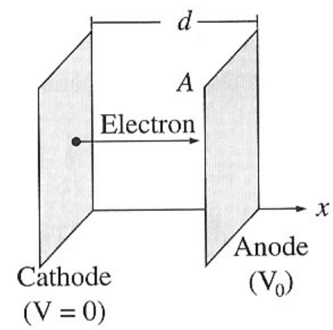
Q1 In a vacuum diode, electrons are boiled off a hot cathode, at potential zero, and accelerated across a gap to the anode, which is held at positive potential V_0 (see the figure below). The cloud of moving electrons within the gap (called the space charge) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then a steady current I flows between the plates. Suppose the plates are large relative to the separation ($A >> d^2$), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

a. Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$? (5 points)
b. In the steady state, I is independent of x. What is the relation between ρ and v? (5 points)
c. Use the results in (a) and (b) to obtain a differential equation for V, by eliminating ρ and v. (5 points)
d. Solve this equation for V as a function of x, V_0 and d. (Hint: you may use the identity, $\frac{d\Phi}{dx}\frac{d^2\Phi}{dx^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{d\Phi}{dx}\right)^2$.) (10 points)
e. Show that $I = kV_0^{3/2}$ and find the constant k. This equation is called the *Child-Langmuir law*. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is nonlinear and doesn't obey the *Ohm's law*.) (5 points)



Disc 210A, Q1:

a) $\frac{1}{2} m v^2(x) = q\phi(x)$

$\Rightarrow v(x) = \sqrt{\frac{2q\phi(x)}{m}}$

b) $I = \rho(x) A v(x) = \text{const.}$, where $\rho(x) =$ charge density

$\Rightarrow \rho(x) = \frac{I}{A v(x)} = \frac{I}{A\sqrt{\frac{2q}{m}}} \phi^{-\frac{1}{2}}(x)$

c) Need one more equation$(\rho(x), \phi(x))$, so take $-\partial_x^2 \phi(x) = \rho/\varepsilon_0$.

$$\partial_x^2 \phi(x) = \underbrace{\frac{-I}{A\sqrt{\frac{2q}{m}}\,\varepsilon_0}}_{-k = I k_0} \phi^{-1/2}(x)$$

d) $\exists$ two methods:

(I) Guess:

$\phi(x) = c x^n$ (Power Law)

Find n by "shaking" $\phi(x)$ with double derivative until x-dependence falls out:

$$\partial_x^2 \phi(x) = c\, n(n-1)\, x^{n-2} = n(n-1)\frac{\phi(x)}{x^2} = -k\,\phi^{-1/2}(x) \Rightarrow \phi^{3/2}(x) = -\frac{k}{n(n-1)} x^2$$

$\Rightarrow \phi(x) \propto x^{4/3}$, $n = 4/3$

Find c by second B.C. (first B.C. already satisfied by guess, $\phi(0) = 0$):

$\phi(d) = \phi_0$, so $c = \frac{\phi_0}{d^{4/3}}$

$$\Rightarrow \boxed{\phi(x) = \phi_0 \left(\frac{x}{d}\right)^{4/3}}$$

Scanned by CamScanner

(II) Use Identity:

$$\partial_x \phi \, \partial_x^2 \phi = \underbrace{\tfrac{1}{2} \partial_x (\partial_x \phi)^2 = -k \, \phi^{-1/2} \, \partial_x \phi}_{\text{Integrate } \int dx}$$

$$\Rightarrow \quad \frac{1}{2} \int d(\partial_x \phi)^2 = \frac{1}{2}(\partial_x \phi)^2 = -2k \, \phi^{1/2}$$

Take sqrt and integrate after rearranging

$$\Rightarrow \quad \partial_x \phi = 2\sqrt{-k} \, \phi^{1/4}$$

$$\Rightarrow \quad \int \phi^{-1/4} \, d\phi = 2\sqrt{-k} \int dx$$

$$\Rightarrow \quad \frac{4}{3} \phi^{3/4} = 2\sqrt{-k} \, x$$

$$\Rightarrow \quad \phi(x)^{3/2} = \frac{-k}{4/9} x^2$$ (squared to check with Method (I))

$$\Rightarrow \quad \phi(x) \propto x^{4/3}, \quad n = 4/3$$

2^{nd} B.C. $\phi(d) = \phi_0$ gives $\boxed{\phi(x) = \phi_0 \left(\frac{x}{d}\right)^{4/3}}$

e) Show $I = k_2 V_0^{3/2}$, find k_2.

⇒ plug in $\phi(x)$ to result from c) to give

$$\phi(x)^{3/2} = -\frac{k_0 I}{4/9} x^2 = \phi_0^{3/2} \frac{x^2}{d^2}$$

$$\Rightarrow \quad \boxed{I = \underbrace{-\frac{9A\sqrt{\frac{2q}{m}}\,\varepsilon_0}{4d^2}}_{k_2} \phi_0^{3/2}}$$