left[(S_a+S_b)^2\right]_K}_{\text{total spin is zero}} = (S_a^2)_K + (S_b^2)_K + 2\,(\vec{S}_a\cdot\vec{S}_b)_K$$

$$\Rightarrow \quad (\vec{S}_a\cdot\vec{S}_b)_K = -\frac{1}{2}\left[(S_a^2)_K + (S_b^2)_K\right]$$

Now, take advantage of us having spin half particles. That is,

$$S_i = \frac{\sigma_i}{2}$$

$$\Rightarrow \quad (\vec{S}_a\cdot\vec{S}_b)_K = -\frac{1}{8}\left[(\sigma_a^2)_K + (\sigma_b^2)_K\right]$$

we know that a property of the Pauli matrices is that $\sigma_i^2 = 1$ (Abers 4.77). So,

$$(\vec{S}_a\cdot\vec{S}_b)_K = -\frac{1}{8}\left[1+1\right] = -\frac{1}{4}$$

Summing over the three coordinates, we get

$$T_{ij} = \frac{1}{3}\sum_K (\vec{S}_a\cdot S_b)_K = \frac{1}{3}\left(-\frac{1}{4}-\frac{1}{4}-\frac{1}{4}\right) = -\frac{1}{4}$$

substituting this result into eq (1) yields

$$\sum_{i,j}(n_a)_i(n_b)_j \langle\psi|(S_a)_i(S_b)_j|\psi\rangle = \sum_i (n_a)_i(n_b)_i\left(-\frac{1}{4}\right)\underbrace{\langle\psi|\psi\rangle}_{\text{normalized orthogonal vectors}}$$

$$= -\frac{1}{4}\,\hat{n}_a\cdot\hat{n}_b = -\frac{1}{4}|\hat{n}_a||\hat{n}_b|\cos(\theta_{ab})$$

where θ_{ab} is the angle between $\hat{n}_a$ and $\hat{n}_b$. Thus,

$$\boxed{\langle\psi|\vec{S}_a\cdot\hat{n}_a\,\vec{S}_b\cdot\hat{n}_b|\psi\rangle = -\frac{1}{4}\cos(\theta_{ab})}$$

Fall 2004 #1 (p 1 of 2)

Abers solution (#4.11)

1. *Quantum Mechanics*

Two spin-half particles are in a state with total spin zero. Let $\hat{n}_a$ and $\hat{n}_b$ be unit vectors in two arbitrary directions. Calculate the expectation value of the *product* of the spin of the first particle along $\hat{n}_a$ and the spin of the second along $\hat{n}_b$. That is, if $\mathbf{s}_a$ and $\mathbf{s}_b$ are the two spin operators, calculate

$$\langle\psi|\,\mathbf{s}_a\cdot\hat{n}_a\mathbf{s}_b\cdot\hat{n}_b\,|\psi\rangle$$

Hint: Because the state is spherically symmetric the answer can depend only on the angle between the two directions.

This is an example of the selection rules from section 5.2. Since the state ψ is symmetric,

$$\begin{aligned}\langle\psi|\left(\mathbf{s}_a\cdot\hat{n}_a\right)\left(\mathbf{s}_b\cdot\hat{n}_b\right)|\psi\rangle &= \sum_{ij}\left(\hat{n}_a\right)_i\left(\hat{n}_b\right)_j\left\langle\psi\right|\left(s_a\right)_i\left(s_b\right)_j\left|\psi\right\rangle \\ &= \sum_{ij}\left(\hat{n}_a\right)_i\left(\hat{n}_b\right)_j\left\langle\psi\left|\frac{1}{3}\delta_{ij}\sum_k\left(s_a\right)_k\left(s_b\right)_k\right|\psi\right\rangle+\cdots\end{aligned} \tag{S10.28}$$

The remaining terms are matrix elements of (linear combinations of) components of spherical tensors of ranks 1 and 2 between spin-zero states, so vanish:

$$\langle\psi|\left(\mathbf{s}_a\cdot\hat{n}_a\right)\left(\mathbf{s}_b\cdot\hat{n}_b\right)|\psi\rangle = \frac{1}{3}\hat{n}_a\cdot\hat{n}_b\langle\psi|\,\mathbf{s}_a\cdot\mathbf{s}_b\,|\psi\rangle \tag{S10.29}$$

But since $\mathbf{s}^2 = 0$ between these states, where $\mathbf{s} = \mathbf{s}_a + \mathbf{s}_b$,

$$\mathbf{s}_a\cdot\mathbf{s}_b = \frac{1}{2}\left(\mathbf{s}^2 - \mathbf{s}_a^2 - \mathbf{s}_b^2\right) = -\frac{3}{4} \tag{S10.30}$$

and

$$\langle\psi|\left(\mathbf{s}_a\cdot\hat{n}_a\right)\left(\mathbf{s}_b\cdot\hat{n}_b\right)|\psi\rangle = -\frac{1}{3}\cos\theta\frac{3}{4} = -\frac{1}{4}\cos\theta \tag{S10.31}$$

Note: It is of course possible to get the answer without using the theorem: Since the matrix element depends only on the angle between these two directions, let $\hat{n}_a = \hat{n}_z$. Then with $\hat{n}_b = \cos\theta\hat{n}_z + \sin\theta\hat{n}_x$, the correlation is

$$\begin{aligned}E(\theta) = \langle\psi|\left(\mathbf{s}_a\cdot\hat{n}_z\right)\left(\mathbf{s}_b\cdot\hat{n}_c\right)|\psi\rangle &= \frac{1}{4}\langle\psi|\left(\boldsymbol{\sigma}_a\cdot\hat{n}_z\right)\left(\boldsymbol{\sigma}_b\cdot\hat{n}_c\right)|\psi\rangle \\ &= \langle\psi|\left(\sigma_a\right)_z\left[\left(\sigma_b\right)_z\cos\theta + \left(\sigma_b\right)_x\sin\theta\right]|\psi\rangle\end{aligned} \tag{S10.32}$$

[2] see Equation (A.27) in the appendix.

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(Aber's solution)

[redacted]

Now

$$(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)\,|\psi\rangle = (\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)\,\frac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z \frac{|++\rangle - |--\rangle}{\sqrt{2}} = \frac{|++\rangle + |--\rangle}{\sqrt{2}} \tag{S10.33}$$

so that

$$\langle\psi|\,(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_x)\,|\psi\rangle = 0 \tag{S10.34}$$

Similarly

$$(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)\,|\psi\rangle = (\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)\,\frac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \frac{-|+-\rangle + |-+\rangle}{\sqrt{2}} = -|\psi\rangle \tag{S10.35}$$

so that

$$\langle\psi|\,(\boldsymbol{\sigma}_a \cdot \hat{\boldsymbol{n}}_z)\,(\boldsymbol{\sigma}_b \cdot \hat{\boldsymbol{n}}_z)\,|\psi\rangle = -1 \tag{S10.36}$$

So again

$$E(\theta) = -\frac{1}{4}\cos\theta \tag{S10.37}$$

F'04 Q.M. #3

a) (Prob. of both being within "b" of the origin)

+ (Prob. of one or both being outside of "b") = 1

=> Prob. of at least one particle farther than b = 1 − Prob. of both within b

$$= 1 - \int_0^b \int_0^{2\pi} \int_0^{\pi} (\psi^* \psi)\, dv_1\, dv_2$$

b) $$H = \frac{p_+^2}{2m_+} + \frac{p_-^2}{2m_-} - \frac{e^2}{|r_+ - r_-|}$$

c) $$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{e^2}{|r|}$$ with $\vec{P} = \vec{p}_+ + \vec{p}_-$

$$\vec{p} = \frac{m_- \vec{p}_+ - m_+ \vec{p}_-}{M}$$

$$M = m_+ + m_-$$

$$\mu = \frac{m_+ m_-}{m_+ + m_-}$$

d) $P = 0 \Rightarrow$ $$H = \frac{p^2}{2\mu} - \frac{e^2}{|r|}$$

which is similar to the equation for the hydrogen atom.

We have the relationship

$$E_n = \frac{-1}{2} \mu c^2 \left(\frac{Z\alpha}{n}\right)^2$$

now $\mu = \frac{m_e m_e}{m_e + m_e} = \frac{m_e}{2}$

$Z = 1$

So $E_n = -\frac{1}{2}\frac{m_e}{2}c^2\left(\frac{\alpha}{n}\right)^2 = \frac{1}{2}\underbrace{\left(-\frac{1}{2}m_e c^2\left(\frac{\alpha}{n}\right)^2\right)}_{\text{hydrogen atom energy.}}$

So for the ground state $n=1$

$$\boxed{E_1 = \frac{1}{2}(-13.6\,eV) = -6.8\,eV}$$

e) By inspection

$$C\,H = \frac{\overbrace{(P_- + P_+)^2}^{p^2 \text{ (unchanged)}}}{2M} + \frac{\left(\frac{m_+ p_- - \overbrace{m p_+}^{(-p)^2 = p^2}}{M}\right)^2}{2\mu} + \frac{-e^2}{|r|}$$

$|r_- - r_+| = |r_+ - r_-|$

so the Hamiltonian is unchanged. So $[C, H] = 0$.

As the Hamiltonian is unchanged and hence also the energy eigenvalues, the eigenvalue of C on the ground state is $+1$.

Let H be the Hamiltonian for the hydrogen atom, including spin, $\hbar\vec{L} = \vec{r}\times\vec{p}$ and $\hbar\vec{S}$ are the orbital and spin angular momentum, respectively, and $\vec{J} = \vec{L}+\vec{S}$. Conventionally the states are labeled $|n,\ell,j,m\rangle$ and they are eigenstates of H, $\vec{L}^2$, $\vec{J}^2$, and J_z.

a) If the electron is in the state $|n,\ell,j,m\rangle$, what values will be measured for these four observables in terms of $\hbar$, c, the fine-structure constant α, and the electron mass m?

$$H|n,\ell,j,m\rangle = E_n|n,\ell,j,m\rangle \quad ; \quad E_n = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$$

$$\vec{L}^2|n,\ell,j,m\rangle = \hbar\,\ell(\ell+1)|n,\ell,j,m\rangle$$
$$\vec{J}^2|n,\ell,j,m\rangle = \hbar\,j(j+1)|n,\ell,j,m\rangle$$
$$J_z|n,\ell,j,m\rangle = \hbar m|n,\ell,j,m\rangle$$

b) What are the restrictions on the possible values of n, ℓ, j, and m?

- n can be any non-zero positive integer
- ℓ can be any integer in the range $n-1, \ldots, 0$
- j can be any value in the range $|\ell+m|, |\ell+m|-1, \ldots, |\ell-m|$
- m can be any value in the range $+j, j-1, \ldots, -j$

c) Let $\vec{J}_{\pm} = J_x \pm J_y$. What are

i) $\langle 3,1,3/2,3/2 | J_+ | 3,1,3/2,-1/2 \rangle = ?$

$J_+ |n,\ell,j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|n,\ell,j,m+1\rangle$

so $J_+ |3,1,3/2,-1/2\rangle = \frac{4\hbar}{2} |3,1,3/2,1/2\rangle = 2\hbar |3,1,3/2,1/2\rangle$

$= \frac{2\hbar}{\cancel{2}} \langle 3,1,3/2,3/2 | 3,1,3/2,1/2\rangle = 0$

ii) $\langle 3,1,3/2,3/2 | J_+ | 3,1,3/2,1/2 \rangle = \hbar\sqrt{3}\,\langle 3,1,3/2,3/2 | 3,1,3/2,3/2\rangle = \sqrt{3}\,\hbar$

iii) $\langle 2,1,3/2,3/2 | p_z | 2,1,3/2,1/2 \rangle = 0$

as $[L_z, p_z] = 0$ so

$\langle 2,1,3/2,3/2 | [L_z,p_z] | 2,1,3/2,1/2\rangle = \langle 2,1,3/2,3/2 | L_z p_z | 2,1,3/2,1/2\rangle - \langle 2,1,3/2,3/2 | p_z L_z | 2,1,3/2,1/2\rangle$

$= \frac{3}{2}\hbar \langle p_z\rangle - \frac{\hbar}{2}\langle p_z\rangle = 0$

so $\langle p_z\rangle = 0$

iv) $\langle 2,1,1/2,-1/2 | L^2 | 2,1,1/2,-1/2\rangle = 2\hbar^2$

v) $\langle 3,2,3/2,-1/2 | J^2 | 3,2,3/2,-1/2\rangle = \hbar^2 \frac{3}{2}\left(\frac{3}{2}+1\right) = \frac{15}{4}\hbar^2$

vi) $\langle 3,1,3/2,3/2 | J_z | 3,1,3/2,1/2\rangle = \frac{1}{2}\hbar\,\underbrace{\langle 3,1,3/2,3/2 | 3,1,3/2,1/2\rangle}_{=0} = 0$

d) What is

$$\langle 1,0,\tfrac{1}{2},\tfrac{1}{2} | p_i p_j | 1,0,\tfrac{1}{2},\tfrac{1}{2} \rangle = ?$$

$i \neq j$ As we are dealing with the spherically symmetric ground state we end up with an integral over an odd function (with symmetric integration limits) so that integral gives us 0.

$i = j$ then we have something like

$$\langle 1,0,\tfrac{1}{2},\tfrac{1}{2} | p_x^2 | 1,0,\tfrac{1}{2},\tfrac{1}{2} \rangle$$

now via the Virial Theorem $\langle T \rangle = +|E| = \langle \frac{p^2}{2m} \rangle = -E_1$

$\Rightarrow \langle p^2 \rangle = -2mE_1$

by symmetry $\langle p_x^2 \rangle = \frac{1}{3}(\langle p_x^2 \rangle + \langle p_y^2 \rangle + \langle p_z^2 \rangle) = \frac{1}{3}\langle p^2 \rangle$

so $\langle p_x^2 \rangle = -\frac{2}{3} m E_1$

so $\langle 1,0,\tfrac{1}{2},\tfrac{1}{2} | p_i p_j | 1,0,\tfrac{1}{2},\tfrac{1}{2} \rangle = -\frac{2}{3} m E_1 \delta_{ij}$

e) For given n, ℓ, j, and m, what are the conditions on n', ℓ', j', and m' so that

$$\langle n', \ell', j', m' | \vec{S} \cdot \vec{r} | n, \ell, j, m \rangle \neq 0$$

For the restriction on j:

$\vec{S}\cdot\vec{r}$ is a scalar so $[J_{x,y,z}, \vec{S}\cdot\vec{r}]=0 \Rightarrow [\vec{J}^2, \vec{S}\cdot\vec{r}]=0$

so $\langle [\vec{J}^2, \vec{S}\cdot\vec{r}] \rangle = 0$ which implies $j \neq j'$

and $\langle [J_{x,y,z}, \vec{S}\cdot\vec{r}] \rangle = 0$ which implies $m \neq m'$

As for ℓ: Parity (P) on a state $|\ell, m\rangle$: $P|\ell, m\rangle = (-1)^{\ell}|\ell, m\rangle$

so $\langle \vec{S}\cdot\vec{r} \rangle = \langle P^2 (\vec{S}\cdot\vec{r}) P^2 \rangle = \langle P[P(\vec{S}\cdot\vec{r})P]P \rangle = (-1)^{\ell+\ell'} \langle \underbrace{P[\vec{S}\cdot\vec{r}]P}_{= -(\vec{S}\cdot\vec{r})} \rangle$

$\Rightarrow \langle \vec{S}\cdot\vec{r} \rangle = (-1)^{\ell+\ell'} \left(- \langle \vec{S}\cdot\vec{r} \rangle \right)$ so $\ell' = \ell \pm 1$

As for n: for the hydrogen atom there is no restriction on the transitions that the electron can make so there is no restriction on n'.

Fall 2004 #5

$$H = \frac{p^2}{2m} + \frac{m\omega x^2}{2}$$ $|\psi_n\rangle$; $n = 0,1,2 \ldots$ usual energy eigenstates

$H|\psi_n\rangle = \omega(n+1/2)$

a) $C_1|\phi\rangle = C_0|\psi_0\rangle + C_1|\psi_1\rangle$

$\hbar = 1$

$\langle\phi|H|\phi\rangle = \omega = |C_0|^2\frac{\omega}{2} + |C_1|^2\frac{3\omega}{2}$

$2 = |C_0|^2 + 3|C_1|^2 \qquad |C_0|^2 = 1 - |C_1|^2$

$\langle\phi|\phi\rangle = 1 = |C_0|^2 + |C_1|^2$

$2 = 1 - |C_1|^2 + 3|C_1|^2$

$1 = 2|C_1|^2 \qquad |C_1|^2 = 1/2 \quad |C_0|^2 = 1/2$

b) pick $C_0 > 0$

$C_1 = |C_1| e^{i\theta}$

$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{m\omega} \qquad \langle\phi|H|\phi\rangle = \omega \qquad$ what is θ_1

$x = \frac{1}{\sqrt{2m\omega}}(a + a^{+}) \qquad a|\psi_n\rangle = \sqrt{n}\psi_{n-1} \quad a^{+}|\psi_n\rangle = \sqrt{n+1}\,\psi_{n+1}$

$x|\phi\rangle = \frac{1}{\sqrt{2m\omega}}\left[C_0(\cancel{a}+a^{+})|\psi_0\rangle + C_1(a+a^{+})|\psi_1\rangle\right]$

$= \frac{1}{\sqrt{2m\omega}}\left[C_0|\psi_1\rangle + C_1|\psi_0\rangle + \sqrt{2}C_2|\psi_2\rangle\right]$

$\frac{1}{2}\sqrt{m\omega} = \langle\phi|x|\phi\rangle = \frac{1}{\sqrt{2m\omega}}\left[C_0^{*}C_1 + C_0 C_1^{*}\right]$

$\frac{1}{2} \qquad = \frac{1}{\sqrt{2m\omega}}\left[C_0|C_1|\left(e^{i\theta_1} + e^{-i\theta_1}\right)\right]$

$$= \frac{1}{\sqrt{2m\omega}} \frac{1}{2} |c_1| \cos\theta_1 = \frac{1}{2}\frac{1}{\sqrt{m\omega}}$$

$$\frac{1}{\sqrt{2}} \cos\theta_1 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\theta_1 = \pi/4$$

d) $|\psi(0)\rangle = |\phi\rangle$

$$|\psi(t)\rangle = c_0 e^{-iE_0 t} |\psi_0\rangle + c_1 e^{-iE_1 t} |\psi_1\rangle$$

$$x|\psi(t)\rangle = \frac{1}{\sqrt{2m\omega}} \left[c_0(t) |\psi_1\rangle + c_1(t)\left(|\psi_0\rangle + \sqrt{2}\, |\psi_2\rangle\right)\right]$$

$$\langle\psi(t)|x|\psi(t)\rangle = \frac{1}{\sqrt{2m\omega}} \left[c_0(t)\, c_1^*(t) + c_1(t)\, c_0^*(t) \right]$$

$$c_1 = |c_1| e^{i\pi/4}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2m\omega}} \left[e^{-iE_0 t} e^{iE_1 t - i\pi/4} + e^{-iE_1 t} e^{iE_0 t + i\pi/4} \right]$$

$$c_1(t) = |c_1| e^{i\theta_1} e^{-iE_1 t}$$

$$= \frac{1}{2\sqrt{2m\omega}} \left[e^{i(E_1 - E_0)t} e^{-i\pi/4} + e^{-i(E_1 - E_0)t} e^{+i\pi/4} \right]$$

$$= \frac{1}{2\sqrt{2m\omega}} \left[e^{i\omega t} e^{-i\pi/4} + e^{-i\omega t} e^{+i\pi/4} \right]$$

$$= \frac{1}{\sqrt{2m\omega}} \cos(\omega t - \pi/4)$$

Fall 2004 #8

$\hat{M z}$ (uniform)

What is B, and H inside?

$$\vec{B} = \frac{2}{3}\mu_0 M \hat{z}$$ see # 12 S '03

$$B = \mu_0(H + M)$$

$$2/3\,\mu_0 M = \mu_0 H + \mu_0 M$$

$$2/3\,M - M = H$$

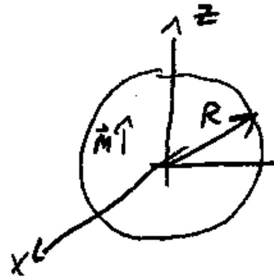
$$\vec{H} = -1/3\,M\,\hat{z}$$

Fall 2004 #8 (p 1 of 3)

Consider a sphere of radius a with uniform magnetization $\vec{M}$, pointing in the z-direction. What are the magnetic induction $\vec{B}$ and magnetic field $\vec{H}$ inside the sphere?

(see spring 2003 #12 and Jackson problem 5.13)

First, recognize that this is an identical problem to the field of a spinning spherical shell with $\sigma R \omega \to M$ (see Griffiths' examples 6.1 and 5.11). That is, we have



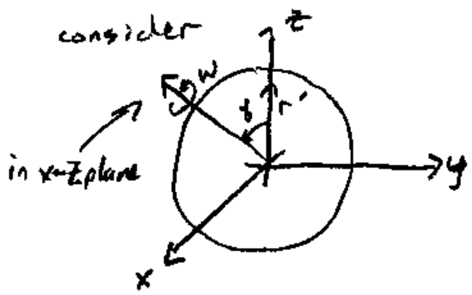
$$\left.\begin{aligned} \vec{J}_m &= \nabla \times \vec{M} = 0 \\ |\vec{K}| &= M \sin\theta \end{aligned}\right\} \text{ mks units}$$

if rotating spherical shell of uniform charge, then
$K = \sigma \omega R \sin\theta$

So let's solve the rotating spherical shell of uniform charge. First we must find the vector potential since

$$\vec{A} = \frac{1}{c} \int \frac{\vec{K}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} da' \qquad (1)$$

consider



$$\text{so,} \quad \underset{\uparrow\ \vec{v} = \vec{\omega}\times\vec{r}\,'}{\vec{K} = \sigma\vec{v}} = \sigma(\vec{\omega}\times\vec{r}\,') \underset{\substack{\uparrow \\ \text{axis of rotation} \\ x\text{-}z \text{ plane}}}{=} \sigma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\gamma & 0 & \omega\cos\gamma \\ a\sin\theta'\sin\phi' & a\sin\theta'\sin\phi' & a\cos\theta \end{vmatrix}$$

$$\Rightarrow \vec{K} = \sigma\left[\hat{x}(-\omega a\cos\gamma\sin\theta'\sin\phi') + \hat{y}(\omega a\cos\gamma\sin\theta'\sin\phi' - \omega a\sin\gamma\cos\theta') + \hat{z}(a\omega\sin\gamma\sin\theta'\sin\phi')\right]$$

Now note; $|\vec{r}-\vec{r}\,'| = \left[r^2 + (r')^2 - 2rr'\cos\theta'\right]^{1/2}\Big|_{r'=a} = \left[r^2 + a^2 - 2ra\cos\theta'\right]^{1/2}$

and $da' = a^2\sin\theta'\,d\theta'\,d\phi'$

Substituting these results into eq (1) yields

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi} d\phi' \int_0^{\pi}\sin\theta'\,d\theta' \left[\frac{-\sin\theta'\sin\phi'\cos\gamma\,\hat{x} + (\sin\theta'\cos\phi'\cos\gamma - \sin\gamma\cos\theta')\hat{y} + \sin\gamma\sin\theta'\sin\phi'}{\left[r^2 + a^2 - 2ra\cos\theta'\right]^{1/2}}\right.$$

since $\int_0^{2\pi}\sin\phi'\,d\phi' = -\left[\cos\phi'\right]_0^{2\pi} = 0$

and $\int_0^{2\pi} d\phi'\cos\phi' = \left[\sin\phi'\right]_0^{2\pi} = 0$

the integration over ϕ' in the x and z direction vanish as well as the first term in the y direction.

So, this messy integral reduces to

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \left[\frac{-\sin\gamma\cos\theta'}{[r^2+a^2-2ra\cos\theta']^{1/2}}\right]$$

$$= \frac{-2\pi\sin\gamma\,\sigma\omega a^3}{c}\int_0^{\pi} d\theta' \frac{\sin\theta'\cos\theta'}{[r^2+a^2-2ra\cos\theta']^{1/2}}$$

let $u = \cos\theta' \Rightarrow du = -\sin\theta' d\theta'$

So, we have

$$\vec{A}(\vec{r}) = \frac{-2\pi\sigma\omega a^3}{c}\sin\gamma\int_{-1}^{1}\frac{u\,du}{[r^2+a^2-2rau]^{1/2}}$$

note: $\displaystyle\int\frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$

So,

$$\vec{A}(\vec{r}) = \frac{-2\pi\sigma\omega a^3}{c}\sin\gamma\left(\frac{-2r}{3a^2}\right)\hat{\phi} \qquad r<a$$

(→ see p 4 and 5 of the Spring 2003 #12 for details of this calculation)

recall that $\vec{\omega}\times\vec{r} = -\omega r\sin\gamma$ from figure on p 1. So,

$$\vec{A}(\vec{r}) = \frac{2\pi\sigma\omega a^3}{c}\sin\theta\left(\frac{2r}{3a^2}\right) = \frac{4\pi\sigma\omega ra}{3c}\sin\theta\,\hat{\phi}$$

where we re-oriented our coordinate system such that $\vec{\omega}$ is aligned with the z-axis,

Now, we are ready to find $\vec{B}$ where $\vec{B} = \nabla\times\vec{A} = \nabla\times(A_\phi\hat{\phi})$

$$\Rightarrow \vec{B} = \begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi}\\ \partial_r & \partial_\theta & \partial_\phi\\ 0 & 0 & r\sin\theta A_\phi\end{vmatrix}\frac{1}{r^2\sin\theta} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}$$

$$= \frac{4\pi\sigma\omega a}{3c}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin^2\theta\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}r^2\,\hat{\theta}\right] = \frac{4\pi\sigma\omega a}{3c}\left[\frac{2\sin\theta\cos\theta}{\sin\theta}\hat{r} - \frac{2r}{r}\sin\theta\right.$$

$$= \frac{8\pi\sigma\omega a}{3c}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right] = \frac{8\pi\sigma\omega a}{3c}\hat{z}$$

Thus,

$$\boxed{\vec{B} = \frac{8\pi\sigma\omega a}{3c}\,\hat{z}} \qquad r < a$$

Now, find $\vec{H}$ inside.

$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad , \quad \vec{M} = M_0\,\hat{z}$$

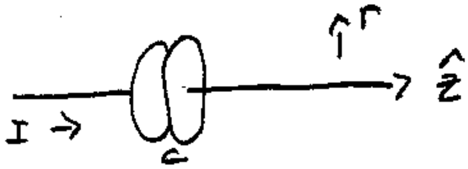
Thus,

$$\boxed{\vec{H} = \hat{z}\left[\frac{8\pi\sigma\omega a}{3c} - 4\pi M_0\right]}$$

Fall 2004 #9 (p 1 of 2)

A wire carrying current I is connected to a circular capacitor of capacitance C. What is the magnetic field outside the wire, far from the capacitor? Using Maxwell's equations, explain why there is a magnetic field outside the capacitor, what is this magnetic field?

(see Spring 2002 #12)



(i) $\vec{B}$ outside wire far away

Far from the wire, the field can simply be found from Ampère's law without a displacement current. That is,

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \cancel{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}^{\,\to 0} \qquad (1)$$

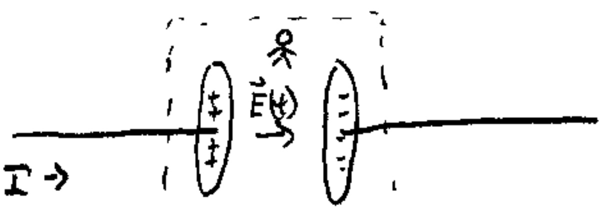
$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$$

from symmetry,

$$|\vec{B}| \cdot 2\pi r = \frac{4\pi}{c} I$$

$$\therefore \boxed{\vec{B} = \frac{2I}{rc}\,\hat{\phi}}$$

(ii) why is there a $\vec{B}$-field outside capacitor?



if you put an observer in the indicated region, the observer will "observe" a magnetic field even though the space between the capacitors is "vacuum" and thus no current density, $\vec{J}$, is possible without any material.

But, from Maxwell's equations, matter and fields play identical roles.

Fall 2004 # 9 (p 2 of 2)

That is, between the plates, there is a displacement current. So Ampere's equation has the form

$$\nabla \times \vec{B} = \cancel{\frac{4\pi}{c}\vec{j}}^{\,0} + \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{\text{displacement current}}$$

An observer in the dotted region observes a magnetic field as though there was a physical conducting wire carrying a current in that region. So, in this case the field (displacement current) plays the role of matter (a wire).

Then the magnetic field observed outside the capacitor is the same as the one far from the wire.

$$\boxed{\vec{B} = \frac{2I}{rc}\,\hat{\phi}}$$

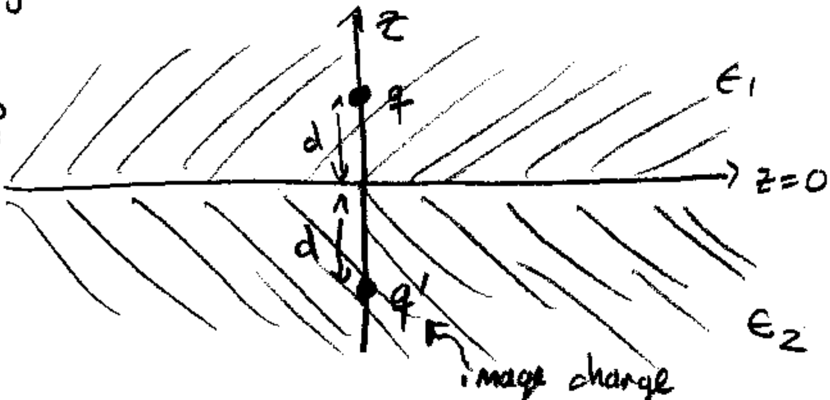
Fall 2004 #10 (p 1 of 3)

The upper half-space is filled with a material permittivity ϵ_1, while the lower half-space is filled with a different material permittivity ϵ_2. Their interface is located at the $z=0$ plane. A point charge q is located at $\vec{r}_q = d\,\hat{z}$ on the z-axis in medium 1. Find the electrostatic potential everywhere.

(I took solution from Jackson!)

(see Lim Yung-Kuo #1078 and Jackson 3rd ed section 4.4 p 154-156)

$$\nabla\times\vec{E}=0 \begin{cases} \epsilon_1 \nabla\cdot\vec{E} = 4\pi\rho \\ \epsilon_2 \nabla\cdot\vec{E} = 0 \end{cases}$$



The boundary condition at $z=0$ is

$$\lim_{z\to 0^+} \begin{Bmatrix} \epsilon_1 E_z \\ E_x \\ E_y \end{Bmatrix} = \lim_{z\to 0^-} \begin{Bmatrix} \epsilon_2 E_z \\ E_x \\ E_y \end{Bmatrix}$$

In cylindrical coordinates, the potential for $z>0$ at some point P is given by

$$\Phi = \frac{1}{\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) \qquad z>0$$

where $R_1 = \sqrt{r^2+(d-z)^2}$ and $R_2 = \sqrt{r^2+(d+z)^2}$

Now, there is no charge in the region $z<0$. So, it must be a solution of the Laplace equation without singularities in the region. So, assume the potential is equivalent to that of a charge q'' at the same position of the actual charge. That is,

$$\Phi = \frac{1}{\epsilon_2}\frac{q''}{R_1} \qquad z<0$$

Now, apply boundary conditions at $z=0$ to find q' and q''.

B.C. I) $\left[\epsilon_1 E_z\right]_{z=0^+} = \left[\epsilon_2 E_z\right]_{z=0^-} \Rightarrow \epsilon_1 \frac{\partial \Phi}{\partial z}\Big|_{z=0^+} = \epsilon_2 \frac{\partial \Phi}{\partial z}\Big|_{z=0^+}$

$$\Rightarrow \left[\frac{+2(d-z)q}{2[r^2+(d-z)^2]^{3/2}} + \frac{-2(d+z)q'}{2[r^2+(d+z)^2]^{3/2}}\right]_{z=0} = +\frac{2(d-z)q''}{2[r^2+(d-z)^2]^{3/2}}\Bigg|_{z=0}$$

$$\Rightarrow \quad q - q' = q'' \qquad (1)$$

B.C. II) $\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{z=0^+} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}_{z=0^-} \Rightarrow \frac{\partial \Phi}{\partial r}\Big|_{z=0^+} = \frac{\partial \Phi}{\partial r}\Big|_{z=0^-}$

$$\Rightarrow \left[\frac{q}{\epsilon_1}\frac{-2r}{2[r^2+(d-z)^2]^{3/2}} + \frac{q'}{\epsilon_1}\frac{-2r}{2[r^2+(d+z)^2]^{3/2}} = \frac{q''}{\epsilon_2}\frac{-2r}{2[r^2+(d-z)^2]^{3/2}}\right]_{z=0}$$

$$\Rightarrow \quad \frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}q'' \qquad (1.5)$$

substituting eq (1) into the expression above yields

$$\frac{1}{\epsilon_1}(q+q') = \frac{1}{\epsilon_2}(q-q')$$

$$\Rightarrow \epsilon_2 q - \epsilon_1 q = -\epsilon_1 q' - \epsilon_2 q'$$

$$\therefore \quad q' = q\left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right) \qquad (2)$$

Fall 2004 #10 (p 3 of 3)

solving eq (1) for q' $\Rightarrow$ $q' = q - q''$ and substituting this into eq (1.5) yields

$$\frac{1}{\epsilon_1}(q + q - q'') = \frac{1}{\epsilon_2} q''$$

$$\Rightarrow 2\epsilon_2 q = \epsilon_1 q'' + \epsilon_2 q''$$

$$\Rightarrow \quad q'' = q \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} \qquad (3)$$

Thus the electrostatic potential everywhere is

$$\boxed{\Phi(r,z) = q \begin{cases} \frac{1}{\epsilon_1}\left[\frac{1}{\sqrt{r^2+(d-z)^2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(d+z)^2}}\right. & z>0 \\ \left(\frac{1}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(d-z)^2}} & z<0 \end{cases}}$$

Fall 2004 #11 (p 1 of 2)

Using general principles, find the radiated power in vacuum of a non-relativistic point charge q whose position is $r(t)$. You do not need to find dimensionless constants (i.e. find the dependence on q, $\vec{r}(t)$, and universal constants).

for $\vec{v} \ll c$,
the fields of a point charge q in arbitrary motion (from Griffiths' section 11.2.1) are

$$\vec{E}(\vec{r},t) = \frac{q\,|r-r'|}{((\vec{r}-\vec{r}')\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + (\vec{r}-\vec{r}')\times(\vec{u}\times\ddot{\vec{r}})\right]$$

where $\vec{u} = c(\vec{r}-\vec{r}') - \vec{v}$ and $\ddot{\vec{r}} = \vec{a}$

$$\vec{B}(\vec{r},t) = (\vec{r}-\vec{r}')\times\vec{E}(\vec{r},t)$$

Now, we know that only accelerated fields represent true radiation. So, the $\vec{E}$-field from radiation is just

$$\vec{E}_{rad}(\vec{r},t) = \frac{q\,|\vec{r}-\vec{r}'|}{[(\vec{r}-\vec{r}')\cdot\vec{u}]^3}\left[(\vec{r}-\vec{r}')\times(\vec{u}\times\ddot{\vec{r}})\right]$$

→ the velocity fields carry energy

So, the poynting vector is

$$\vec{S}_{rad} = \frac{c}{4\pi}(\vec{E}\times\vec{B})_{rad} = \frac{c}{4\pi}|\vec{E}_{rad}|^2 \underbrace{\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}}_{\equiv\hat{n}}$$

If the charge is instantaneously at rest (at time t_r, retarded time), then $\vec{u} = c\hat{n}$

So,

$$\vec{E}_{rad} = \frac{q\,|\vec{n}|}{[\vec{n}\cdot c\hat{n}]^3}\left[\vec{n}\times(c\hat{n}\times\vec{a})\right] = \frac{qc}{c^3 n^2}\left[\hat{n}\times(\hat{n}\times\vec{a})\right]$$

$$\Rightarrow \quad \vec{E}_{rad} = \frac{q}{c^2 r}\left[\hat{r} \times (\hat{r} \times \vec{a})\right]$$

note: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

So, $\hat{r} \times (\hat{r} \times \vec{a}) = \hat{r}(\hat{r} \cdot \vec{a}) - \vec{a}(\hat{r} \cdot \hat{r}) = \hat{r}(\hat{r} \cdot \vec{a}) - \vec{a}$

$$\Rightarrow \vec{E}_{rad} = \frac{q}{c^2 r}\left[(\hat{r} \cdot \vec{a})\hat{r} - \vec{a}\right]$$

So, the Poynting vector is

$$\vec{S}_{rad} = \frac{c}{4\pi}|\vec{E}_{rad}|^2 \hat{r} = \frac{c q^2}{4\pi c^4 r^2}\left[(\hat{r} \cdot \vec{a})\hat{r} - \vec{a}\right]^2 \hat{r}$$

$$= \frac{q^2}{4\pi c^3 r^2}\hat{r}\left(a^2 + (\hat{r} \cdot \vec{a})^2 - 2(\vec{a} \cdot \hat{r})(\hat{r} \cdot \vec{a})\right)$$

$$= \frac{q^2}{4\pi c^3 r^2}\hat{r}\left[a^2 - (\vec{a} \cdot \hat{r})^2\right] \quad , \vec{a} \cdot \hat{r} = |\vec{a}||\hat{r}|\sin\theta$$

$$\Rightarrow \quad \vec{S}_{rad} = \frac{q^2 a^2}{4\pi c^3} \frac{\sin^2\theta}{r^2} \hat{r}$$

, θ is angle between $\vec{a}$ and $\hat{r}$

Then the total power radiated is

$$P = \oint \vec{S}_{rad} \cdot d\vec{a} = \frac{q^2 a^2}{4\pi c^3} \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$$

note: $\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$

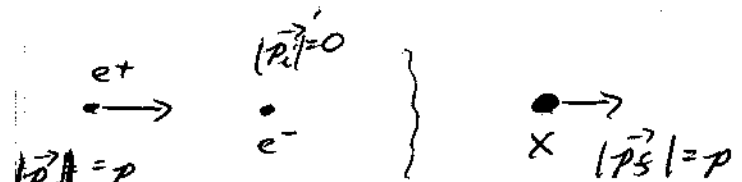
$$\therefore \boxed{P = \frac{2q^2 a^2}{3c^3}}$$

Larmor Formula

EM F'04 #12

a) If one goes into the center of momentum (CM) frame then the total initial momentum will be zero. Hence by momentum conservation the final momentum also has to be zero. For a massive particle X this is possible. But for a photon it is not as it is massless and hence travels at the speed of light. This then would violate momentum conservation. Hence we need another photon traveling in the opposite direction to give a total final momentum of zero.

b) In the lab frame:



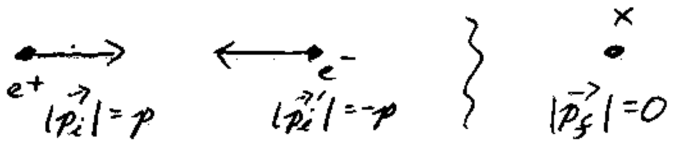
$$E_X^2 = |\vec{p}_f|^2 + M_X^2 \Rightarrow M_X^2 = E_X^2 - |\vec{p}_f|^2$$

now $\quad E_X = E_{e^+} + E_{e^-} = E_{e^+} + m_e \quad$ (leave E_{e^+} as is as we want minimum incident energy).

hence
$$\begin{aligned} M_X^2 &= (E_{e^+} + m_e)^2 - p^2 \\ &= E_{e^+}^2 + 2m_e E_{e^+} + m_e^2 - p^2 \\ &= \underbrace{E_{e^+}^2 - p^2}_{m_e^2} + 2m_e E_{e^+} + m_e^2 = 2m_e E_{e^+} + 2m_e^2 \end{aligned}$$

$$\Rightarrow \quad E_{e^+} = \frac{M_X^2 - 2m_e^2}{2m_e} \approx \frac{M_X^2}{2m_e} = \underline{\frac{M_X}{2}\,\frac{M_X}{m_e}}$$

as for the CM frame



$$E_X^2 = \vec{p}_f^{\,2} + m_X^2 \Rightarrow m_X^2 = E_X^2 - \cancel{p_f^2}^{\,\to 0} = E_X^2$$

so $m_X = E_X$; $E_X = E_{e^+} + E_{e^-} = 2E_{e^+}$ as $E_{e^+} = E_{e^-}$

$$(|\vec{p}_i|^2 + m_e^2)^{1/2} = (\underbrace{|\vec{p}_i'|^2}_{|\vec{p}_i|^2} + m_e^2)^{1/2}$$

so $\boxed{E_{e^+} = \dfrac{m_X}{2}}$

Stat. Mech. F'04 # 14

Assuming 3-D photon gas, need $\frac{\langle E\rangle}{\langle n\rangle}$

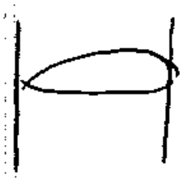
First thing is to find $D(\nu)$ (the energy density) as

$$\langle E\rangle = \int_0^\infty \frac{h\nu}{e^{h\nu/kT}-1} D(\nu)\,d\nu$$

and

$$\langle n\rangle = \int_0^\infty \frac{1}{e^{h\nu/kT}-1} D(\nu)\,d\nu$$

So to find $D(\nu)$



$$\frac{L}{\lambda/2} = n \Rightarrow \lambda = \frac{2L}{n} \quad \text{and} \quad p = \frac{h}{\lambda} = \frac{h}{2L}n$$

also $E = pc \Rightarrow E = h\nu \Rightarrow h\nu = pc \Rightarrow \nu = \frac{cp}{h}$

$$p = (p_x^2 + p_y^2 + p_z^2)^{1/2} \Rightarrow \nu = \frac{c}{h}(p_x^2 + p_y^2 + p_z^2)^{1/2}$$

$$= \frac{c}{h}\frac{h}{2L}(n_x^2 + n_y^2 + n_z^2)^{1/2} = \frac{c}{2L}(n_x^2 + n_y^2 + n_z^2)^{1/2} = \frac{c}{2L}n$$

$$\Rightarrow \nu = \frac{c}{2L}n \Rightarrow n = \frac{2L}{c}\nu$$

polarization

$$\frac{2}{8}\int_0^\infty 4\pi n^2\,dn = \frac{2}{8}\int_0^\infty 4\pi\left(\frac{4L^2}{c^2}\right)\nu^2\,\frac{2L}{c}\,d\nu = \int_0^\infty \frac{8\pi L^3}{c^3}\nu^2\,d\nu$$

So $D(\nu) = \frac{8\pi V}{c^3}\nu^2$

hence

$$\langle E\rangle = \int_0^\infty \frac{h\nu}{e^{h\nu/kT}-1}\,\frac{8\pi V}{c^3}\,\nu^2 d\nu = \frac{8\pi V h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}\,d\nu$$

$$x \equiv \frac{h\nu}{kT} \Rightarrow \nu = \frac{kT}{h}x \Rightarrow d\nu = \frac{kT}{h}dx$$

$$= \frac{8\pi V}{c^3}h\left(\frac{kT}{h}\right)^3\left(\frac{kT}{h}\right)\underbrace{\int_0^\infty \frac{x^3}{e^x-1}\,dx}_{\frac{\pi^4}{15}} = \frac{8\pi^5 V h}{15c^3}\left(\frac{kT}{h}\right)^4$$

$$\langle N\rangle = \int_0^\infty \frac{1}{e^{h\nu/kT}-1}\,\frac{8\pi V}{c^3}\,\nu^2 d\nu = \frac{8\pi V}{c^3}\int_0^\infty \frac{\nu^2}{e^{h\nu/kT}-1}\,d\nu$$

$$x \equiv \frac{h\nu}{kT} \Rightarrow \nu = \frac{kT}{h}x \Rightarrow d\nu = \frac{kT}{h}dx$$

$$= \frac{8\pi V}{c^3}\left(\frac{kT}{h}\right)^3\underbrace{\int_0^\infty \frac{x^2}{e^x-1}\,dx}_{2.404} = \frac{8\pi V}{c^3}\left(\frac{kT}{h}\right)^3 2.404$$

hence

$$\frac{\langle E\rangle}{\langle N\rangle} = \frac{\frac{\cancel{8\pi}^{4}\cancel{V}}{15\cancel{c^3}}\cancel{\left(\frac{kT}{h}\right)^3}kT}{\frac{\cancel{8\pi V}}{\cancel{c^3}}\cancel{\left(\frac{kT}{h}\right)^3}2.404} = \frac{\pi^4}{15\cdot 2.404}kT \approx 2.7\,kT$$