1. *Quantum Mechanics* (Spring 2005)

Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

(a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory.

(b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

2. *Quantum Mechanics* (Spring 2005)

Show that in one space dimension any attractive potential, no matter how weak, always has at least one bound state. *Hint:* Use the variational principle with some appropriate trial wave function such as the normalized Gaussian

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where b is a parameter.

3. *Quantum Mechanics* (Spring 2005)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r = a$.

(a) What is the S-wave ($l = 0$) phase shift as a function of the incident energy or momentum?

(b) What is the total cross section in the limit of zero incident kinetic energy?

4. *Quantum Mechanics* (Spring 2005)

An electron is at rest in a constant magnetic field pointing along the z direction. The Hamiltonian is

$$H = -\mu \cdot \mathbf{B} = g\mu_o \frac{\mathbf{s}}{\hbar} \cdot \mathbf{B}$$

where $\mathbf{B} = B_o \hat{n}_z$. $\mathbf{s}$ is the electron spin. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

(a) What are the eigenstates of the Hamiltonian, and what is the energy difference between them?

(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. Calculate $|\psi(t)\rangle$ for any t.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

5. *Quantum Mechanics* (Spring 2005)

An electron moves in a hydrogen atom potential – ignoring spin and relativity – in a state $|\psi\rangle$ that has the wave function

$$\psi(r,\theta,\phi) = NR_{21}(r)\left[2iY_1^{-1}(\theta,\phi) + (2+i)Y_1^0(\theta,\phi) + 3iY_1^1(\theta,\phi)\right]$$

where the $Y_l^m(\theta,\phi)$ are the spherical harmonics, $R_{nl}(r)$ are the normalized hydrogen atom wave functions, and N is a positive real number.

(a) Calculate N.

(b) What is the expectation value of L_z? ($\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$)

(c) What is the expectation value of $\mathbf{L}^2$?

(d) What is the expectation value of the kinetic energy in terms of $\hbar, c$, the electron charge e or the fine-structure constant α, and the electron mass m?

Note: The explicit forms of the functions that appear in $\psi(r,\theta,\phi)$ above are

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{r}{a^{5/2}}e^{-r/2a} \qquad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi} \qquad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$

6. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A closed container is divided by a wall into two equal parts (A and B), each of volume $V/2$. Part A contains an ideal gas with $N/2$ molecules of mass M_1 while part B contains an ideal gas with $N/2$ molecules of mass M_2. The container is kept at a fixed temperature T. The molecules of each kind are all identical, but distinguishable from the molecules of the other kind.

(a) The partition function $Z(N)$ of an ideal gas of N particles of mass M in a volume V is given by

$$Z(N) = \frac{1}{N!}\left(\frac{V}{\sqrt{2\pi\hbar^2/Mk_BT}}\right)^N$$

Give the partition function of the gas in the container before and after the wall is removed. What are the entropy and pressure before and after the wall is removed?

(b) How much heat is absorbed or released following the removal of the wall? Is the removal of the wall a reversible or irreversible process?

(c) Same question as (b), but now for the case that the two kinds of molecules are indistinguishable from each other (so $M_1 = M_2$). Compare your answers for (b) and (c) and provide a physical explanation for the difference in entropy between the two cases.

7. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A (nearly) ideal gas with a temperature T and pressure P contains atoms of mass M that are either in the ground state or in the first excited state. An atom that returns to the ground state from the first excited state emits a photon of frequency f_o. For a stationary observer observing the spectral line emitted by a *moving* atom, this frequency is shifted by the Doppler effect to

$$f(v_{||}) = f_o(1 + v_{||}/c)$$

where c is the velocity of light and $v_{||}$ is the projection of the velocity of the atom on the line of sight from the observer to the atom.

(a) What is the statistical distribution $P(f)$ of the frequency of the spectal line? Assume the atoms obey the Maxwell-Boltzmann distribution.

(b) Obtain from $P(f)$ the contribution by the Doppler effect to the width $\sqrt{\langle (f - f_o)^2 \rangle}$ of the spectral line. Can you think of a way this effect could be exploited in the study of stellar atmospheres?

(c) The *natural* line shape $P(f)$ of an atomic spectral line is, according to quantum mechanics, given by

$$P(f) \sim \frac{1}{(f - f_o)^2 + \tau^{-2}}$$

where τ is the *lifetime* of the excited state. For atoms in a dense gas, the actual lifetime of the excited state is not intrinsic, but instead determined by the time interval between successive collisions between atoms. Let the cross section of an atom equal σ. Obtain an expression for τ in terms of σ, the pressure P and the temperature T. Under which conditions will this "collisional" broadening of the spectral line dominate over the Doppler broadening as computed under (b)?

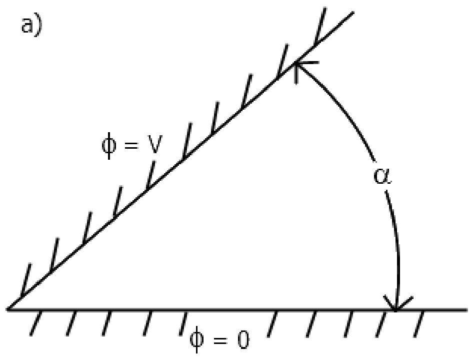
8. *Electricity and Magnetism* (Spring 2005)

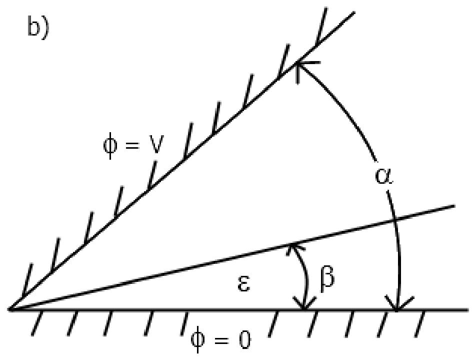
Consider a two-dimensional (r, θ) electrostatic problem consisting of two infinite plates making an angle α with each other and held at a potential difference V, as shown below:

(a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.

Now insert a wedge dielectric, of dielectric coefficient ϵ, and angle β, resting on the bottom plate as shown below:

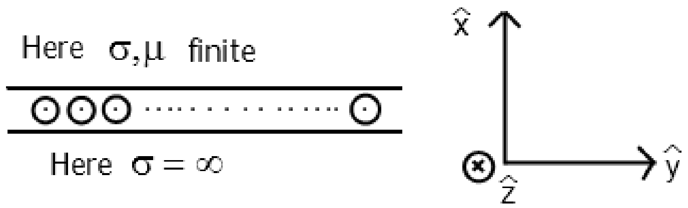
(b) Find the pressure experienced by the bottom plate at a distance r from the apex (from the line joining the two plates).





9. *Electricity and Magnetism* (Spring 2005)

An infinitely thin current sheet carrying a surface current $\lambda = \lambda_o \hat{z} \cos(\omega t)$ is sandwiched between a perfect conductor ($\sigma = \infty$) and a material having finite conductivity σ and magnetic permeability μ. The angular frequency ω is sufficiently low that magnetostatic conditions prevail. λ_o is a constant, $\hat{z}$ is a unit vector parallel to the interface located at $x = 0$, and t is the time.



(a) Find the appropriate partial differential equation that governs the behavior of the magnetic field **H** for $x > 0$ (above the current sheet). Do not solve.

(b) What is the appropriate boundary condition for **H** in this system?

(c) Find the magnetic field **H** at an arbitrary distance $x > 0$ at time t.

10. *Electricity and Magnetism* (Spring 2005)

A relativistic charged particle of charge q and rest-mass m_o is in a region of uniform magnetic field $B_o\hat{z}$. At time $t = 0$ the particle has zero velocity along $\hat{z}$ (that is $\beta_z = v_z/c = 0$) and finite transverse speed $\beta_\perp = \beta_o$, with

$$\beta_\perp = \sqrt{v_x^2 + v_y^2}/c$$

Here, x, y, and z are Cartesian coordinates in the lab frame.

(a) What is the value of $\beta_\perp(t)$ for $t > 0$?

(b) What is the angular frequency Ω of rotation (that is, the gyrofrequency)? No need for a calculation, just identify Ω.

(c) Now apply a uniform electric field $E_o\hat{z}$, parallel to $\mathbf{B}$, starting at $t = 0$. Without solving the detailed equations, conclude what happens to the $\beta_\perp$ in part (a). Does it change?

11. *Electricity and Magnetism* (Spring 2005)

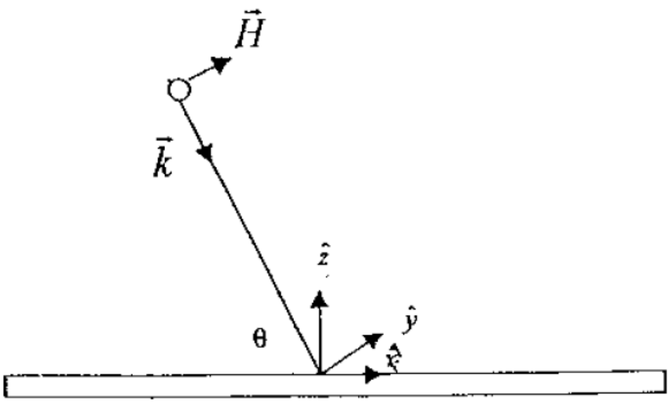
A linearly polarized electromagnetic wave propagating through the vacuum falls on a flat metallic surface. The wavelength of the incident wave is λ. The angle between the wave vector $\mathbf{k}$ and the metal surface is equal to θ. The electric field has a magnitude E_o and a direction normal to the page (positive y direction, see Figure). Assume that the metal surface has infinite conductivity.

(a) Show that the boundary conditions can be obeyed by adding a *reflected* wave to the incident plane wave. Draw, in the Figure, the directions of the electric and magnetic field vectors of the reflected wave such that the boundary conditions hold at the surface.

(b) Calculate the *time-averaged* Poynting vector of the incident plus the reflected wave in terms of E_o. Along what direction is the electromagnetic energy being transported by the two waves?

(c) Show that the repeat length of the interference pattern of the two waves along the surface of the plates is given by $\lambda/\cos(\theta)$, while the repeat length perpendicular to the surface is given by $\lambda/\sin(\theta)$. It follows from this that one can insert a *second* metal plate at a height $D(m) = m(\lambda/2)\sin(\theta)$ above the first metal plate, with m an integer, without perturbing the wave pattern.

(d) Using (c) compute the *phase velocity* $v(f)$ of an electromagnetic wave trapped between two parallel plates with spacing D as a function of the frequency f of the wave. This phase velocity should *diverge* as you reduce f. Demonstrate that the fact that $v(f)$ exceeds the velocity of light for some f is not a violation of the principle of special relativity (even though $v > c$ for small f).



12. *Electricity and Magnetism* (Spring 2005)

A thin copper ring (conductivity σ, density ρ) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. At time $t = 0$ the ring is set rotating with frequency ω_o. Calculate the time it takes the frequency to decrease to $1/e$ of its original value, assuming the energy goes into Joule heating.

13. *Statistical Mechanics and Thermodynamics* (Spring 2005)

Consider the one-dimensional Ising model on a periodic lattice, that is, a chain of N spins, with sping $s_i = \pm 1$ residing on the i-th site, $i = 1, ..., N$, forming a closed loop. The partition function in the presence of an external magnetic field H is then

$$Z_N = \sum_{\{s_i = \pm 1\}} \exp\left(\beta J \sum_{i=1}^{N} s_i s_{i+1} + \beta H \sum_{i=1}^{N} s_i\right)$$

where $\beta = 1/kT$. Define the 2×2 transfer matrix $\mathbf{T}$ with elements

$$T(s, s') = \exp\left[\nu s s' + \frac{B}{2}(s + s')\right] \qquad (s, s' = \pm 1)$$

where we let $\nu = \beta J$ and $B = \beta H$.

(a) Show that

$$Z_N = \mathrm{Tr}(\mathbf{T}^N)$$

and hence

$$Z_N = \lambda_1^N + \lambda_2^N$$

where λ_1 and λ_2 are the two eigenvalues of $\mathbf{T}$.

(b) Determine λ_1, λ_2. If λ_1 denotes the larger eigenvalue, observe that λ_2/λ_1 is strictly less than one for all $\nu > 0$. Hence show that the free energy per spin in the thermodynamic limit $N \to \infty$ is given by

$$-F/kT = \ln(\lambda_1)$$

(c) What is the spontaneous magnetization per spin for any $\nu > 0$?

14. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A photon gas in thermal equilibrium is contained within a box of volume V at temperature T.

(a) Use the partition function to find the average number of photons $\bar{n}_r$ in the state having energy E_r.

(b) Find a relationship between the radiation pressure P and the energy density u (i.e. the average energy per unit volume).

(c) If the volume containing the photon gas is decreased adiabatically by a factor of 8, what is the final pressure if the initial pressure is P_o?

Spring 2005 #1

$H' = -qEx$

a) Calculate correction to second order pertubation theory

$H = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m\omega^2 x^2}{2}\psi$ or $\frac{-1}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m\omega^2x^2}{2}\psi$ in natural units.

$E = \hbar\omega(n+\frac{1}{2})$

$|\psi\rangle = |n\rangle$

1st $\langle n|H'|n\rangle = -qE\langle n|x|n\rangle$ $\quad x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$

$= -\frac{eE}{\sqrt{2m\omega}}[\underbrace{\langle n|a|n\rangle}_{\cancel{\langle n|n-1\rangle}^{0}} + \underbrace{\langle n|a^+|n\rangle}_{\cancel{\langle n|n+1\rangle}^{0}}]$

$a|n\rangle = \sqrt{n}|n-1\rangle$
$a^+|n\rangle = \sqrt{n+1}|n+1\rangle$

$=0$

2nd $\sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_n^0 - E_m^0}$

$\langle m|H'|n\rangle = 0$ onles $m = n+1$ or $m-1$

$\langle n+1|H'|n\rangle = \frac{-eE}{\sqrt{2m\omega}}[\cancel{\langle n+1|a|n\rangle} + \langle n+1|a^+|n\rangle]$

$= \frac{-eE}{\sqrt{2m\omega}}\sqrt{n}$

$\langle n-1|H'|n\rangle = \frac{-eE}{\sqrt{2m\omega}}\sqrt{n+1}$

$\frac{\frac{e^2E^2 n}{2m\omega}}{E_n^0 - E_{n+1}} + \frac{\frac{e^2E^2(n+1)}{2m\omega}}{E_n^0 - E_{n+1}} = \frac{\frac{e^2E^2 n}{2\omega\omega}}{\omega} + \frac{\frac{e^2E^2(n+1)}{2m\omega}}{\omega}$

$\omega(n+\frac{1}{2}) - \omega(n+1+\frac{1}{2})$
$\omega n + \frac{\omega}{2} - \omega n - \frac{3}{2} = -\omega$

$$E^{(2)} = -\frac{q^2 E^2}{2m\omega^2}$$

b) Find Exact energy

$$-\frac{1}{2m}\frac{d^2\psi}{dx^2} + \underbrace{\frac{m\omega^2 x^2}{2}\psi - qEx\psi}_{} = E\psi$$

rewrite to make it look like the harmonic oscillator

$$\frac{m\omega^2}{2}\left[x^2 - 2\frac{qE}{m\omega^2}x\right] = \frac{m\omega^2}{2}\left[x^2 - 2\frac{qEx}{m\omega^2} + \left(\frac{qE}{m\omega^2}\right)^2 - \left(\frac{qE}{m\omega^2}\right)^2\right]$$

$$= \frac{m\omega^2}{2}\left[x - \frac{qE}{m\omega^2}\right]^2 - \underbrace{\frac{m\omega^2}{2}\frac{q^2E^2}{m^2\omega^4}}_{\frac{q^2E^2}{2m\omega^2}}$$

$$-\frac{1}{2m}\frac{d^2\psi}{dx^2} + \frac{m\omega^2}{2}\left(x - \frac{qE}{m\omega^2}\right)^2\psi = \underbrace{\left(E + \frac{q^2E^2}{2m\omega^2}\right)}_{E'}\psi$$

$$u = x - \frac{qE}{m\omega^2} \qquad du = dx$$

$$-\frac{1}{2m}\frac{d^2\psi}{du^2} + \frac{m\omega^2 u^2}{2}\psi = E'\psi \Rightarrow E' = \omega(n+1/2) = E + \frac{q^2E^2}{2m\omega^2}$$

$$E = \omega(n+1/2) - \frac{q^2E^2}{2m\omega^2} = E^0 + E^2$$

Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

Part a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory. (see Yang-Kuo Lim # 5011)

(i) 1st-order

$$E_n^{(1)} = \langle H' \rangle = \langle n|H'|n\rangle = -qE\langle n|x|n\rangle = 0$$

note: we immediately know that this is zero since the only way for it to be non-zero is if the wave functions have opposite parity (since x is odd).

(ii) 2nd-order

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_n^\circ - E_m^\circ}$$

where $E_n^\circ = \omega(n+\frac{1}{2})$ & $E_m^\circ = \omega(m+\frac{1}{2}) \Rightarrow E_n^0 - E_m^\circ = \omega(n-m)$

and

$$\langle m|H'|n\rangle = -qE\langle m|x|n\rangle = \frac{-qE}{\sqrt{2m\omega}}\langle m|(a+a^+)|n\rangle$$

$$= \frac{-qE}{\sqrt{2m\omega}}\left[\langle m|a|n\rangle + \langle m|a^+|n\rangle\right]$$

$$\Rightarrow \langle m|H'|n\rangle = \frac{-qE}{\sqrt{2m\omega}}\left[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right]$$

So,

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m|H'|n\rangle|^2}{E_n^0 - E_m^0} = \frac{q^2E^2}{2m\omega}\left[\frac{n}{\omega(n-n+1)} + \frac{n+1}{\omega(n-n-1)}\right]$$

$$\Rightarrow \quad E_n^{(2)} = \frac{q^2 E^2}{2m\omega}\left[\frac{n}{\omega} - \frac{n+1}{\omega}\right]$$

$$\therefore \quad \boxed{E_n^{(2)} = -\frac{q^2 E^2}{2m\omega^2}}$$

part b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

(see Zettili example 9.1 (p473))

the total Hamiltonian is given by

$$H = H_0 + H' = \left(-\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right) - qEx$$

let $y = x - \frac{qE}{m\omega^2} \Rightarrow y^2 = x^2 + \frac{q^2E^2}{(m\omega^2)^2} - \frac{2qE}{m\omega^2}x$

so,

$$\frac{1}{2}m\omega^2 y^2 - \frac{q^2E^2}{2m\omega^2} = \frac{1}{2}m\omega x^2 + \frac{q^2E^2}{2m\omega^2} - qEx - \frac{q^2E^2}{2m\omega^2}$$

$$= \frac{1}{2}m\omega^2 x^2 - qEx \quad \checkmark$$

Thus, by this substitution our Hamiltonian is

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 y^2 - \frac{q^2E^2}{2m\omega^2}$$

This is now the Hamiltonian of a harmonic oscillator from which a constant is subtracted. So, the exact eigenstates are

$$\boxed{E_n = \omega\left(n + \frac{1}{2}\right) - \frac{q^2E^2}{2m\omega^2}}$$

→ this agrees exactly with the result of part (a).

Spring 2005 #2

$H = T+V \quad H_s = T + V_s \quad H_s, V_s$ belong to a one dim. attractive square well. which always has a bound state.

$\Rightarrow \int \psi_s^* (T+V_s)\psi_s \, dx = E_s \quad (1)$

E_0 is ground state for V, Lets use ψ_s as a trail wave function

(2) $\int \psi_s^* (T+V) \psi_s \, dx \geq E_0$ [main point of variation method]

Sutract (1) from (2)

$\int \psi_s^* (V-V_s)\psi_s \, dx \geq E_0 - E_s$ Since V is negative for all x and V_s always has a bound state no matter what size it is.

$\Rightarrow (V-V_s)$ negative for all x

$\Rightarrow$ negative amount $\geq E_0 - E_s$, or

$E_0 \leq$ negative amount $+ E_s$ And E_s is negative

$\Rightarrow E_0$ is always negative and therefor bound.

Show that in 1-D any attractive potential, no matter how weak, always has at least one bound state. Hint: use variational principle with some appropriate trial wave function such as

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where b is a parameter.

(see Yung-Kuo Lim QM # 8020) ... this solution is a bit suspect.

attractive potential $\Rightarrow V(x) < 0$.

The Hamiltonian is given by

$$H = \frac{-1}{2m}\frac{d^2}{dx^2} + V(x) = \frac{-1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 + V(x) - \frac{1}{2}m\omega^2x^2$$

so,

$$E = \langle\psi(x)|H|\psi(x)\rangle = -\frac{1}{2m}\left\langle\frac{d^2}{dx^2}\right\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle + \langle V(x)\rangle - \frac{1}{2}m\omega^2\langle x^2\rangle$$

$$= \langle\psi(x)|H_0|\psi(x)\rangle + \langle\psi(x)|H'|\psi(x)\rangle$$

The reason we wrote the Hamiltonian in this way is because we already know $\langle H_0\rangle$ which is the ground state of the Harmonic oscillator. So, we have

$$E = \frac{\omega}{2} + \langle V(x)\rangle - \frac{1}{2}m\omega\langle x^2\rangle$$

where

$$\langle x^2\rangle = \left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} x^2 e^{-2bx^2}dx = \left(\frac{2b}{\pi}\right)^{1/2}\frac{\sqrt{\pi}}{2(2b)^{3/2}} = \frac{1}{2}\left(\frac{1}{2b}\right) = \frac{1}{4b}$$

so,

$$-\frac{1}{2}m\omega^2\langle x^2\rangle = \frac{-m\omega^2}{8b} = \frac{-\omega}{4}$$

$\Rightarrow b = \frac{m\omega}{2}$ to fit the Harmonic oscillator treatment.

So,

$$E = \frac{\omega}{4} + \langle V(x) \rangle = \frac{b}{2m} + \langle V(x) \rangle \qquad (1)$$

Then,

$$\frac{\partial E}{\partial b} = \frac{1}{2m} + \frac{\partial}{\partial b}\left[\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-2bx^2} V(x)\,dx\right]$$

$$= \frac{1}{2m} + \underbrace{\frac{1}{2}\sqrt{\frac{2}{\pi b}}\int_{-\infty}^{\infty} e^{-2bx^2} V(x)\,dx}_{\equiv I_1} - \underbrace{2\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-2bx^2} V(x)\,dx}_{\equiv I_2}$$

(since $\frac{\sqrt{b}}{b} = \frac{1}{\sqrt{b}}$)

$$\therefore \frac{\partial E}{\partial b} = \frac{1}{2m} + \frac{1}{2b}\langle V(x) \rangle - 2\langle x^2 V(x) \rangle$$

note: For attractive well, we must have

$$\int_{-\infty}^{\infty} V(x)\,dx \text{ is finite}$$

and

$$\int_{-\infty}^{\infty} x^2 V(x)\,dx \text{ is finite}$$

So,

$$\lim_{b\to 0} I_1 \to \frac{1}{0}\int_{-\infty}^{\infty} V(x)\,dx \to -\infty \qquad (\text{since } V(x) < 0)$$

$$\lim_{b\to 0} I_2 \to 0\int_{-\infty}^{\infty} x^2 V(x)\,dx \to 0$$

Thus,

$$\boxed{\lim_{b\to 0} \frac{\partial E}{\partial b} \to -\infty}$$

Now,
$$\lim_{b\to\infty} I_1 \to \frac{1}{\infty}\int_{-\infty}^{\infty} 0\cdot V(x)\,dx \to 0$$

$$\lim_{b\to\infty} I_2 \to \infty \int_{-\infty}^{\infty} 0\cdot x^2 V(x)\,dx \to 0$$

goes to zero faster than goes to ∞

Thus,
$$\boxed{\lim_{b\to\infty} \frac{\partial E}{\partial b} \to \frac{1}{2m} > 0}$$

The range of these limits shows us that $\frac{\partial E}{\partial b} = 0$ for some value of b. let us call that value of b, b_0. That is,

$$\frac{\partial E}{\partial b} = 0 = \frac{1}{2m} + \frac{1}{2b_0}\langle V(x)\rangle - 2\langle x^2 V(x)\rangle$$

Solving for $\langle V(x)\rangle$ yields

$$\langle V(x)\rangle = 2b_0\left[2\langle x^2 V(x)\rangle - \frac{1}{2m}\right] \qquad (2)$$

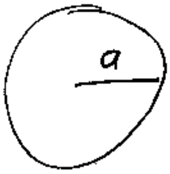
Thus, the energy is (substitute eq (2) into eq (1))

$$E = \langle H(b_0)\rangle = \frac{b_0}{2m} + 4b_0\langle x^2 V(x)\rangle - \frac{b_0}{m}$$

$$\Rightarrow \boxed{E = -\frac{b_0}{2m} + 4b_0\langle x^2 V(x)\rangle < 0} \quad , V(x) < 0$$

∴ the system has at least one bound state!

Spring 2005 #3



$V(r) = V_0\, \Theta(a-r)$ with $V_0 = \infty$

Radial wave equation (with $\hbar = 1$)

$U = r R(r)$

$\lambda = 2mV(r)$

$k^2 = 2mE$

$$U''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) U - \lambda U = 0$$

$0 \quad r<a$

$$U''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) U = 0 \quad r>a$$

$$R_\ell(r) = a_\ell\, j_\ell(kr) + b_\ell\, n_\ell(kr)$$

j_ℓ = spherical Bessel function

n_ℓ = neuman functions

$$R_\ell(a) = 0 = a_\ell\, j_\ell(ka) + b_\ell\, n_\ell(ka)$$

$$a_\ell\, j_\ell(ka) = -b_\ell\, n_\ell(ka)$$

$$\underset{r\to\infty}{j_\ell(kr)} = \frac{\cos\left[kr - (\ell+1)\frac{\pi}{2}\right]}{kr} = \frac{\sin(kr - \frac{\pi\ell}{2})}{kr}$$

$$\underset{r\to\infty}{n_\ell(kr)} = \frac{\sin\left[kr - (\ell+1)\frac{\pi}{2}\right]}{kr} = -\frac{\cos(kr - \frac{\ell\pi}{2})}{kr}$$

$$R_\ell(r) = \frac{a_\ell \sin(kr - \frac{\pi\ell}{2})}{kr} - \frac{b_\ell \cos(kr - \frac{\ell\pi}{2})}{kr}$$

But then we know that another form of the solution is

$$R_\ell(r) = \frac{e^{i\delta_\ell}}{kr}\sin\left(kr - \frac{\pi\ell}{2} + \delta_\ell\right) \qquad \text{with } \tan\delta_\ell = \frac{-b_\ell}{a_\ell} = \frac{j_\ell(ka)}{n_\ell(ka)}$$

arbitrary phase constant.

This works since $\sin(kr - \frac{\pi\ell}{2} + \delta_\ell) = \sin(kr - \frac{\pi\ell}{2})\cos\delta_\ell + \cos(kr - \frac{\pi\ell}{2})\sin\delta_\ell$

Then by comarison $\cos\delta_\ell = a_\ell$

$\sin\delta_\ell = -b_\ell$

But now, we want $\ell = 0$ case.

$$\Rightarrow \tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = -\tan(ka). \quad \Rightarrow \delta_0 = -ka$$

For small k

$$\sigma_{tot} = \frac{4\pi}{k^2}\sin^2\delta_0 \, ka \approx \frac{4\pi(ka)^2}{k^2} = 4\pi a^2$$

small angle approximation.

Spring 2005 #3 (p 1 of 2)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is zero outside the sphere, infinite inside. The wave function must vanish at $r=a$.

What is the S-wave ($\ell=0$) phase shift as a function of the incident energy or momentum? What is the total cross section in the limit of zero incident kinetic energy?

see spring 1999 #13. & Abers p 283, 284

The potential we are given has the form

$$V(r) = V_0 \,\Theta(a-r) \quad \text{in the limit } V_0 \to \infty$$

we know that the radial wave equation is

$$\frac{d^2u}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)u - \mathcal{V}u = 0 \quad , \; k^2 = 2mE \;\; \& \;\; \mathcal{V} = 2mV(r)$$

where $u = rR(r)$.

substituting in for the value of the potential yields

$$\frac{d^2u}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right]u = 0 \qquad r>a$$

there is no equation for $r<a$ since it is an impenetrable sphere. The solutions to this D.E. are

$$R_\ell(r) = a_\ell\, j_\ell(kr) + b_\ell\, n_\ell(kr) \qquad (1)$$

we are told that

$$R_\ell(r=a) = 0 = a_\ell\, j_\ell(ka) + b_\ell\, n_\ell(ka)$$

$$\Rightarrow \boxed{\frac{-b_\ell}{a_\ell} = \frac{j_\ell(ka)}{n_\ell(ka)} \equiv \tan\delta_\ell} \qquad (2)$$

Spring 2005 #3 (p 2 of 2)

as $r \to \infty$, our solution given by eq (1) becomes

$$R_\ell(r) = \frac{a_\ell \sin(kr - \frac{\pi \ell}{2})}{kr} - \frac{b_\ell \cos(kr - \pi\ell/2)}{kr}$$

We know from solutions to D.E.s that we can also write this solution as

$$R_\ell(r) = \frac{e^{i\delta_\ell}}{kr} \sin\left(kr - \frac{\pi \ell}{2} + \delta_\ell\right)$$

where $e^{i\delta_\ell}$ is a phase shift to make sure that the outgoing wave is only due to the scattering and not the plane wave.

Now, we want to find the s-wave ($\ell = 0$) phase shift. From eq 2) this becomes

$$\tan\delta_0 = \frac{j_0(ka)}{n_0(ka)} = \frac{\frac{\sin ka}{ka}}{-\frac{\cos ka}{ka}} = -\tan ka$$

$$\therefore \boxed{\delta_0 = -ka} \quad , \; k = \sqrt{2mE}$$

Now, we want the total cross section. From Abers eq 8.121 we know that in the limit as $k \to 0$, σ_{tot} is given by

$$\sigma_{tot} = \frac{4\pi}{k^2} \sin^2\delta_0$$

since $\delta_0 \ll 1$ for our case (since $k \to 0$) we have

$$\boxed{\sigma_{tot} = \frac{4\pi}{k^2}\delta_0^2 = 4\pi a^2}$$

Spring 2005 # 4

$H = g\mu_0 \frac{S}{\hbar}\cdot B \qquad B = B_0 \hat{z} \qquad \hbar = 1 \qquad \sigma = 2S$

a) $H = g\mu_0 B_0 S_z = \frac{g\mu_0 B_0 \sigma_z}{2}$

$\lambda = \pm \frac{g\mu_0 B_0}{2} \qquad |+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\Delta E = g\mu_0 B_0$

b) $t=0 \qquad |\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]$

$\psi(t) = e^{-iHt}|\psi(0)\rangle$

$= \frac{e^{-iHt}}{\sqrt{2}}|+\rangle + \frac{e^{-iHt}}{\sqrt{2}}|-\rangle = \begin{pmatrix} e^{-ig\frac{\mu_0}{2}B_0 t} \\ e^{ig\frac{\mu_0 B_0}{2}t}\end{pmatrix}\frac{1}{\sqrt{2}} \qquad \omega = \frac{g\mu_0 B_0}{2}$

$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t}\end{pmatrix}$

c) $\langle S_x\rangle = \langle \frac{\sigma_x}{2}\rangle = \frac{1}{\sqrt{2}}\left(e^{+i\omega t} \; e^{-i\omega t}\right)\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t}\end{pmatrix}$

$= \frac{1}{4}\left(e^{+i\omega t} \; e^{-i\omega t}\right)\begin{pmatrix} e^{i\omega t} \\ e^{-i\omega t}\end{pmatrix} = \frac{1}{4}\left(e^{i2\omega t} + e^{-i2\omega t}\right) = \frac{1}{2}\cos(2\omega t)$

$$\langle S_y \rangle = \frac{i}{4}\left(e^{i\omega t}\ \ e^{-i\omega t}\right)\underbrace{\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{-i\omega t}\\ e^{i\omega t}\end{pmatrix}}$$

$$= \frac{i}{4}\left(e^{i\omega t}\ \ e^{-i\omega t}\right)\begin{pmatrix}-e^{i\omega t}\\ e^{-i\omega t}\end{pmatrix} = \frac{i}{4}\left(-e^{i2\omega t} + e^{-i2\omega t}\right)$$

$$= \frac{1}{4i}\left(e^{i2\omega t} - e^{-i2\omega t}\right) = \frac{1}{2}\sin(2\omega t)$$

$$\langle S_z \rangle \underset{\substack{\uparrow\\ \text{should}}}{=} 0 \quad \frac{1}{4}\left(e^{i\omega t}\ \ e^{-i\omega t}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}e^{-i\omega t}\\ e^{i\omega t}\end{pmatrix}$$

$$= \frac{1}{4}\left(e^{i\omega t}\ \ e^{-i\omega t}\right)\begin{pmatrix}e^{-i\omega t}\\ -e^{i\omega t}\end{pmatrix} = \frac{1}{4}\left(1 - 1\right) = 0 \checkmark$$

Spring 2005 #4 (p 1 of 2)

A electron is at rest in a constant magnetic field pointing in the z-direction. The Hamiltonian is then

$$H = -\vec{\mu}\cdot\vec{B} = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B}$$

where $\vec{B} = B_0\hat{z}$.

Let $|\psi_\pm\rangle$ be the eigenstates of S_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

part a) What are the eigenstates of the Hamiltonian, and what is the energy difference between them?

(See spring 1999 #11)

$\vec{B} = B_0\hat{z}$ $$H = \frac{g\mu_0 B_0}{2}\sigma_z = \frac{g\mu_0 B_0}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Since this is a diagonal matrix, the eigenvalues are just the elements along the diagonal. That is,

$$\lambda = \pm\frac{g\mu_0 B_0}{2}$$

So the eigenstates are

$$|\lambda = -g\tfrac{\mu_0 B_0}{2}\rangle : \begin{pmatrix} g\mu_0 B_0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = 0 \Rightarrow |\lambda_-\rangle = \begin{pmatrix}0\\1\end{pmatrix} \equiv |\psi_-\rangle$$

$$|\lambda = +g\tfrac{\mu_0 B_0}{2}\rangle : \begin{pmatrix} 0 & 0\\ 0 & -g\mu_0 B_0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = 0 \Rightarrow |\lambda_+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \equiv |\psi_+\rangle$$

the energy difference is $\boxed{\Delta E = g\mu_0 B_0}$

(b) At time $t=0$ the electron is in an eigenstate of S_x with eigenvalue $\hbar/2$. Calculate $|\psi(t)\rangle$ for any t.

$$S_x = \frac{\sigma_x}{2}, \quad \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \Rightarrow \lambda_{\sigma_x} = \pm 1 \;;\; |\lambda = +1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

(see S99 #11 for details)

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|\psi_+\rangle + \frac{1}{\sqrt{2}}|\psi_-\rangle$$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-iH_+ t}|\psi_+\rangle + \frac{1}{\sqrt{2}} e^{-iH_- t}|\psi_-\rangle$$

$$\therefore \boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-ig\frac{\mu_0 B_0}{2}} |\psi_+\rangle + \frac{1}{\sqrt{2}} e^{+ig\frac{\mu_0 B_0}{2}} |\psi_-\rangle}$$

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t? let $\alpha = g\frac{\mu_0 B_0}{2}$

(i) $\langle S_x \rangle$

$$\langle S_x\rangle = \langle\psi(t)|S_x|\psi(t)\rangle = \frac{1}{2}\langle\sigma_x\rangle = \frac{1}{4}\left(e^{i\alpha t}\ e^{-i\alpha t}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix}$$

$$= \frac{1}{4}\left(e^{i\alpha t}\ e^{-i\alpha t}\right)\begin{pmatrix}e^{i\alpha t}\\ e^{-i\alpha t}\end{pmatrix} = \frac{1}{4}\left(e^{i2\alpha t} + e^{-i2\alpha t}\right)$$

$$\therefore \boxed{\langle S_x\rangle = \frac{1}{2}\cos(2\alpha t) = \frac{1}{2}\cos(g\mu_0 B_0 t)}$$

(ii) $\langle S_y \rangle$

$$\langle S_y\rangle = \frac{1}{2}\langle\sigma_y\rangle = \frac{1}{4}\left(e^{i\alpha t}\ e^{-i\alpha t}\right)\begin{pmatrix}0&-i\\+i&0\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix}$$

$$\Rightarrow \boxed{\langle S_y\rangle = \frac{1}{2}\sin(g\mu_0 B_0 t)}$$

(iii) $\langle S_z \rangle$

$$\boxed{\langle S_z\rangle = 0} \quad , \langle S_z\rangle = \frac{1}{4}\left(e^{i\alpha t}\ e^{-i\alpha t}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}e^{-i\alpha t}\\ e^{i\alpha t}\end{pmatrix} = 0$$

Spring 2005 #5 natural units $\hbar = c = 1$

$$\psi(r,\theta,\phi) = N\, R_{21}(r)\left[2i\, Y_1^{-1} + (2+i)\, Y_1^{0} + 3i\, Y_1^{1}\right]$$

Y_ℓ^m, $R_{n\ell}$ Remember Y_ℓ^m's are normalized

$\int Y_\ell^{*m} Y_\ell^m \sin\theta\, d\theta\, d\phi = 1$

a) Find N

$$1 = N^2 \int \psi^* \psi\, d^3r = N^2 \Big[\int_0^\infty R_{21}^2(r)\,(-2i)(2i)\, Y_\ell^{*-1} Y_\ell\, r^2 \sin\theta\, d\theta\, d\phi$$

$$+ 5\int_0^\infty R_{21}^2 r^2 dr + 9\int_0^\infty R_{21}^2 r^2 dr\Big]$$

$$= 18 N^2 \int_0^\infty |R_{21}|^2 r^2 dr$$ But Radial wave equations are also normalized.

$18N^2 = 1 \quad N^2 = \frac{1}{18} \quad N = \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$

b) $L_z Y_\ell^m = m\, Y_\ell^m$

$$\langle\psi|L_z|\psi\rangle = N^2 \int |R_{21}|^2 \left[(4)\, Y_1^{*-1} L_z Y_1^{-1} + 5\, Y_1^{*0} L_z Y_1^0 + 9\, Y_1^{1*} L_z Y_1^1\right] d^3r$$

$$= N^2 \int |R_{21}|^2 \left[4\cdot(-1) + 5\cdot(0) + 9\cdot(1)\right] r^2 dr$$

$$= N^2 5 \int |R_{21}|^2 r^2 dr \qquad -4 + 9 = 5$$

$$= N^2 5$$

$$= \frac{5}{18}$$

c) $L^2 Y_\ell^m = \ell(\ell+1) Y_\ell^m$

$$\langle \psi | L^2 | \psi \rangle = N^2 \int |R_{21}|^2 r^2 dr \left[\underbrace{4(1(1+1)) + 5(1(1+1)) + 9(1(1+1))}_{8 + 10 + 18 = 36} \right]$$

$$= N^2 36$$

$$= \frac{36}{18} = 2$$

d) $V(r) = \frac{-e^2}{4\pi\varepsilon_0} \frac{1}{r}$ $\qquad r \frac{\partial}{\partial r}\left(\frac{-e^2}{4\pi\varepsilon_0} \frac{1}{r}\right) = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r} = -V$

$$2\langle T \rangle = \langle r \frac{\partial}{\partial r} V(r) \rangle = \langle -V \rangle$$

$$\langle T \rangle = \frac{-1}{2} \langle V \rangle$$

$$\langle H \rangle = \langle T \rangle + \langle V \rangle = \frac{-1}{2} \langle V \rangle + \langle V \rangle = \frac{1}{2} \langle V \rangle$$

$$\langle H \rangle = -\frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{1}{n^2} = \frac{1}{2} \langle V \rangle$$

Seting $\hbar = \hbar$ for this section.

$$\langle V \rangle = \frac{-m_e}{\hbar^2} \left(\frac{e^2}{4\pi\varepsilon_0}\right) \frac{1}{n^2}$$

$$\Rightarrow \langle T \rangle = \frac{m_e}{2\hbar^2} \left(\frac{e^2}{4\pi\varepsilon_0}\right) \frac{1}{n^2}$$

Spring 2005 #5 (p 1 of 2)

An election moves in a hydrogen atom potential - ignoring spin and relativity - in a State $|\psi\rangle$ that has the wave function

$$\psi(r,\theta,\phi) = N\, R_{21}(r)\left[2i\, Y_1^{-1}(\theta,\phi) + (2+i)\, Y_1^{0}(\theta,\phi) + 3i\, Y_1^{1}(\theta,\phi)\right]$$

where the $Y_\ell^m(\theta,\phi)$ are the spherical harmonics, $R_{n\ell}(r)$ are the normalized hydrogen atom wave functions, and N is a positive real number.

(See Abers Final from 221A #2)

we can re-write this wave function for simplicity. That is,

$$|\psi\rangle = N\left[C_{m=-1}\, |2,1,-1\rangle + C_{m=0}\, |2,1,0\rangle + C_{m=1}\, |2,1,1\rangle\right]$$

where $C_{m=-1} = 2i$, $C_{m=0} = (2+i)$, and $C_{m=1} = 3i$

(a) Calculate N.

$$1 = |N|^2\left[\,|2i|^2 + |2+i|^2 + |3i|^2\,\right] = |N|^2\left[4+5+9\right]$$

$$\therefore\ \boxed{N = \frac{1}{\sqrt{18}}}$$

(b) what is the expectation value of L_z?

$$L_z\,|n,\ell,m\rangle = m\,|n,\ell,m\rangle$$

so

$$\langle L_z\rangle = |N|^2\left[\,|2i|^2\langle 2,1,-1|L_z|2,1,-1\rangle + |2+i|^2\langle 2,1,0|L_z|2,1,0\rangle + \right.$$

$$\left. + |3i|^2\langle 2,1,1|L_z|2,1,1\rangle\right]$$

$$= \frac{1}{18}\left[4(-1) + 5(0) + 9(1)\right] = \boxed{\frac{5}{18}}$$

(c) what is the expectation value of L^2?

$$L^2 |n,\ell,m\rangle = \ell(\ell+1)\,|n,\ell,m\rangle$$

so,

$$\langle L^2\rangle = |N|^2\Big[\,|2i|^2\langle 2,1,-1|L^2|2,1,-1\rangle + |2+i|^2\langle 2,1,0|L^2|2,1,0\rangle + $$
$$+\ |3i|^2\langle 2,1,1|L^2|2,1,1\rangle\Big]$$

$$= \frac{1}{18}\Big[4\cdot 1(1+1) + 5\cdot 1(1+1) + 9\cdot 1(1+1)\Big]$$

$$= \frac{1}{9}\cdot 18 = \boxed{2}$$

(d) what is the expectation value of the kinetic energy in terms of $\hbar$, c, the fine structure constant α, and the electron mass m?

From the virial theorem we know that

$$\langle T\rangle = -E_n$$ see Griffiths' QM 4.41

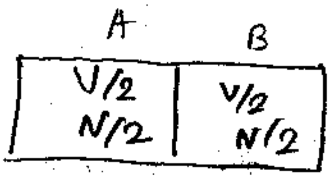
the energy levels for a hydrogen atom are (in natural units)

$$E_n = \frac{-\alpha^2 m}{2n^2}$$

so, for $n=2$

$$\boxed{\langle T\rangle = \frac{+\alpha^2 m}{8}}$$

Spring 2005 #6



a) $Z = \frac{1}{N!}\left(\frac{V}{(\sqrt{2\pi\hbar^2/mk_bT})^3}\right)^N = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3}\right)^N$

Before

$Z_A = \frac{1}{N_A!}\left(\frac{V_A}{\lambda_{th}^3}\right)^{N_A} \quad Z_B = \frac{1}{N_B!}\left(\frac{V_B}{\lambda_{th}^3}\right)^{N_B}$

After

$Z_{AB} = \frac{1}{N_A!}\left(\frac{V}{\lambda_{th}^3}\right)^{N_A}\frac{1}{N_B!}\left(\frac{V}{\lambda_{th}^3}\right)^{N_B}$

$S = k(\ln Z + \beta \bar{E})$ — $\bar{E}$: Internal energy $\frac{3}{2}NKT$

If $Z = \frac{Z_1^N}{N!}$

$\ln Z = N\ln Z_1 - \ln N!$
$= N\ln Z_1 - N\ln N + N$

$= KN[\ln V + \frac{3}{2}\ln T + \sigma] + K(-N\ln N + N) \qquad \sigma = \frac{3}{2}\ln\left(\frac{2\pi m k}{h^2}\right) + \frac{3}{2}$

$S = KN[\ln\frac{V}{N} + \frac{3}{2}\ln T + \sigma_0] \qquad \sigma_0 = 1 + \sigma$

Before

$S = S_A + S_B = \underbrace{K\frac{N}{2}[\ln\frac{V/2}{N/2} + \frac{3}{2}\ln T + \sigma_0]}_{S_A} + \underbrace{\frac{KN}{2}[\ln\frac{V/2}{N/2} + \frac{3}{2}\ln T + \sigma_0]}_{S_B}$

$= KN[\ln\frac{V}{N} + \frac{3}{2}\ln T + \sigma_0]$

After

$S = S_A + S_B = K\frac{N}{2}[\ln\frac{V}{N/2} + \frac{3}{2}\ln T + \sigma_0] + \frac{KN}{2}[\ln\frac{V}{N/2} + \frac{3}{2}\ln T + \sigma_0]$

$= KN[\ln\frac{V}{N} + \frac{3}{2}\ln T + \sigma_0] + NK\ln 2$

$S_{after} - S_{before} = Nk \ln 2$

b) If you remove the wall, no work is done

$\Rightarrow \Delta E = \Delta Q$ But since the energy depends only on the temperature for ideal gases and T is fixed $\Delta E = 0 \Rightarrow \Delta Q = 0$

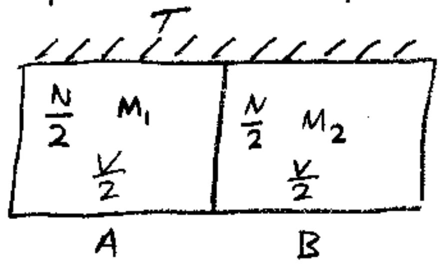
The process is irreversable though since $\Delta S \neq 0$.

c) $\Delta Q = 0$ still for the same reason.

$\Delta S = 0$ Now so the process is reversable which makes sence with the particles all being identical. Basicly nothing happened.

Spring 2005 #6 (p 1 of 4)

A closed container is divided by a wall into two equal parts like the figure



where M_1 & M_2 are identical types of particles but distinguishable from each other and they both make up ideal gases.

a) the partition function $Z(N) = \frac{1}{N!}\left(\frac{V}{\sqrt{2\pi\hbar^2/MKT}}\right)^N$ is for an ideal gas

of N particles of mass M in a volume V. Give the partition function of the gas in the container before and after the wall is removed. What are the entropy and pressure before and after the wall is removed?

Before wall is removed, we simply have

$$Z_A = \frac{1}{(N/2)!}\left(\frac{V/2}{\sqrt{2\pi\hbar^2/M_1KT}}\right)^{N/2}$$

$$Z_B = \frac{1}{(N/2)!}\left(\frac{V/2}{\sqrt{2\pi\hbar^2/M_2KT}}\right)^{N/2}$$

Set $\hbar = 1$.

So, the partition function for the system is

$$Z = Z_A Z_B = \frac{1}{[(N/2)!]^2}\left[\frac{V^2/4}{\frac{2\pi}{KT}\sqrt{\frac{1}{M_1M_2}}}\right]^{N/2}$$

$$\therefore \boxed{Z_{before} = \frac{1}{[(N/2)!]^2}\left[\frac{V(M_1M_2)^{1/4}}{2\sqrt{\beta 2\pi}}\right]^N}$$

→ note partition functions of uncorrelated systems are multiplied by each other

After the wall is removed, the partition function is (now $\frac{V}{2} \to V$)

$$\boxed{Z_{after} = \frac{1}{[(N/2)!]^2}\left[V\sqrt{\frac{(M_1M_2)^{1/2}}{2\pi\beta}}\right]^N}$$

Now, we want to find the entropy and pressure. To do this, let's first find the free energy. So,

$$F = -KT\ln Z \quad \text{and} \quad p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \quad \& \quad S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

where

$$\ln Z_{before} = -2\ln\left(\frac{N}{2}\right)! + N\ln\frac{V(M_1M_2)^{1/4}}{2\sqrt{\beta 2\pi}} \qquad , N \gg 1$$

$$\cong -N\ln\frac{N}{2} + N + N\ln V + N\ln\frac{(M_1M_2)^{1/4}}{2\sqrt{2\pi\beta}} + \frac{1}{2}N\ln KT$$

and

$$\ln Z_{after} \simeq -N\ln\frac{N}{2} + N + N\ln V + \frac{N}{2}\ln\frac{(M_1M_2)^{1/2}}{2\pi\beta} + \frac{N}{2}\ln KT$$

So,

$$P_{before} = KT\left(\frac{\partial\ln Z_{before}}{\partial V}\right)_{T,N} = \boxed{\frac{KTN}{V}}$$

and

$$P_{after} = KT\left(\frac{\partial\ln Z_{after}}{\partial V}\right)_{T,N} = \boxed{\frac{KTN}{V}}$$

} pressure is constant!

Also, we know that $S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$. But the following will be easier to use

$$S = k(\ln Z + \beta\bar{E}) \qquad \text{(Ref eq 6.6.5)}$$

So,

$$S_i = k\left(\ln Z_i + \beta \bar{E}\right) \quad , \quad \bar{E} = \frac{3}{2}\left(\frac{N}{2}\right)kT$$

Then,

$$S_{before} = S_A + S_B = k\left(\ln Z_{A\,before} + \frac{3}{2}\frac{N}{2}\right) + k\left(\ln Z_{B\,before} + \frac{3}{2}\left(\frac{N}{2}\right)\right)$$

$$= k\ln Z_{A\,before} Z_{B\,before} + \frac{3}{2}Nk = k\ln Z_{before} + \frac{3}{2}Nk$$

$$\boxed{S_{before} = kN\left[-\ln\frac{N}{2} + \frac{5}{2} + N\ln\frac{V(m_1 m_2)^{1/4}}{2\sqrt{\beta 2\pi}}\right]}$$

And

$$\boxed{S_{after} = S_{before} + kT\ln 2}$$

Thus,

$$\Delta S = S_{after} - S_{before} = kT\ln 2$$

(b) How much heat is absorbed or released following the removal of the wall? Is it reversible or irreversible process?

Since $\Delta S > 0$, then the process is irreversible. Since no heat is exchanged with the environment $dS \neq \frac{dQ}{T}$, so, we must consider 1st law of thermo (no work is done)

$$dE = dQ$$

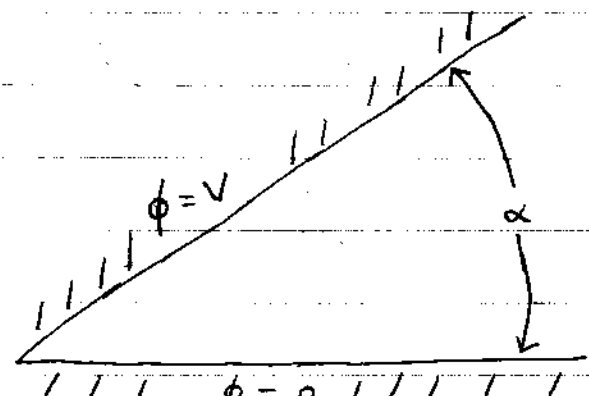
Since $dE = 0$, $dQ = 0$.

Spring 2005 #6 (p 4 of 4)

(c) let $M_1 = M_2$ and answer the same question as part (b).

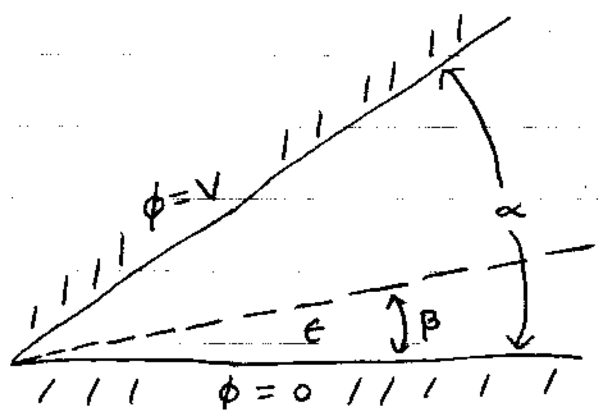
Since the particles are no identicle, $\Delta S = 0$. The process is reversible, we still have $dQ = 0$.

Consider a two dimensional (r, θ) electrostatic problem consisting of two infinite plates making an angle α with each other and held at a potential difference V, as shown below:



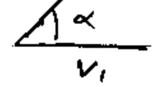
Part a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.

Now insert a wedge dielectric, of dielectric coefficient ϵ, and angle β, resting on the bottom plate as shown below:



Part b) Find the pressure experienced by the bottom plate at a distance r from the apex (from the line joining the two plates).

a) From Prof. Wong's lecture notes (p. 21)



$$\phi = V_1 - \frac{V_2 - V_1}{\alpha}\theta$$

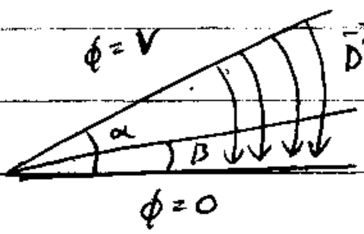
so in our case $V_1 = 0$; $V_2 = V$:

$$\boxed{\phi = \frac{V}{\alpha}\theta}$$

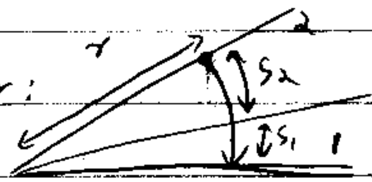
(This was assuming the first image was incorrect and there was actually no dielectric material present - original comp. image was identical to the second image).

b) dielectric coefficient $\epsilon = \epsilon_r = \frac{\varepsilon}{\varepsilon_0} = \varepsilon$ by definition (Griffiths p. 180 4.34) as $\varepsilon_0 \equiv 1$

Now the field lines go like:



so at a point r:



$$\phi = E_1 s_1 + E_2 s_2 \; ; \quad S_1 = r\beta \; ; \quad S_2 = r(\alpha - \beta)$$
$$= V \qquad \varepsilon E_1 = D \qquad E_2 = D$$
or
$$\epsilon E_1 = D$$

$$\Rightarrow \phi = \frac{D}{\epsilon} r\beta + Dr(\alpha - \beta) \quad \text{or} \quad D = \frac{\phi}{r(\beta/\epsilon + \alpha - \beta)}$$

now $|\vec{D}| = \sigma_f$ and Pressure, $P = \frac{1}{2} \underset{\uparrow \frac{D}{\epsilon}}{E_1} \sigma$ (Griffiths p. 102 2.50)

$$\text{so } P = \underbrace{\frac{1}{2} \frac{D}{\epsilon} \overbrace{\frac{\phi}{r(\alpha - \beta + \beta/\epsilon)}}^{D}}_{\frac{D^2}{\epsilon}} = \frac{1}{2} \frac{\phi^2}{\epsilon r^2} \frac{1}{[\alpha - \beta + \beta/\epsilon]^2} = \boxed{\frac{1}{2} \frac{V^2}{\epsilon r^2} \frac{1}{[\alpha - \beta + \beta/\epsilon]^2}}$$

and $\phi = V$

A relativistic charged particle of charge q and rest-mass m_0 is in a region of uniform magnetic field $B_0\hat{z}$. At time $t=0$ the particle has zero velocity along $\hat{z}$ (that is $\beta_z = v_z/c = 0$) and finite transverse speed $\beta_\perp = \beta_0$, with

$$\beta_\perp = \sqrt{v_x^2 + v_y^2}/c$$

Here x, y, and z are Cartesian coordinates in the lab frame.

Part a) What is the value of $\beta_\perp(t)$ for $t>0$?

Part b) what is the angular frequency Ω of rotation (that is, the gyrofrequency). No need for a calculation, just identify Ω.

Part c) Now apply a uniform electric field $E_0\hat{z}$, parallel to $\vec{B}$, starting at $t=0$. Without solving detailed equations, conclude what happens to the $\beta_\perp$ in part (a). Does it change?

a) $\beta_\perp = \sqrt{\beta_x^2 + \beta_y^2} =$ Const. as a magnetic field can do no work. Hence $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is also constant.

The only difference between this case and the nonrelativistic one is: instead of using regular momentum $m_0 v$ one needs to use the relativistic one $c\gamma m_0 \vec{\beta}$

So $\quad \frac{d\vec{p}}{dt} = q(\vec{v}\times\vec{B}) = qc(\vec{\beta}\times\vec{B}) = \underbrace{\frac{qB_0}{\gamma m_0}(\vec{p}\times\vec{B})}$ ← from $\vec{B} = B_0\hat{z}$

$$\cancel{c}\gamma m_0 \frac{d\vec{\beta}}{dt} = q\cancel{c}B_0(\vec{\beta}\times\hat{z})$$

This form will be useful for (c) – here you can already see $\Omega = \frac{qB}{\gamma m_0}$

so $$\frac{d\vec{\beta}}{dt} = \frac{q B_0}{\gamma m_0} \begin{vmatrix} \beta_x & \beta_y & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underbrace{\frac{q B_0}{\gamma m_0}}_{= \Omega} \left[\hat{x}\beta_y - \hat{y}\beta_x \right]$$

hence $$\frac{d\beta_x}{dt} = \Omega\, \beta_y \quad ; \quad \frac{d\beta_y}{dt} = \Omega \beta_x$$

taking time derivative of second equation and pluging into first one:

$$\frac{d^2 \beta_y}{dt^2} = \Omega \frac{d}{dt} \beta_x = \Omega^2 \beta_y$$

So $\beta_y = A \cos(\Omega t) + C \sin(\Omega t)$

similarly $\beta_x = D \cos(\Omega t) + E \sin(\Omega t)$

From the initial condition $\beta^2 = \beta_x^2 + \beta_y^2 = \beta_0^2$

$$\beta_x = \beta_0 \cos(\Omega t) \; ; \; \beta_y = \beta_0 \sin(\Omega t)$$ (just need to give it circular motion)

So $\beta_\perp(t) = \underline{\beta_0 \left(\cos(\Omega t), \sin(\Omega t) \right)}$

(b) $\Omega = \frac{qB}{\gamma m_0} = \frac{qB}{E_{rest}}$ which differs from the classical result by $\frac{1}{\gamma}$

(c) With an E-field along the $\hat{z}$ direction there is now an acceleration along $\hat{z}$ - which causes $\vec{\beta} \neq$ constant. This also means γ is no longer constant as well, i.e. $\gamma(t)$

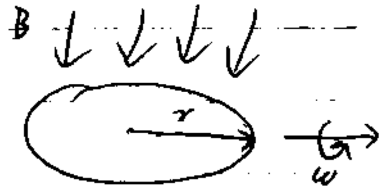
So from the previous page:

$$\frac{d\vec{p}}{dt} = qc \left(\vec{\beta} \times \vec{B} \right) = \frac{q B_0}{\gamma m_0} (\vec{p} \times \hat{z})$$

From dimensions $\frac{q B_0}{\gamma m_0} = 1/\text{seconds}$ hence $\Omega(t) = \frac{q B_0}{m_0 \gamma(t)}$

so as $\vec{\beta}$ increases to 1 $\gamma \to \infty$ hence $\Omega \to 0$ or the particle stops rotating! So $\beta_\perp$ decreases to ≈ 0.

A thin copper ring (conductivity σ, density ρ) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field B perpendicular to the axis of rotation. At time $t=0$, the ring is set rotating with frequency ω_0. Calculate the time it takes the frequency to decrease to 1/e of its original value, assuming the energy goes into Joule heating.



Assuming ω_0 is large enough that it doesn't change much after one ~~revolution~~ rotation.

$T = \frac{1}{2} I\omega^2 \quad I = \text{moment of inertia} = \frac{1}{2} Mr^2$

Joule heating $P = IV = \frac{V^2}{R}$, now as for V!

$\mathcal{E} = -\frac{d}{dt}\Phi$; $\Phi = B\pi r^2 \cos(\omega_0 t)$ « can also be $\sin(\omega_0 t)$ as it is V^2

$\mathcal{E} = -\frac{d}{dt} B\pi r^2 \cos(\omega_0 t) = B\pi r^2 \omega_0 \sin(\omega_0 t)$

so $\frac{\mathcal{E}^2}{R} = \frac{V^2}{R} = \frac{(B\pi r^2\omega_0)^2 \sin^2(\omega_0 t)}{R} = P = \frac{(B\pi r^2\omega_0)^2}{2R}$ as $\langle \sin^2(\omega_0 t)\rangle = \langle\cos^2(\omega_0 t)\rangle = 1/2$

now the change in kinetic energy over one period

$$\frac{\frac{dT}{dt}}{T} = \frac{-P}{T} = -\frac{\frac{(B\pi r^2\omega_0)^2}{2R}}{\frac{1}{4}Mr^2\omega_0^2} = -\frac{(2B^2\pi^2)r^{4}\cancel{\omega_0^2}}{RMr^2\cancel{\omega_0^2}} = -\frac{2(B\pi r)^2}{MR} = \text{Const.}$$

so $\frac{dT}{T} = -C\,dt \Rightarrow T = T_0 e^{-Ct} \Rightarrow \omega^2 = \omega_0^2 e^{-Ct}$

or $\omega = \omega_0 e^{-\frac{C}{2}t}$

so the time constant $\tau^{-1} = \frac{c}{2} - \frac{(B\pi r)^2}{MR}$

now $M = \rho(2\pi r)(\pi r_w^2)$; $R = \frac{1}{\sigma}\frac{2\pi r}{\pi r_w^2}$

so $MR = \rho(2\pi r)(\cancel{\pi r_w^2})\frac{1}{\sigma}\frac{2\pi r}{\cancel{\pi r_w^2}} = \frac{\rho}{\sigma}(2\pi r)^2$

and then $\tau^{-1} = B^2 \frac{\cancel{\pi^2 r^2}}{\rho/\sigma \; 4\cancel{\pi^2 r^2}} = \underline{\frac{\sigma}{4\rho} B^2}$ or $\tau = \frac{4\rho}{\sigma}\frac{1}{B^2}$

Spring 2005 #13

$$Z_N = \sum_{S_i=\pm 1} e^{\beta J \sum_{i=1}^{N} S_i S_{i+1} + \beta H \sum_{i=1}^{N} S_i} \qquad \beta = 1/k_T$$

Assume periodic boundry conditions

$$S_{N+1} = S_1$$

elements

$$\langle S | T | S' \rangle = e^{V S S' + \frac{B}{2}(S + S')} \qquad V = \beta J \quad B = \beta H \quad (S, S' = \pm 1)$$

$$\langle -1 | T | -1 \rangle$$

$$T = \begin{pmatrix} e^{V} e^{-B} & e^{-V} \\ e^{-V} & e^{V} e^{B} \end{pmatrix}$$

$$\langle 1 | T | 1 \rangle$$

$$Z_N = \sum_{S_i=\pm 1}^{S_N} e^{V \sum_{i=1}^{N} S_i S_{i+1} + \frac{B}{2} \sum_{i=1}^{N} S_i + S_{i+1}}$$

$$= \sum_{S_i=\pm 1} e^{V S_1 S_2 + \frac{B}{2}(S_1 + S_2)} e^{V S_2 S_3 + \frac{B}{2}(S_2 + S_3)} \cdots$$

$$= \sum_{S_i=\pm 1} \langle S_1 | T | S_2 \rangle \langle S_2 | T | S_3 \rangle \cdots \langle S_N | T | S_1 \rangle$$

Identity

$$\sum_a \langle x | M | a \rangle \langle a | M | y \rangle = \langle x | M^2 | y \rangle$$

$$Z_N = \sum_{S_1} \langle S_1 | T^N | S_1 \rangle = Tr(T^N)$$

Trace is not effected by diagonalization.

so if Q is diagonlized form of T, and we found eigenvalues of T

$$Tr(Q^N) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^N = \lambda_1^N + \lambda_2^N \quad (\lambda_1, \lambda_2 \text{ eigenvalues of } T)$$

(or more simply, the trace of a matrix is the sum of the eigenvalues

b)

$$\det(T - \lambda I) = 0$$

$$(e^{\beta(J-H)} - \lambda)(e^{\beta(J+H)} - \lambda) - e^{-2\beta J} = 0$$

$$e^{2\beta J} - \lambda e^{\beta(J+H)} - \lambda e^{\beta(J-H)} + \lambda^2 - e^{-2\beta J} = 0$$

$$\lambda^2 - \lambda e^{\beta J}(e^{\beta H} + e^{-\beta H}) + e^{2\beta J} - e^{-2\beta J} = 0$$

$$\lambda^2 - \lambda 2 e^{\beta J}\cosh(\beta H) + 2\sinh(2\beta J) = 0$$

$$\lambda = \frac{2e^{\beta J}\cosh(\beta H) \pm \sqrt{4e^{2\beta J}\cosh^2(\beta H) - 8\sinh(2\beta J)}}{2}$$

$$= e^{\beta J}\cosh(\beta H) \underset{\lambda_2}{\overset{\lambda_1}{\pm}} \sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(2\beta J)}$$

$$\frac{\lambda_2}{\lambda_1} < 1$$

as $N \to \infty$

$$Tr(T) = \lambda^N$$

Good to know

$$\frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\frac{e^x - e^{-x}}{2} = \sinh(x)$$

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$F = -kT \ln Z = -kT \ln \lambda_1^N = \frac{-N}{\beta} \ln \lambda_1$$

$$\boxed{-\frac{F}{kT} = \ln \lambda_1}$$ (per spin)

c) $$M = \frac{\partial F}{\partial H}$$ ★ $$M = \frac{-N}{\beta} \frac{\partial \ln \lambda_1}{\partial H} = \frac{-N \, \partial \ln \lambda_1}{\partial(\beta H)}$$

pulling in constant makes it easier

$$M = -N \left(\frac{e^{\beta J} \sinh(\beta H) + \frac{1}{2} \frac{2\cosh(\beta H)\sinh(\beta H)\, e^{2\beta J}}{\sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(2\beta J)}}}{e^{\beta J}\cosh(\beta H) + \sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(2\beta J)}} \right)$$

$$= -N \left(\frac{\sinh(\beta H) + \frac{\cosh(\beta H)\sinh(\beta H)}{\sqrt{e^{2\beta J}\cosh^2(\beta H) - 2\sinh(2\beta J)}}}{\cosh(\beta H) + \sqrt{\cosh^2(\beta H) - 2e^{-2\beta J}\sinh(2\beta J)}} \right)$$

Spring 2005 #14:

photon gas

a) partition function, $\bar{n}_{state}$

$$Z\bar{n}_s = \frac{-1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}$$

$$Z = \sum_R e^{-\beta E_R} = \sum_{R - n_1, n_2, n_3 \ldots} e^{-\beta(n_1 \epsilon_1 + n_2 \epsilon_2 + \ldots)}$$

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

all other terms go away since direvative selects ϵ_s state

$$= -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

easier seen

$$\bar{n}_s = \frac{\sum_{n_s} e^{-\beta n_s \epsilon_s} n_s \cancel{\sum_{n_1 n_2 \ldots}^{(\text{excluding } s)} e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 \ldots)}}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \cancel{\sum_{n_1 n_2 n_3 \ldots}^{(s \text{ excluded})} e^{-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 \ldots)}}}$$

$$\sum_{n_s = 0}^{\infty} e^{-\beta n_s \epsilon_s} = 1 + e^{-\beta \epsilon_s} + e^{-2\beta \epsilon_s} + \ldots$$

$$= \frac{1}{1 - e^{-\beta \epsilon_s}}$$

$$\bar{n}_s = \frac{\sum_{n_s} e^{-\beta n_s \epsilon_s} n_s}{\sum e^{-\beta n_s \epsilon_s}} = -\frac{1}{\beta}\frac{d}{d\beta}\ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

$$\Rightarrow \bar{n}_s = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln\left(\frac{1}{1 - e^{-\beta n_s \epsilon_s}}\right)$$

$$= \frac{1}{\beta}\frac{\partial}{\partial \epsilon_s}\ln(1 - e^{-\beta \epsilon_s}) = \frac{e^{-\beta \epsilon_s}}{1 - e^{-\beta \epsilon_s}} = \frac{1}{e^{\beta \epsilon_s} - 1}$$

b) find relation between radiation pressure and mean energy density ($\bar{U}$)

Reif 9.13.20

$\bar{p} = \sum_s \bar{n}_s \left(-\frac{\partial \epsilon_s}{\partial V}\right)$ follows from $\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}$ $\bar{p} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}\frac{\partial \epsilon_s}{\partial V}$

$-\bar{n}_s$

Consider cube $L_x = L_y = L_z$ $V = L^3$

$\epsilon_s = \hbar\omega = cK = \hbar c (K_x^2 + K_y^2 + K_z^2)^{1/2}$ $K_i = \frac{2\pi}{L_i} n_i$

wave vector

$= \hbar c \left(\frac{2\pi}{L}\right)(n_x^2 + n_y^2 + n_z^2)^{1/2}$

$\Rightarrow \epsilon_s = B L^{-1} = B V^{-1/3}$ B = constant

$\frac{\partial \epsilon_s}{\partial V} = -\frac{1}{3} B V^{-4/3} = -\frac{1}{3}\frac{\epsilon_s}{V}$

$\bar{p} = \sum_s \bar{n}_s \left(\frac{1}{3}\frac{\epsilon_s}{V}\right) = \frac{1}{3V}\sum_s \bar{n}_s \epsilon_s = \frac{1}{3V}\bar{E} = \frac{1}{3}\bar{u}$

c) Adiabatic process $dQ = 0 \Rightarrow dE = -p\,dV$

$p = -\frac{\partial E}{\partial V}$

$\Rightarrow \frac{\partial \epsilon_s}{\partial V} = -p_s = -\frac{1}{3} B V^{-4/3}$

$\Rightarrow pV^{4/3} \propto$ constant in general

$p_0 V_0^{4/3} = p_f V_f^{4/3}$ $V_f = \frac{1}{8} V_0$

$p_f = \bar{p}_0 \left(\frac{V_0}{V_f}\right)^{4/3} = p_0 (8)^{4/3}$

1. *Quantum Mechanics* (Spring 2005)

Consider a particle of charge q in a one-dimensional harmonic oscillator potential. Suppose there is also a weak electric field E so that the potential is shifted by

$$H' = -qEx$$

(a) Calculate the correction to the simple harmonic oscillator energy levels through second order in perturbation theory.

(b) Now solve the problem exactly. How do the exact energy levels compare with the perturbative result in (a)?

a. Recall $x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right)$

$$\Delta_n^{(1)} = H'_{nn} = \langle \Psi_n | H' | \Psi_n \rangle = -qE \langle \Psi_n | x | \Psi_n \rangle = 0$$

$$\Delta_n^{(2)} = \sum_{k\neq n} \frac{|H'_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} = \sum_{k \neq n} \frac{|\langle \Psi_n | H' | \Psi_k \rangle|^2}{(n+\frac{1}{2})\hbar\omega - (k+\frac{1}{2})\hbar\omega}$$

$$= \frac{q^2E^2}{\hbar\omega} \sum_{k\neq n} \frac{|\langle \Psi_n | x | \Psi_k \rangle|^2}{n-k}$$

$$= \frac{q^2E^2}{\hbar\omega} \frac{\hbar}{2m\omega} \sum_{k\neq n} \frac{|\langle \Psi_n | a + a^+ | \Psi_k \rangle|^2}{n-k}$$

$$= \frac{q^2E^2}{2m\omega^2} \sum_{k \neq n} \frac{|\sqrt{k}\,\delta_{n,k-1} + \sqrt{k+1}\,\delta_{n,k+1}|^2}{n-k}$$

$$= \frac{q^2E^2}{2m\omega^2}\left(\frac{n+1}{-1} + \frac{n}{1}\right) = -\frac{q^2E^2}{2m\omega^2}$$

b. Complete the square in the potential.

$$V = \frac{1}{2}m\omega^2x^2 - qEx \Rightarrow \frac{2}{m\omega^2}V = x^2 - \frac{2}{m\omega^2}qEx + \left(\frac{q^2E^2}{m^2\omega^4} - \frac{q^2E^2}{m^2\omega^4}\right)$$

$$\Rightarrow \frac{2}{m\omega^2}V = \left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{m^2\omega^4}$$

$$\Rightarrow V = \frac{1}{2}m\omega^2\left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

So the exact shift is $\Delta_n = -\frac{q^2E^2}{2m\omega^2}$, which is exactly the perturbative result found in part a.

1. *Quantum Mechanics* (Spring 2006)

An electron is at rest in a constant magnetic field pointing along the z-direction. The Hamiltonian is

$$H = -\boldsymbol{\mu}\cdot\mathbf{B} = g\mu_0\frac{\mathbf{s}}{\hbar}\cdot\mathbf{B}$$

where $\mathbf{B} = B_0\hat{\boldsymbol{n}}_z$. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

(a) What are the eigenstates of the Hamiltonian in terms of $|\psi_\pm\rangle$, and what is the energy difference between them?

(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. What is $|\psi(0)\rangle$ in terms of $|\psi_\pm\rangle$? Calculate $|\psi(t)\rangle$ for any later time t in terms of these same two states.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

a. $H = g\mu_0 \frac{\vec{S}}{\hbar}\cdot\vec{B} = \frac{g}{2}\mu_0\vec{\sigma}\cdot\vec{B} = \frac{g}{2}\mu_0\sigma_z B_0 \cong \mu_0\sigma_z B_0$

The eigenvalues of σ_z are ± 1, so the eigenvalues of H are $\pm\mu_0 B_0$ $\Rightarrow \Delta E = \mu_0 B_0 - (-\mu_0 B_0) = 2\mu_0 B_0$

b. $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow |\Psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ since $\sigma_x|\Psi_{x+}\rangle = (+1)|\Psi_{x+}\rangle$

So $|\Psi(0)\rangle = |\Psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|\Psi_{z+}\rangle + \frac{1}{\sqrt{2}}|\Psi_{z-}\rangle$

$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\mu_0B_0t/\hbar}|\Psi_{z+}\rangle + \frac{1}{\sqrt{2}}e^{i\mu_0B_0t/\hbar}|\Psi_{z-}\rangle$

c. $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix} \Rightarrow a = \pm b$

$\Rightarrow |\Psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|\Psi_{-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-ib\\ia\end{pmatrix} \Rightarrow b = \pm ia$

$\Rightarrow |\Psi_{y+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ and $|\Psi_{y-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\-b\end{pmatrix} \Rightarrow a=0$ or $b=0$

$\Rightarrow |\Psi_{z+}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\Psi_{z-}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\langle S_x\rangle = \frac{\hbar}{2}|\langle\Psi_{x+}|\Psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\Psi_{x-}|\Psi(t)\rangle|^2$

$= \frac{\hbar}{2}|\cos(\mu_0B_0t/\hbar)|^2 + \left(-\frac{\hbar}{2}\right)|-\sin(\mu_0B_0t/\hbar)|^2$

$= \frac{\hbar}{2}\cos(2\mu_0B_0t/\hbar)$

$\langle S_y\rangle = \frac{\hbar}{2}|\langle\Psi_{y+}|\Psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\Psi_{y-}|\Psi(t)\rangle|^2$

$= \frac{\hbar}{2}\left|\frac{1}{2}\left(e^{-i\mu_0B_0t/\hbar} + ie^{i\mu_0B_0t/\hbar}\right)\right|^2 + \left(-\frac{\hbar}{2}\right)\left|\frac{1}{2}\left(e^{-i\mu_0B_0t/\hbar} - ie^{i\mu_0B_0t/\hbar}\right)\right|^2$

$= \frac{\hbar}{2}\frac{1}{4}\left[(\cos-\sin)^2 + (-\sin+\cos)^2\right] + \left(-\frac{\hbar}{2}\right)\frac{1}{4}\left[(\cos+\sin)^2 + (-\sin-\cos)^2\right]$

$= \frac{\hbar}{2}\frac{1}{2}(\cos-\sin)^2 - \frac{\hbar}{2}\frac{1}{2}(\cos+\sin)^2$

$= -\frac{\hbar}{2}\frac{1}{2}4\sin\cos = -\frac{\hbar}{2}\sin(2\mu_0B_0t/\hbar)$

$\langle S_z\rangle = \frac{\hbar}{2}|\langle\Psi_{z+}|\Psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\Psi_{z-}|\Psi(t)\rangle|^2 = \frac{\hbar}{2}\left(\frac{1}{2}\right) - \frac{\hbar}{2}\left(\frac{1}{2}\right) = 0$

2. *Quantum Mechanics* (Spring 2005)

Show that in one space dimension any attractive potential, no matter how weak, always has at least one bound state. *Hint:* Use the variational principle with some appropriate trial wave function such as the normalized Gaussian

$$\psi(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$$

where b is a parameter.

This is Shankar 5.2.2 b

We must assume as Shankar does that an attractive potential is everywhere less than its limiting values as $x \to \pm\infty$.

Then we define $V(\pm\infty) = 0$ so that $V(x) = -|V(x)|$ for all x.

$$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)|$$

$$\Rightarrow E(b) = \langle \psi_b | H | \psi_b \rangle = \langle \psi_b | -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - |V(x)| | \psi_b \rangle$$

where $\psi_b(x) = \left(\frac{2b}{\pi}\right)^{1/4} e^{-bx^2}$

$$\Rightarrow E(b) = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left(\frac{2b}{\pi}\right)^{1/2} e^{-bx^2}\frac{\partial^2}{\partial x^2} e^{-bx^2}dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-bx^2}\frac{\partial}{\partial x}\left(-2bx e^{-bx^2}\right)dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty} e^{-bx^2}\left(-2b e^{-bx^2} + 4b^2x^2 e^{-bx^2}\right)dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\left(4b^2x^2 - 2b\right) e^{-2bx^2}dx - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= -\frac{\hbar^2}{2m}\cancel{\left(\frac{2b}{\pi}\right)^{1/2}}\left(4b^2\frac{1}{4b}\cancel{\sqrt{\frac{\pi}{2b}}} - 2b\cancel{\sqrt{\frac{\pi}{2b}}}\right) - \langle \psi_b | |V(x)| | \psi_b \rangle$$

$$= \frac{\hbar^2 b}{2m} - \int_{-\infty}^{\infty}\left(\frac{2b}{\pi}\right)^{1/2} e^{-2bx^2}|V(x)|\,dx$$

In the limit of very small b,

$$E(b) \cong \frac{\hbar^2 b}{2m} - \left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}|V(x)|\,dx$$

We have a bound state iff $E(b) < 0$ iff

$$\frac{\hbar^2 b}{2m} - \left(\frac{2b}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}|V(x)|\,dx < 0$$

$$\Leftrightarrow \frac{\hbar^2 b}{2m}\left(\frac{\pi}{2b}\right)^{1/2} < \int_{-\infty}^{\infty}|V(x)|\,dx$$

$$\Leftrightarrow \frac{\hbar^2\sqrt{\pi}}{2\sqrt{2}\,m}\, b^{1/2} < \int_{-\infty}^{\infty}|V(x)|\,dx$$

$$\Leftrightarrow b < \frac{8m^2}{\pi\hbar^4}\left[\int_{-\infty}^{\infty}|V(x)|dx\right]^2$$

So whenever b satisfies this condition and is very small we have a bound state. Then the variational principle tells us that the ground state has an energy less than this, so the ground state also has an energy less than zero, so the ground state must be a bound state.

3. *Quantum Mechanics* (Spring 2005)

A beam of particles scatters off an impenetrable sphere of radius a. That is, the potential is zero outside the sphere, and infinite inside. The wave function must therefore vanish at $r = a$.

(a) What is the S-wave ($l = 0$) phase shift as a function of the incident energy or momentum?

(b) What is the total cross section in the limit of zero incident kinetic energy?

See Sakurai pages 406-408

a. Recall $A_\ell(r) = e^{i\delta_\ell}\left[\cos(\delta_\ell) j_\ell(kr) - \sin(\delta_\ell) n_\ell(kr)\right]$

which is Sakurai (7.6.33)

The wavefunction must vanish at $r=a \Rightarrow A_\ell(r)|_{r=a} = 0$

$$\Rightarrow \cos(\delta_\ell) j_\ell(KR) - \sin(\delta_\ell) n_\ell(KR) = 0$$

$$\Rightarrow \tan(\delta_\ell) = \frac{j_\ell(KR)}{n_\ell(KR)}$$

We are trying to find the S-wave ($\ell=0$) phase shift so

$$\Rightarrow \tan(\delta_0) = \frac{j_0(KR)}{n_0(KR)} = \frac{\sin(KR)/KR}{-\cos(KR)/KR} = -\tan(KR)$$

$$\Rightarrow \delta_0 = -KR$$

b. Recall $\sigma_{tot} = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{K^2}\sum_\ell (2\ell+1)\sin^2(\delta_\ell)$

which is Sakurai (7.6.18)

In the limit of zero incident kinetic energy only the $\ell=0$ term contributes so

$$\lim_{K\to 0} \sigma_{tot} = \lim_{K\to 0} \frac{4\pi}{K^2} \sin^2(\delta_0) = \lim_{K\to 0} \frac{4\pi}{K^2} \sin^2(-KR)$$

$$= \lim_{K\to 0} \frac{4\pi}{K^2} K^2 R^2 = 4\pi R^2$$

5. *Quantum Mechanics* (Spring 2005)

An electron moves in a hydrogen atom potential – ignoring spin and relativity – in a state $|\psi\rangle$ that has the wave function

$$\psi(r,\theta,\phi) = NR_{21}(r)\left[2iY_1^{-1}(\theta,\phi) + (2+i)Y_1^0(\theta,\phi) + 3iY_1^1(\theta,\phi)\right]$$

where the $Y_l^m(\theta,\phi)$ are the spherical harmonics, $R_{nl}(r)$ are the normalized hydrogen atom wave functions, and N is a positive real number.

(a) Calculate N.

(b) What is the expectation value of L_z? ($\hbar\mathbf{L} = \mathbf{r}\times\mathbf{p}$)

(c) What is the expectation value of $\mathbf{L}^2$?

(d) What is the expectation value of the kinetic energy in terms of $\hbar, c$, the electron charge e or the fine-structure constant α, and the electron mass m?

Note: The explicit forms of the functions that appear in $\psi(r,\theta,\phi)$ above are

$$R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{r}{a^{5/2}}e^{-r/2a} \quad Y_1^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{\pm i\phi} \quad Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$$

a. By normalization, $1 = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty}\int \psi^*(r,\theta,\phi)\psi(r,\theta,\phi)r^2 dr\, d\Omega$

$= |N|^2\int_0^\infty R_{21}^*(r)R_{21}(r)r^2dr\,(4+5+9)$ since $\int (Y_{l'}^{m'})^* Y_l^m d\Omega = \delta_{ll'}\delta_{mm'}$

$= 18|N|^2\int_0^\infty \frac{1}{24}\frac{r^4}{a^5}e^{-r/a}dr$

$= \frac{3}{4}|N|^2 a^{-5}(a^5 4!) = 18|N|^2 \Rightarrow N = \frac{1}{\sqrt{18}}$ up to a phase

b. $|\psi\rangle = \frac{2i}{\sqrt{18}}|21-1\rangle + \frac{2+i}{\sqrt{18}}|210\rangle + \frac{3i}{\sqrt{18}}|211\rangle$

and $L_z|nlm\rangle = \hbar m|nlm\rangle$

$\Rightarrow \langle\psi|L_z|\psi\rangle = \frac{4}{18}\langle 21-1|L_z|21-1\rangle + \frac{5}{18}\langle 210|L_z|210\rangle$
$+ \frac{9}{18}\langle 211|L_z|211\rangle = -\frac{4}{18}\hbar + \frac{9}{18}\hbar = \frac{5}{18}\hbar$

c. $L^2|nlm\rangle = \hbar^2 l(l+1)|nlm\rangle$

$\Rightarrow \langle\psi|L^2|\psi\rangle = \frac{4}{18}\langle 21-1|L^2|21-1\rangle + \frac{5}{18}\langle 210|L^2|210\rangle$
$+ \frac{9}{18}\langle 211|L^2|211\rangle = \frac{8}{18}\hbar^2 + \frac{10}{18}\hbar^2 + \frac{18}{18}\hbar^2 = 2\hbar^2$

d. We know the total energy because this is a Hydrogen atom in an $n=2$ energy eigenstate, so $E = -\frac{me^4}{2\hbar^2(2)^2} = -\frac{me^4}{8\hbar^2}$

So we can use the virial theorem to get the kinetic energy

$\langle T\rangle = -\langle E\rangle = \frac{me^4}{8\hbar^2}$

6. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A closed container is divided by a wall into two equal parts (A and B), each of volume $V/2$. Part A contains an ideal gas with $N/2$ molecules of mass M_1 while part B contains an ideal gas with $N/2$ molecules of mass M_2. The container is kept at a fixed temperature T. The molecules of each kind are all identical, but distinguishable from the molecules of the other kind.

(a) The partition function $Z(N)$ of an ideal gas of N particles of mass M in a volume V is given by

$$Z(N) = \frac{1}{N!}\left(\frac{V}{\sqrt{2\pi\hbar^2/Mk_BT}}\right)^N$$

Give the partition function of the gas in the container before and after the wall is removed. What are the entropy and pressure before and after the wall is removed?

(b) How much heat is absorbed or released following the removal of the wall? Is the removal of the wall a reversible or irreversible process?

(c) Same question as (b), but now for the case that the two kinds of molecules are indistinguishable from each other (so $M_1 = M_2$). Compare your answers for (b) and (c) and provide a physical explanation for the difference in entropy between the two cases.

a. Before: $Z_0(N) = \sum_r e^{-\beta E_r} \cong \int\cdots\int e^{-\beta E_r} dx_1^{(1)}\cdots dx_{N/2}^{(1)} dp_1^{(1)}\cdots dp_{N/2}^{(1)} dx_1^{(2)}\cdots dx_{N/2}^{(2)} dp_1^{(2)}\cdots dp_{N/2}^{(2)}$

$= \int\cdots\int e^{-\beta E_r^{(1)}} dx_1^{(1)}\cdots dx_{N/2}^{(1)} dp_1^{(1)}\cdots dp_{N/2}^{(1)} \int\cdots\int e^{-\beta E_r^{(2)}} dx_1^{(2)}\cdots dx_{N/2}^{(2)} dp_1^{(2)}\cdots dp_{N/2}^{(2)}$

$= Z_1(\frac{N}{2}) Z_2(\frac{N}{2}) = \frac{1}{(\frac{N}{2}!)(\frac{N}{2}!)}\left(\frac{V^2/4}{2\pi\hbar^2/\sqrt{M_1M_2}KT}\right)^{N/2}$

After the partition is removed, the only difference is the volume doubles.

After: $Z_0'(N) = \frac{1}{(\frac{N}{2}!)(\frac{N}{2}!)}\left(\frac{V^2}{2\pi\hbar^2/\sqrt{M_1M_2}KT}\right)^{N/2}$

The pressure is $p = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V} = \frac{1}{\beta}\frac{\partial}{\partial V}(N\ln(V)) = \frac{1}{\beta}\frac{N}{V} = \frac{NKT}{V}$ before & after

The entropy is $S = K(\ln(Z) + \beta E) = K(\ln(Z) - \beta\frac{\partial \ln(Z)}{\partial\beta})$

Before: $\ln(Z_0) = N\ln(\frac{V}{2}) - \frac{N}{2}\ln(\beta) + \frac{N}{2}\ln\left(\frac{\sqrt{M_1M_2}}{2\pi\hbar^2}\right) - 2\ln(\frac{N}{2}!)$

$\Rightarrow S = K\left(N\ln(\frac{V}{2}) - \frac{N}{2}\ln(\beta) + \frac{N}{2}\ln\left(\frac{\sqrt{M_1M_2}}{2\pi\hbar^2}\right) - 2\ln(\frac{N}{2}!) + \frac{N}{2}\right)$

and $S' = K\left(N\ln(V) - \frac{N}{2}\ln(\beta) + \frac{N}{2}\ln\left(\frac{\sqrt{M_1M_2}}{2\pi\hbar^2}\right) - 2\ln(\frac{N}{2}!) + \frac{N}{2}\right)$

b. $\Delta S = KN(\ln(V) - \ln(\frac{V}{2})) = KN\ln(2)$ so it is irreversible, but $\Delta Q \neq T\Delta S$ because a free expansion of a gas is not quasistatic. In fact, for an adiabatic free expansion, $Q=0$ and $W=0 \Rightarrow \Delta E = 0$ and for an ideal gas $E = \frac{3}{2}NKT$ so $\Delta T = 0$. Therefore the reservoir at temperature T never exchanges any heat because it is always at the same temperature.

c. Before: $Z_0(N) = \frac{1}{(\frac{N}{2}!)^2}\left(\frac{V^2/4}{2\pi\hbar^2/MKT}\right)^{N/2}$ After: $Z_0'(N) = \frac{1}{N!}\left(\frac{V}{\sqrt{2\pi\hbar^2/MKT}}\right)^N$

$\Delta S = K\Delta(\ln(Z)) = K[N\ln(V) - \ln(N!) - N\ln(\frac{V}{2}) + 2\ln(\frac{N}{2}!)]$

If N is large, $\Delta S = K[N\ln(2) - N\ln(N) + \cancel{N} + 2(\frac{N}{2}\ln(\frac{N}{2}) - \cancel{\frac{N}{2}})]$

$= K[N\ln(2) - N\ln(N) + N\ln(N) - N\ln(2)] = 0$

So it is reversible and $\Delta Q = T\Delta S = 0$. The difference in entropy is due to the decrease in entropy due to indistinguishability.

7. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A (nearly) ideal gas with a temperature T and pressure P contains atoms of mass M that are either in the ground state or in the first excited state. An atom that returns to the ground state from the first excited state emits a photon of frequency f_o. For a stationary observer observing the spectral line emitted by a *moving* atom, this frequency is shifted by the Doppler effect to

$$f(v_{\|}) = f_o(1 + v_{\|}/c)$$

where c is the velocity of light and $v_{\|}$ is the projection of the velocity of the atom on the line of sight from the observer to the atom.

(a) What is the statistical distribution $P(f)$ of the frequency of the spectal line? Assume the atoms obey the Maxwell-Boltzmann distribution.

(b) Obtain from $P(f)$ the contribution by the Doppler effect to the width $\sqrt{\langle(f - f_o)^2\rangle}$ of the spectral line. Can you think of a way this effect could be exploited in the study of stellar atmospheres?

(c) The *natural* line shape $P(f)$ of an atomic spectral line is, according to quantum mechanics, given by

$$P(f) \sim \frac{1}{(f - f_o)^2 + \tau^{-2}}$$

where τ is the *lifetime* of the excited state. For atoms in a dense gas, the actual lifetime of the excited state is not intrinsic, but instead determined by the time interval between successive collisions between atoms. Let the cross section of an atom equal σ. Obtain an expression for τ in terms of σ, the pressure P and the temperature T. Under which conditions will this "collisional" broadening of the spectral line dominate over the Doppler broadening as computed under (b)?

a. We start from $P_f(f)\,df = P_{v_{\|}}(v_{\|})\,dv_{\|}$. Then $f = f_o(1+v_{\|}/c)$

$\Rightarrow f - f_o = \frac{f_o}{c} v_{\|} \Rightarrow v_{\|} = \frac{c}{f_o}(f - f_o)$

Therefore $P_f(f) = P_{v_{\|}}\left(\frac{c}{f_o}(f-f_o)\right)\frac{dv_{\|}}{df} = P_{v_{\|}}\left(\frac{c}{f_o}(f-f_o)\right)\frac{c}{f_o}$

By the Maxwell-Boltzmann distribution, $P_{v_{\|}}(v_{\|})dv_{\|} = \left(\frac{m}{2\pi KT}\right)^{1/2} e^{-mv_{\|}^2/2KT} dv_{\|}$

$\Rightarrow P_f(f) = \frac{c}{f_o}\left(\frac{m}{2\pi KT}\right)^{1/2} e^{-m\left(\frac{c}{f_o}\right)^2(f-f_o)^2/2KT}$

b. The equation for a Gaussian is $G(x) = A\,e^{-(x-x_o)^2/2\sigma^2}$

$\Rightarrow m\left(\frac{c}{f_o}\right)^2/2KT = \frac{1}{2\sigma^2} \Rightarrow \sigma^2 = \frac{KT}{m}\left(\frac{f_o}{c}\right)^2 \Rightarrow \sigma = f_o\sqrt{\frac{KT}{mc^2}}$

So the linewidth can be used to determine the temperature of a stellar atmosphere.

c. $n\bar{v}\sigma$ particles scatter per unit time off one particle

$\Rightarrow \tau^{-1} = n\bar{v}\sigma$ and for an ideal gas $pV = NKT \Rightarrow n = \frac{p}{KT}$

$\Rightarrow \tau^{-1} = \frac{p}{KT}\bar{v}\sigma \Rightarrow \tau = \frac{KT}{p\bar{v}\sigma}$ and $\bar{v} = \sqrt{\frac{8KT}{\pi M}}$

$\Rightarrow \tau = \frac{KT}{p\sigma}\sqrt{\frac{\pi M}{8KT}} = \frac{\sqrt{M\pi/8}}{p\sigma}(KT)^{1/2}$

The broadening is largest when $P(f)$ is large when f is far from f_o, which happens when τ^{-2} is large, so τ is small, so either T is small or p is large.

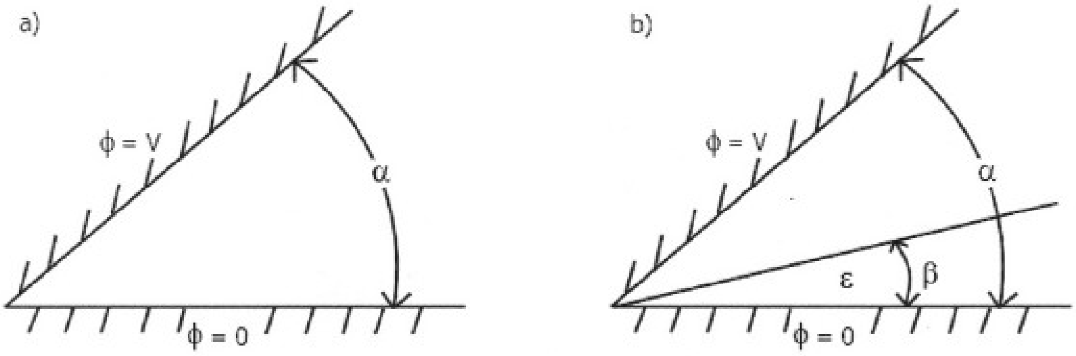
8. *Electricity and Magnetism* (Spring 2005)

Consider a two-dimensional (r, θ) electrostatic problem consisting of two infinite plates making an angle α with each other and held at a potential difference V, as shown below:

(a) Find the potential $\phi(r, \theta)$ in the vacuum region between the plates.

Now insert a wedge dielectric, of dielectric coefficient ϵ, and angle β, resting on the bottom plate as shown below:

(b) Find the pressure experienced by the bottom plate at a distance r from the apex (from the line joining the two plates).



a. We must solve $\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} = 0$ in a cylindrical geometry

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \cancel{\frac{\partial^2 \Phi}{\partial z^2}} = 0 \qquad \frac{\partial^2 \Phi}{\partial z^2} = 0 \text{ by symmetry}$$

We seek solutions of the form $\Phi(r,\theta) = R(r)\,Q(\theta)$

$$\Rightarrow \frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \theta^2} = 0 \quad \text{and dividing by } \Phi,$$

$$\Rightarrow \frac{1}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Qr^2}\frac{\partial^2 Q}{\partial \theta^2} = 0 \quad \text{and multiplying by } r^2,$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \theta^2} = 0$$

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda^2 R \quad \text{and} \quad \frac{\partial^2 Q}{\partial \theta^2} = -\lambda^2 Q$$

$$\Rightarrow \begin{cases} \lambda \neq 0: & R(r) = Ar^{\lambda} + Br^{-\lambda} \quad \text{and} \quad Q(\theta) = C\sin(\lambda\theta) + D\cos(\lambda\theta) \\ \text{or } \lambda = 0: & R(r) = A\ln(r) + B \quad \text{and} \quad Q(\theta) = C\theta + D \end{cases}$$

B.C $\Rightarrow \lambda = 0$ and $A = 0$ and $D = 0$, then $BC = \frac{V}{\alpha} \Rightarrow \Phi(r,\theta) = \frac{V}{\alpha}\theta$

b. The pressure is caused by the force from the wedge due to its polarization charge at $\theta = 0$ and $\theta = \beta$ interacting with the field.

$$-V = \int_0^{\alpha} \vec{E}\cdot d\vec{l} = \int_0^{\beta} E_\theta\, dl + \int_\beta^{\alpha} E_\theta\, dl \quad \text{where } dl = r\,d\theta$$

$$-V = r\beta E_\theta^{(\epsilon)} + r(\alpha-\beta)E_\theta^{(\epsilon_0)} = r\beta\frac{D_\theta}{\epsilon} + r(\alpha-\beta)\frac{D_\theta}{\epsilon_0}$$

$$\Rightarrow D_\theta = -\frac{V}{r}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-1}$$

Let $E_\theta^0 = E_\theta(\theta<0) = 0$, $E_\theta^1 = E_\theta(0<\theta<\beta) = \epsilon D_\theta$, $E_\theta^2 = E_\theta(\beta<\theta<\alpha) = \epsilon_0 D_\theta$

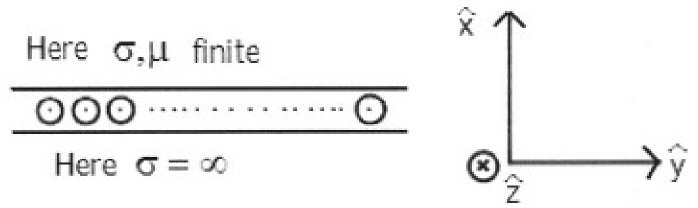
$$P_\beta = \sigma_\beta E_{avg}(\beta) = \frac{E_\theta^2 - E_\theta^1}{\epsilon_0}\frac{1}{2}(E_\theta^2 + E_\theta^1) = \frac{1}{2\epsilon_0}\left[(E_\theta^2)^2 - (E_\theta^1)^2\right] = \frac{1}{2\epsilon_0}(\epsilon_0^2 - \epsilon^2)\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2}$$

$$P_0 = \sigma_0 E_{avg}(0) = \frac{E_\theta^1 - E_\theta^0}{\epsilon_0}\frac{1}{2}(E_\theta^1 + E_\theta^0) = \frac{1}{2\epsilon_0}\left[(E_\theta^1)^2 - (E_\theta^0)^2\right] = \frac{1}{2\epsilon_0}\epsilon^2\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2}$$

$$P = P_\beta + P_0 = \frac{1}{2}\epsilon_0\frac{V^2}{r^2}\left[\beta\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_0}\right) + \frac{\alpha}{\epsilon_0}\right]^{-2} = \frac{1}{2}\frac{V^2}{r^2}\frac{\epsilon_0^3}{(\alpha - \beta + \beta\epsilon_0/\epsilon)^2}$$

9. *Electricity and Magnetism* (Spring 2005)

An infinitely thin current sheet carrying a surface current $\lambda = \lambda_o \hat{z}\cos(\omega t)$ is sandwiched between a perfect conductor ($\sigma = \infty$) and a material having finite conductivity σ and magnetic permeability μ. The angular frequency ω is sufficiently low that magnetostatic conditions prevail. λ_o is a constant, $\hat{z}$ is a unit vector parallel to the interface located at $x = 0$, and t is the time.



(a) Find the appropriate partial differential equation that governs the behavior of the magnetic field $\mathbf{H}$ for $x > 0$ (above the current sheet). Do not solve.

(b) What is the appropriate boundary condition for $\mathbf{H}$ in this system?

(c) Find the magnetic field $\mathbf{H}$ at an arbitrary distance $x > 0$ at time t.

See Griffiths Ex 5.8

a. $\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \lambda_o \hat{z}\cos(\omega t)\delta(x)$ since magnetostatics is the study of steady currents, so $\frac{\partial \vec{D}}{\partial t} = 0$.

$\Rightarrow \left|\begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ H_x & H_y & H_z \end{matrix}\right|_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \lambda_o \cos(\omega t)\,\delta(x)$

b. $H_\perp^2 - H_\perp^1 = 0 \Rightarrow H_x(0,y,z) = 0$ since it must be zero inside a perfect conductor, so it is also zero on the other side.

$\vec{H}_\parallel^2 - \vec{H}_\parallel^1 = \vec{K}_f \times \hat{n} \Rightarrow \vec{H}_\parallel(0,y,z) = \vec{\lambda}\,\hat{y}$

Combining these two we get $\vec{H}(0,y,z) = \lambda_o \cos(\omega t)\,\hat{y}$

c. The problem is totally symmetric in the y direction.

so $\frac{\partial H_x}{\partial y} = 0 \Rightarrow \frac{\partial H_y}{\partial x} = \lambda_o \cos(\omega t)\,\delta(x)$

$\Rightarrow H_y(x,y,z,t) = \int_0^x \lambda_o \cos(\omega t)\,\delta(x')\,dx' + C$

$= \lambda_o \cos(\omega t) + C$

Now $C = 0$ because of the B.C., so $H_y(x,y,z) = \lambda_o \cos(\omega t)$

The field can't have a z-component because the field must be perpendicular to the current by the Biot-Savart law. It also can't have an x-component because contributions from $-y$ cancel those from y.

Therefore $\vec{H}(x,y,z,t) = \lambda_o \cos(\omega t)\,\hat{y}$

10. *Electricity and Magnetism* (Spring 2005)

A relativistic charged particle of charge q and rest-mass m_o is in a region of uniform magnetic field $B_o\hat{z}$. At time $t = 0$ the particle has zero velocity along $\hat{z}$ (that is $\beta_z = v_z/c = 0$) and finite transverse speed $\beta_\perp = \beta_o$, with

$$\beta_\perp = \sqrt{v_x^2 + v_y^2}/c$$

Here, x, y, and z are Cartesian coordinates in the lab frame.

(a) What is the value of $\beta_\perp(t)$ for $t > 0$?

(b) What is the angular frequency Ω of rotation (that is, the gyrofrequency)? No need for a calculation, just identify Ω.

(c) Now apply a uniform electric field $E_o\hat{z}$, parallel to $\mathbf{B}$, starting at $t = 0$. Without solving the detailed equations, conclude what happens to the $\beta_\perp$ in part (a). Does it change?

a. Magnetic forces are always perpendicular to the direction of the field, so $B_z(t) = 0$. Also, magnetic forces do no work, so $B(t) = B(0) \Rightarrow B_\perp(t) = B_o$

b. Basically we just have to use the relativistic momentum $\vec{p} = \gamma m_o \vec{v}$. The force is $\vec{F} = q\vec{v}\times\vec{B} = qc\vec{\beta}\times\vec{B}$

$$\Rightarrow \vec{F} = qcB_o(\beta_y\hat{x} - \beta_x\hat{y})$$

$$\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \frac{dp_x}{dt} = qcB_o\beta_y \quad \text{and} \quad \frac{dp_y}{dt} = -qcB_o\beta_x$$

$$\gamma \text{ is constant} \Rightarrow \frac{dv_x}{dt} = \frac{qcB_o}{\gamma m_o}\beta_y \quad \text{and} \quad \frac{dv_y}{dt} = -\frac{qcB_o}{\gamma m_o}\beta_x$$

$$\Rightarrow \frac{d^2v_x}{dt^2} = \frac{qB_o}{\gamma m_o}\frac{dv_y}{dt} = -\left(\frac{qB_o}{\gamma m_o}\right)^2 v_x$$

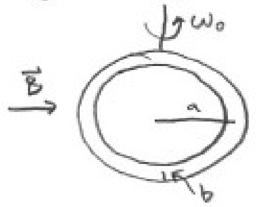
$$\Rightarrow v_x(t) = A\sin\left(\frac{qB_o}{\gamma m_o}t + \phi\right)$$

$$\Rightarrow \Omega = \frac{qB_o}{\gamma m_o}$$

c. There is the constraint $B^2 = B_\perp^2 + B_z^2 \leq 1$, so as the electric field accelerates B_z toward 1, $B_\perp$ must approach 0.

12. *Electricity and Magnetism* (Spring 2005)

A thin copper ring (conductivity σ, density ρ) is suspended so it can rotate freely about one diameter. There is a uniform magnetic field $\mathbf{B}$ perpendicular to the axis of rotation. At time $t = 0$ the ring is set rotating with frequency ω_o. Calculate the time it takes the frequency to decrease to $1/e$ of its original value, assuming the energy goes into Joule heating.



$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = -\frac{\partial}{\partial t}\left(\int_S \vec{B}\cdot d\vec{a}\right) = -\frac{\partial}{\partial t}\left(\int_S B\,da\cos(\omega t)\right)$$

$$= -B\frac{\partial}{\partial t}\left(\cos(\omega t)\int_S da\right) = +\pi a^2 B\omega\sin(\omega t)$$

$$\frac{dE_J}{dt} = P = \frac{\mathcal{E}^2}{R} \quad \text{where } R = \rho_R\frac{l}{A} = \frac{1}{\sigma}\frac{2\pi a}{\pi b^2} = \frac{2a}{\sigma b^2}$$

$$\Rightarrow \frac{dE_J}{dt} = \frac{\sigma b^2}{2a}\pi^2 a^4 B^2\omega^2\sin^2(\omega t) = \frac{1}{2}\pi^2\sigma a^3 b^2 B^2\omega^2\sin^2(\omega t)$$

We must assume ω is large enough that we can use the average power

$$\left\langle\frac{dE_J}{dt}\right\rangle = \frac{1}{4}\pi^2\sigma a^3 b^2 B^2\omega^2$$

Now, the kinetic energy is $T = \frac{1}{2}I\omega^2$ where $I = \int r^2 dm$

with $r = a\sin(\theta)$, $m(\theta) = (\pi b^2\rho)(a\theta) \Rightarrow \frac{dm}{d\theta} = \pi a b^2\rho$

So $I = 2\int_0^\pi a^2\sin^2(\theta)\pi a b^2\rho\,d\theta = 2\pi a^3 b^2\rho\int_0^\pi \sin^2(\theta)d\theta = \pi^2 a^3 b^2\rho$

Therefore $T = \frac{1}{2}I\omega^2 = \frac{1}{2}\pi^2 a^3 b^2\rho\omega^2$

$$\Rightarrow \frac{dT}{dt} = \pi^2 a^3 b^2\rho\omega\frac{d\omega}{dt}$$

By conservation of energy, the kinetic energy is decreasing at the instantaneous average rate that Joule heating energy is increasing.

$$\Rightarrow \frac{dT}{dt} = -\left\langle\frac{dE_J}{dt}\right\rangle \Rightarrow \cancel{\pi^2}\cancel{a^3}\cancel{b^2}\rho\omega\frac{d\omega}{dt} = -\frac{1}{4}\cancel{\pi^2}\sigma\cancel{a^3}\cancel{b^2}B^2\omega^2$$

$$\Rightarrow \frac{d\omega}{dt} = -\frac{\sigma B^2}{4\rho}\omega$$

$$\Rightarrow \omega(t) = \omega_o e^{-\frac{\sigma B^2}{4\rho}t}$$

So the frequency reaches $\frac{\omega_o}{e}$ when $t = \frac{4\rho}{\sigma B^2}$

14. *Statistical Mechanics and Thermodynamics* (Spring 2005)

A photon gas in thermal equilibrium is contained within a box of volume V at temperature T.

(a) Use the partition function to find the average number of photons $\bar{n}_r$ in the state having energy E_r.

(b) Find a relationship between the radiation pressure P and the energy density u (i.e. the average energy per unit volume).

(c) If the volume containing the photon gas is decreased adiabatically by a factor of 8, what is the final pressure if the initial pressure is P_o?

a. $$Z = \sum_r e^{-\beta E_r} = \sum_r e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$
$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots)}$$
$$= \left(\sum_{n_1=0}^{\infty} e^{-\beta n_1\epsilon_1}\right)\left(\sum_{n_2=0}^{\infty} e^{-\beta n_2\epsilon_2}\right)\cdots$$
$$= \left(\frac{1}{1-e^{-\beta\epsilon_1}}\right)\left(\frac{1}{1-e^{-\beta\epsilon_2}}\right)\cdots$$
$$\Rightarrow \ln(Z) = \sum_{r=1}^{\infty} \ln\left(\frac{1}{1-e^{-\beta\epsilon_r}}\right) = -\sum_{r=1}^{\infty}\ln(1-e^{-\beta\epsilon_r})$$

Therefore $$\bar{n}_r = -\frac{1}{\beta}\frac{\partial \ln(Z)}{\partial \epsilon_r} = +\frac{1}{\cancel{\beta}}\frac{1}{1-e^{-\beta\epsilon_r}}\cancel{\beta} e^{-\beta\epsilon_r} = \frac{1}{e^{\beta\epsilon_r}-1}$$

b. $$P = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial V} = -\frac{1}{\cancel{\beta}}\sum_{r=1}^{\infty}\frac{1}{1-e^{-\beta\epsilon_r}}\cancel{\beta}\frac{\partial\epsilon_r}{\partial V}e^{-\beta\epsilon_r}$$
$$= -\sum_{r=1}^{\infty}\frac{1}{e^{\beta\epsilon_r}-1}\frac{\partial\epsilon_r}{\partial V} = -\sum_{r=1}^{\infty}\bar{n}_r\frac{\partial\epsilon_r}{\partial V}$$

Now $$\epsilon_r = \hbar K c = \hbar c\sqrt{\left(\frac{n_x\pi}{L}\right)^2+\left(\frac{n_y\pi}{L}\right)^2+\left(\frac{n_z\pi}{L}\right)^2} = \frac{\hbar c\pi}{V^{1/3}}\sqrt{n_x^2+n_y^2+n_z^2}$$
$$\Rightarrow \frac{\partial\epsilon_r}{\partial V} = -\frac{1}{3}\frac{\hbar c\pi}{V^{4/3}}\sqrt{n_x^2+n_y^2+n_z^2} = -\frac{\epsilon_r}{3V}$$
$$\Rightarrow P = \sum_{r=1}^{\infty}\bar{n}_r\frac{\epsilon_r}{3V} = \frac{1}{3V}\sum_{r=1}^{\infty}\bar{n}_r\epsilon_r = \frac{E}{3V} = \frac{1}{3}u$$

c. $dE = -p\,dV$ since $đQ=0$ for an adiabatic process
$$\Rightarrow d(uV) = -\frac{u}{3}dV$$
$$\Rightarrow u\,dV + V\,du = -\frac{u}{3}dV$$
$$\Rightarrow \frac{dV}{V}+\frac{du}{u} = -\frac{1}{3}\frac{dV}{V}$$
$$\Rightarrow \frac{du}{u} = -\frac{4}{3}\frac{dV}{V}$$
$$\Rightarrow \ln(u) = -\frac{4}{3}\ln(V)+C$$
$$\Rightarrow u = A\,V^{-4/3}$$
$$\Rightarrow P = A'\,V^{-4/3}$$ so this is a polytropic process
$$\Rightarrow P_o V_o^{4/3} = P_f V_f^{4/3} = A' \Rightarrow P_f = P_o\left(\frac{V_o}{V_f}\right)^{4/3} = P_o(8)^{4/3} = 16P_o$$