14. *Statistical Mechanics and Thermodynamics* (Fall 2005)

Consider a classical system of N nonrelativistic charged particles in the presence of a constant external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ at temperature T.

(a) Write down the partition function for the system.

(b) Compute the induced magnetization of the system along the direction of $\mathbf{B}$. From this you can answer the question whether paramagnetism occurs in classical physics.

(a) Classical Stat Mech! Careful!

$$U = -\vec{m}\cdot\vec{B} = -mB\cos\theta \quad \text{(not QM spin!)}$$

While not specified in the problem, I'm going to assume that this is a rigid rotor (couldn't find a non-rigid rotor method of solving)

$$K_{rot} = \frac{1}{2I}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right)$$

$$Z = \iint \frac{d^2q\,d^2p}{(2\pi\hbar)^2}\, e^{-\beta\left(\frac{1}{2I}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right) - mB\cos\theta\right)}$$

$$= \frac{1}{2\pi\hbar^2}\int_0^\pi d\theta\, e^{\beta mB\cos\theta}\int_{-\infty}^{\infty} dp_\theta dp_\varphi\, e^{-\frac{\beta}{2I}\left(p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta}\right)}$$

$$= \frac{1}{2\pi\hbar^2}\int_0^\pi d\theta\, e^{\beta mB\cos\theta}\sqrt{\frac{2I\pi}{\beta}}\int_{-\infty}^{\infty} dp_\varphi\, e^{-\frac{\beta}{2I}\frac{p_\varphi^2}{\sin^2\theta}} \qquad \sqrt{\pi/(\beta/2I)}$$

$$= \frac{1}{2\pi\hbar^2}\int_0^\pi d\theta\, e^{\beta mB\cos\theta}\left(\frac{2I\pi}{\beta}\right)\sin\theta \qquad \sqrt{\pi/\left(\frac{\beta}{2I\sin^2\theta}\right)}$$

$$= \frac{1}{\cancel{2\pi}\hbar^2}\left(\frac{\cancel{2}I\cancel{\pi}}{\beta}\right)\int_{-1}^{1} dx\, e^{\beta mBx} = \frac{1}{\hbar^2}\left(\frac{I}{\beta}\right)\left(\frac{1}{\beta mB}\right)2\sinh(\beta mB)$$

$$\boxed{Z_1 = \left(\frac{2I}{\beta mB\hbar^2}\right)\sinh(\beta mB) \qquad Z_N = (Z_1)^N}$$

(b) $F = -kT\ln Z$ and $F = U - TS - MB$

$$\left(\frac{dF}{dB}\right)_{T,V,N} = -M$$

$$M_z = kT\frac{\partial \ln Z}{\partial B} = NkT\left(\frac{1}{Z}\right)\frac{\partial Z}{\partial B}$$

$$= NkT\left(\frac{1}{Z}\right)\left(\frac{2I}{\beta m\hbar^2}\right)\left\{\frac{\beta m}{B}\cosh(\beta mB) - \frac{1}{B^2}\sinh(\beta mB)\right\}$$

$$= N\cancel{k}T\left(\frac{\cancel{B}}{\cancel{\sinh(x)}}\right)\left(\frac{\beta m}{\cancel{B}}\right)\cancel{\sinh(x)}\left\{\coth(x) - \frac{1}{x}\right\}$$

$$\boxed{M_z = Nm\left\{\coth(\beta mB) - \frac{1}{\beta mB}\right\}}$$

$\beta mB > 1 \Rightarrow$ $\boxed{mB > kT \text{ paramagnetism occurs classically}}$