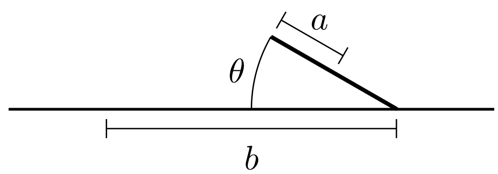
5. (Classical Mechanics)

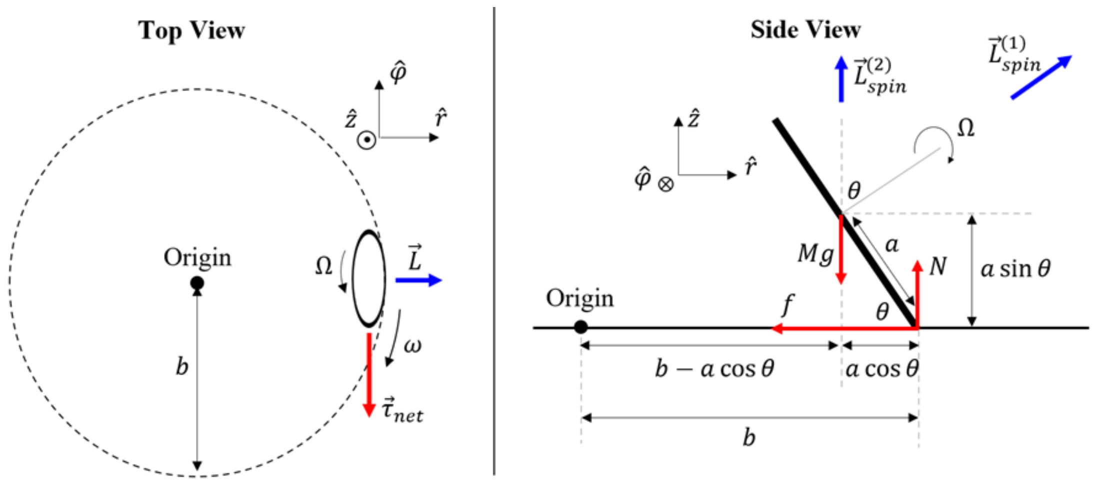
A coin, idealized as a uniform disk of radius a and negligible thickness, is rolling in a circle on a table. The point of contact describes a circle of radius b on the table. The plane of the coin makes an angle θ with the plane of the table. Find the angular velocity ω of the motion of the center of mass of the coin.

Hint: You don't need to use a Lagrangian for this problem, just Newton's laws.



Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This is an example of a gyroscope problem (any problem that has a spinning object precessing in space counts as a gyroscope problem). The first step in any gyroscope problem is to draw a detailed diagram of the setup:



Note that we have introduced the Ω to describe the angular velocity of the coin's rotation in the plane of the coin. We'll discuss the colored arrows in a moment. For now, make sure you understand the lengths marked on the diagram, as well as the orientation of each diagram. We will work in cylindrical coordinates (r, φ, z).

The only major formula that is useful for gyroscope problems is the relation between torque and angular momentum:

$$\boldsymbol{\tau}_{\text{net}} = \frac{d\mathbf{L}}{dt} \tag{12}$$

The challenging feature of gyroscope problems is that we need to use the vector version of this formula, since this is not fixed-axis rotation.

Let's start by calculating the net torque about the origin $\boldsymbol{\tau}_{\text{net}}$. The torque due to a force $\mathbf{F}$ is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{13}$$

where $\mathbf{r}$ is the vector from the origin to the point where the force is applied. There are three forces acting on the coin, all of which are shown in the side-view diagram above:

- The weight of the coin $\mathbf{W} = -Mg\,\hat{\mathbf{z}}$ (where M is the mass of the coin), which points downward from the center of mass of the coin. From the diagram above, we find that vector from the origin to the point where this force is applied is $\mathbf{r}_W = (b - a\cos\theta)\,\hat{\mathbf{r}} + a\sin\theta\,\hat{\mathbf{z}}$.
- The normal force $\mathbf{N} = N\,\hat{\mathbf{z}}$, which points upward (perpendicular to the table) at the point where the coin contacts the table. The vector from the origin to the point where this force is applied is $\hat{\mathbf{r}}_N = b\,\hat{\mathbf{r}}$.
- The friction force $\mathbf{f} = -f\,\hat{\mathbf{r}}$, which provides the centripetal acceleration needed to keep the coin moving in a circle and points toward the center of the circle at the point where the coin contacts the ground. The vector from the origin to the point where this force is applied is $\hat{\mathbf{r}}_f = b\,\hat{\mathbf{r}}$.

Before calculating the torques due to each force, note that since the center of mass of the coin is at rest in the $\hat{\mathbf{z}}$-direction, the net force in the $\hat{\mathbf{z}}$-direction is zero by Newton's second law. Since the weight and the normal force are the only forces in the $\hat{\mathbf{z}}$-direction, this implies that

$$0 = F_{\text{net},z} = N - Mg \qquad \Longrightarrow \qquad N = Mg \tag{14}$$

Now, we can repeatedly apply (13) to find the torque due to each force:

$$\begin{aligned} \text{Weight:} \quad \boldsymbol{\tau}_W &= \mathbf{r}_W \times \mathbf{W} \\ &= [(b - a\cos\theta)\,\hat{\mathbf{r}} + a\sin\theta\,\hat{\mathbf{z}}] \times [-Mg\,\hat{\mathbf{z}}] \\ &= -Mg(b - a\cos\theta)\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} \quad \text{since } \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \\ &= Mg(b - a\cos\theta)\hat{\varphi} \quad \text{since } \hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{r}} = -\hat{\varphi} \end{aligned} \tag{15}$$

$$\begin{aligned} \text{Normal force:} \quad \boldsymbol{\tau}_N &= \mathbf{r}_N \times \mathbf{N} \\ &= b\,\hat{\mathbf{r}} \times N\,\hat{\mathbf{z}} \\ &= Nb\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} \\ &= -Nb\,\hat{\varphi} \quad \text{since } \hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\hat{\mathbf{z}} \times \hat{\mathbf{r}} = -\hat{\varphi} \\ &= -Mgb\,\hat{\varphi} \quad \text{since } N = Mg \text{ by (14)} \end{aligned} \tag{16}$$

$$\begin{aligned} \text{Friction:} \quad \boldsymbol{\tau}_f &= \mathbf{r}_f \times \mathbf{f} \\ &= b\,\hat{\mathbf{r}} \times [-f\,\hat{\mathbf{r}}] \\ &= 0 \quad \text{since } \hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0 \end{aligned} \tag{17}$$

The net torque is just the sum of these three torques:

$$\begin{aligned} \boldsymbol{\tau}_{\text{net}} &= \boldsymbol{\tau}_W + \boldsymbol{\tau}_N + \boldsymbol{\tau}_f \\ &= Mg(b - a\cos\theta)\hat{\varphi} - Mgb\,\hat{\varphi} + 0 \\ \boldsymbol{\tau}_{\text{net}} &= -Mga\cos\theta\,\hat{\varphi} \end{aligned} \tag{18}$$

Note that the direction of $\boldsymbol{\tau}_{\text{net}}$ is marked in the top-view diagram above.

Now for the $\frac{d\mathbf{L}}{dt}$ part of (12). The angular momentum of the coin about the origin can be broken up into orbital and spin angular momentum:

$$\mathbf{L} = \mathbf{L}_{\text{orbit}} + \mathbf{L}_{\text{spin}} \tag{19}$$

The orbital angular momentum is just the angular momentum from the center of mass's motion about the origin:

$$\mathbf{L}_{\text{orbit}} = M\,\mathbf{r}_{\text{COM}} \times \mathbf{v}_{\text{COM}} \tag{20}$$

where $\mathbf{r}$ is the vector from the origin to the center of mass. From the side-view diagram above, we can see that

$$\mathbf{r}_{\text{COM}} = (b - a\cos\theta)\,\hat{\mathbf{r}} + a\sin\theta\,\hat{\mathbf{z}} \tag{21}$$

The center of mass of the coin is precessing about the z-axis in a circle with constant angular frequency ω, as shown on the top-view diagram above. The linear speed of a point a distance $r_\perp$ from the axis that precesses in this manner is given by

$$v = \omega r_\perp \tag{22}$$

In this case, from the side-view diagram above, we see that the distance from the center of mass to the z-axis is

$$r_\perp = b - a\cos\theta \tag{23}$$

so the speed of the center of mass is

$$v_{\text{COM}} = \omega(b - a\cos\theta) \tag{24}$$

The top-view diagram above shows that the center of mass moves in the $-\hat{\varphi}$ direction, so we have

$$\mathbf{v}_{\text{COM}} = -\omega(b - a\cos\theta)\hat{\varphi} \tag{25}$$

Applying (21) and (25) to the orbital angular momentum formula (20), we get (using $\hat{\mathbf{r}} \times \hat{\varphi} = \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \times \hat{\varphi} = -\hat{\varphi} \times \hat{\mathbf{z}} = -\hat{\mathbf{r}}$)

$$\begin{aligned}\mathbf{L}_{\text{orbit}} &= M\left[(b - a\cos\theta)\,\hat{\mathbf{r}} + a\sin\theta\,\hat{\mathbf{z}}\right] \times \left[-\omega(b - a\cos\theta)\hat{\varphi}\right] \\ &= -M\omega(b - a\cos\theta)^2\,(\hat{\mathbf{r}} \times \hat{\varphi}) - M\omega a\sin\theta(b - a\cos\theta)(\hat{\mathbf{z}} \times \hat{\varphi}) \\ \mathbf{L}_{\text{orbit}} &= -M\omega(b - a\cos\theta)^2\,\hat{\mathbf{z}} + M\omega a\sin\theta(b - a\cos\theta)\,\hat{\mathbf{r}} \end{aligned} \tag{26}$$

Now for the spin angular momentum. We will first consider the rotation of the coin that occurs in the plane of the coin, which we'll call $\mathbf{L}^{(1)}_{\text{spin}}$. The spin angular momentum due to this rotation is given by the equation

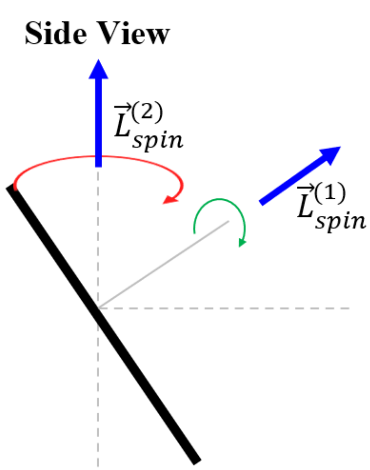
$$\mathbf{L}^{(1)}_{\text{spin}} = I_{\text{coin}}\boldsymbol{\Omega} \tag{27}$$

where $\boldsymbol{\Omega}$ has the magnitude of the angular velocity of the coin's rotation in the plane of the coin. The vector $\boldsymbol{\Omega}$ points along the axis of this rotation; from the side-view diagram above, we can determine that this direction is $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\mathbf{z}}$. Putting all this together, we get

$$\mathbf{L}^{(1)}_{\text{spin}} = I_{\text{coin}}\Omega\,(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\mathbf{z}}) \tag{28}$$

There is one other type of spin angular momentum here: The coin doesn't just rotate in the plane of the coin; it also rotates around the z-axis as it rolls. To understand this, imagine that you film the coin and stabilize the footage so that the center of mass of the coin is always at the center of the image. (This isolates the contribution of the spin angular momentum separately from the contribution of the orbital angular momentum.) You will see both the rotation of the coin in the plane of the coin (i.e. the image on the coin rotates upside-down and right-side-up again) and the rotation of the coin about the z-axis (the side of the coin facing the camera changes from heads to tails and back again). Grab a coin and try it out for yourself!

In the image below, the rotation of the coin in the plane of the coin is indicated by the green arrow. Its contribution to the spin angular momentum is $\mathbf{L}^{(1)}_{\text{spin}}$, which we have already calculated. The rotation of the coin about the z-axis is indicated by the red arrow. Its contribution to the spin angular momentum is $\mathbf{L}^{(2)}_{\text{spin}}$.



So what is $\mathbf{L}^{(2)}_{\text{spin}}$? For now, we won't calculate it, since we will soon see it doesn't matter for this problem. All we will do is note that it is in the $\hat{\mathbf{z}}$-direction and depends only on the parameters M, a, b, ω, Ω, θ, and g, all of which are constants. Therefore, we can write

$$\mathbf{L}^{(2)}_{\text{spin}} = L^{(2)}_{\text{spin}}\,\hat{\mathbf{z}} \tag{29}$$

Adding the orbital angular momentum (20) to the two types of spin angular momentum (28) and (29), we get the total angular momentum of the coin:

$$\begin{aligned}
\mathbf{L} &= \mathbf{L}_{\text{orbit}} + \mathbf{L}^{(1)}_{\text{spin}} + \mathbf{L}^{(2)}_{\text{spin}} \\
&= -M\omega(b - a\cos\theta)^2\,\hat{\mathbf{z}} + M\omega a\sin\theta(b - a\cos\theta)\,\hat{\mathbf{r}} + I_{\text{coin}}\Omega\,(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\mathbf{z}}) + L^{(2)}_{\text{spin}}\,\hat{\mathbf{z}} \\
\mathbf{L} &= [M\omega a\sin\theta(b - a\cos\theta) + I_{\text{coin}}\Omega\sin\theta]\,\hat{\mathbf{r}} + [\ldots]\,\hat{\mathbf{z}}
\end{aligned} \tag{30}$$

Here, $[\ldots]$ represents some quantity that depends on the parameters M, a, b, ω, Ω, θ, and g, all of which are constants.

We now need to find $\frac{d\mathbf{L}}{dt}$ and set it equal to the net torque $\boldsymbol{\tau}_{\text{net}}$ we calculated earlier. The parameters M, a, b, ω, Ω, θ, and g are constant, so they have no time derivative. The only time-dependent part of (30) is the vector $\hat{\mathbf{r}}$. Looking at the top-view diagram from earlier, we can see that

$$\begin{aligned}
\hat{\mathbf{r}}(t) &= \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}} \\
\implies \frac{d\hat{\mathbf{r}}}{dt} &= \frac{d\varphi}{dt}\,(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}) \\
&= \frac{d\varphi}{dt}\,\hat{\varphi} \\
\frac{d\hat{\mathbf{r}}}{dt} &= -\omega\,\hat{\varphi} \quad \text{since the coin rotates clockwise}
\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= [M\omega a\sin\theta(b - a\cos\theta) + I_{\text{coin}}\Omega\sin\theta]\,\frac{d\hat{\mathbf{r}}}{dt} \\
&= -\omega\,[M\omega a\sin\theta(b - a\cos\theta) + I_{\text{coin}}\Omega\sin\theta]\,\hat{\varphi} \\
\frac{d\mathbf{L}}{dt} &= -\omega\sin\theta\,[M\omega a(b - a\cos\theta) + I_{\text{coin}}\Omega]\,\hat{\varphi}
\end{aligned} \tag{32}$$

Note that the $\hat{\mathbf{z}}$-component of the angular momentum, including the spin angular momentum $\mathbf{L}^{(2)}_{\text{spin}}$, does not contribute to $\frac{d\mathbf{L}}{dt}$, since $\hat{\mathbf{z}}$ is a constant unit vector. We were free to ignore this component the whole time.

We can now set the net torque (18) equal to the change in angular momentum (32):

$$\begin{aligned}
-Mga\cos\theta\,\hat{\varphi} &= -\omega\sin\theta\,[M\omega a(b - a\cos\theta) + I_{\text{coin}}\Omega]\,\hat{\varphi} \\
Mga\cos\theta &= \omega\sin\theta\,[M\omega a(b - a\cos\theta) + I_{\text{coin}}\Omega]
\end{aligned} \tag{33}$$

Before solving for ω, we must relate the parameter Ω, the angular velocity of the coin's rotation in the plane of the coin, to the precession frequency ω. To do this, we need to assume the coin rolls without slipping.

The most general condition for the coin's rolling without slipping is that the distance covered by a point on the edge of the coin (relative to the center of the coin) is always equal to the distance covered by the center of mass covered by the coin. If the coin rolls through a positive angle φ on the table while simultaneously rotating in the plane of the coin through a positive angle α, we have

$$\begin{aligned}&\text{Distance covered by a point on the edge of the coin}\\ &\qquad = \text{Distance covered by the center of mass of the coin}\\ a\alpha &= b\varphi \end{aligned} \tag{34}$$

Taking the time derivative of this expression, we get

$$\begin{aligned} a\frac{d\alpha}{dt} &= b\frac{d\varphi}{dt} \\ a\Omega &= b\omega \end{aligned} \tag{35}$$

since Ω is the angular velocity of the coin's rotation in the plane of the coin, and ω is the angular velocity of the center of mass of the coin.

Plugging this relation into (33), we can solve for ω:

$$\begin{aligned} Mga\cos\theta &= \omega\sin\theta\left[M\omega a(b - a\cos\theta) + I_{\text{coin}}\left(\frac{b\omega}{a}\right)\right] \\ Mga\cos\theta &= \omega^2\sin\theta\left[Ma(b - a\cos\theta) + I_{\text{coin}}\frac{b}{a}\right] \\ \omega^2 &= \frac{Mga\cos\theta}{\sin\theta\left[Ma(b-a\cos\theta) + I_{\text{coin}}\frac{b}{a}\right]} \\ \omega &= \sqrt{\frac{Mga\cos\theta}{\sin\theta\left[Ma(b-a\cos\theta) + I_{\text{coin}}\frac{b}{a}\right]}} \end{aligned} \tag{36}$$

All that remains is to find the moment of inertia of the coin about an axis perpendicular to the plane of the coin I_{coin}. Assuming the coin is a uniform-density cylinder of radius, this can be calculated using the integral formula

$$\begin{aligned} I_{\text{coin}} &= \int dm\, r_\perp^2 \quad \text{where } r_\perp \text{ is the distance to the axis} \\ &= \int\left(\frac{M}{\pi a^2}\,d^2r\right)r^2 \quad \text{since } \frac{M}{\pi a^2} \text{ is the coin's density} \\ &= \left(\frac{M}{\pi a^2}\right)\int_{r=0}^{r=a}(2\pi r\,dr)\,r^2 \\ &= \left(\frac{M}{\pi a^2}\right)\left(\frac{2\pi a^4}{4}\right) \\ &= \frac{1}{2}Ma^2 \end{aligned} \tag{37}$$

Plugging into (36) and simplifying, we get

$$\begin{aligned} \omega &= \sqrt{\frac{Mga\cos\theta}{\sin\theta\left[Ma(b-a\cos\theta) + \left(\frac{1}{2}Ma^2\right)\frac{b}{a}\right]}} \\ &= \sqrt{\frac{g\cos\theta}{\sin\theta\left[(b-a\cos\theta)+\frac{b}{2}\right]}} \quad \text{dividing through by } Ma^2 \end{aligned}$$

$$\boxed{\omega = \sqrt{\frac{g}{\tan\theta\left[\frac{3}{2}b - a\cos\theta\right]}}} \tag{38}$$