Question 1: Quantum Mechanics

Consider two particles of masses $m_{1,2}$ in a one-dimensional harmonic oscillator potential $V = \frac{1}{2}m_1\omega_1^2 x_1^2 + \frac{1}{2}m_2\omega_2^2 x_2^2$. In the far past the x_1-oscillator is in the ground state while the x_2-oscillator is in its first excited state. They then experience a perturbation $\Delta V(x_2, x_2, t) = \lambda(x_1 - x_2)^2 e^{-\frac{1}{2}\alpha^2 t^2}$. Compute, to lowest nontrivial order in λ, the probability that in the far future the x_1-oscillator is in the first excited state while the x_2-oscillator is in its ground state.

Solution:

We apply time dependent perturbation theory. The amplitude for a transition from $|i\rangle$ to $|f\rangle$ is

$$A_{fi} = -\frac{i}{\hbar}\int_{-\infty}^{\infty} \langle f|\Delta H(t)|i\rangle e^{i\omega_{fi}t} dt$$

The ground state and first excited states of the harmonic oscillator have opposite parity, so only the cross term in the perturbation will have a nonzero matrix element, so

$$A_{fi} = \frac{2i\lambda}{\hbar}\langle 1|x_1|0\rangle\langle 0|x_2|1\rangle \int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2 t^2} e^{i\omega_{12}t} dt$$

with $\omega_{12} = \omega_1 - \omega_2$. The Gaussian integral is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha^2 t^2} e^{i(\omega_1-\omega_2)t} dt = \frac{\sqrt{2\pi}}{\alpha} e^{-\frac{\omega_{12}^2}{2\alpha^2}}$$

For the matrix element we express x in terms of ladder operators as

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

which gives

$$\langle 1|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|aa^\dagger|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

We thus get

$$A_{fi} = \frac{2i\lambda}{\hbar}\left(\frac{\hbar}{2\sqrt{m_1 m_2 \omega_1 \omega_2}}\right)\frac{\sqrt{2\pi}}{\alpha} e^{-\frac{\omega_{12}^2}{2\alpha^2}} = \frac{\sqrt{2\pi} i\lambda}{\sqrt{m_1 m_2 \omega_1 \omega_2}\alpha} e^{-\frac{\omega_{12}^2}{2\alpha^2}}$$

The probability is

$$P = |A_{fi}|^2 = \frac{2\pi\lambda^2}{m_1 m_2 \omega_1 \omega_2 \alpha^2} e^{-\frac{\omega_{12}^2}{\alpha^2}}$$

Question 2: Quantum Mechanics

A system is described by a Hilbert space spanned by two orthonormal kets $|0\rangle$ and $|1\rangle$. In this basis, the matrix elements of the Hamiltonian H_0 are:

$$\begin{pmatrix} \langle 0| H_0 |0\rangle & \langle 0| H_0 |1\rangle \\ \langle 1| H_0 |0\rangle & \langle 1| H_0 |1\rangle \end{pmatrix} = \begin{pmatrix} 2\hbar\omega & 0 \\ 0 & 0 \end{pmatrix}$$

where ω is real. At time $t = 0$ the system is in state $|0\rangle$, and a perturbation, H_1, is suddenly switched on. The matrix elements of H_1 are:

$$\begin{pmatrix} \langle 0| H_1 |0\rangle & \langle 0| H_1 |1\rangle \\ \langle 1| H_1 |0\rangle & \langle 1| H_1 |1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \hbar\lambda \\ \hbar\lambda & 0 \end{pmatrix}$$

where λ is real.

(a) Find the eigenvalues and (normalized) eigenvectors of the full Hamiltonian $H_0 + H_1$. You may express these eigenvalues, $E_\pm$, and eigenvectors, $|\mu_+\rangle$ and $|\mu_-\rangle$, in terms of ω, λ, Δ and α, where $\Delta^2 \equiv \omega^2 + \lambda^2$ and $\alpha^2 \equiv 2\Delta(\omega + \Delta) = (\omega + \Delta)^2 + \lambda^2$. The normalized eigenvectors can be expressed in the form $(c_1/\alpha,\, c_2/\alpha)$ and $(-c_2/\alpha,\, c_1/\alpha)$. Write down expressions for c_1 and c_2.

(b) Show that the probability of finding the system in state $|1\rangle$ at time t, given that it was in state $|0\rangle$ at time 0, is given by $(\lambda^2/\Delta^2)\sin^2(\Delta t)$

(c) By using time-dependent perturbation theory to first order, find an approximate expression for the probability in part (b).

(d) By Taylor expanding the exact probability in part (b), recover the perturbative result of part (c) in the limit that $\omega \gg \lambda$.

Solution

(a) The eigenvalues are given by diagonalizing the matrix:

$$\begin{pmatrix} 2\hbar\omega & \hbar\lambda \\ \hbar\lambda & 0 \end{pmatrix}$$

These eigenvalues are given by:

$$E_{\pm} = \hbar\omega \pm \frac{\sqrt{4\hbar^2\omega^2 + 4\hbar^2\lambda^2}}{2} = \hbar(\omega \pm \Delta)$$

The eigenkets are given by solving:

$$\begin{pmatrix} \hbar\omega - \hbar\Delta & \hbar\lambda \\ \hbar\lambda & -\hbar\omega - \hbar\Delta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \hbar\omega + \hbar\Delta & \hbar\lambda \\ \hbar\lambda & -\hbar\omega + \hbar\Delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

These equations give $v_1 = \omega + \Delta$, $v_2 = \lambda$, $w_1 = -\lambda$ and $w_2 = \omega + \Delta$. Normalizing gives:

$$|\mu_+\rangle = \begin{pmatrix} v_1/\sqrt{v_1^2+v_2^2} \\ v_2/\sqrt{v_1^2+v_2^2} \end{pmatrix} = \begin{pmatrix} (\omega+\Delta)/\alpha \\ \lambda/\alpha \end{pmatrix} = ((\omega+\Delta)/\alpha)\,|0\rangle + (\lambda/\alpha)\,|1\rangle$$

and

$$|\mu_-\rangle = \begin{pmatrix} w_1/\sqrt{w_1^2+w_2^2} \\ w_2/\sqrt{w_1^2+w_2^2} \end{pmatrix} = \begin{pmatrix} -\lambda/\alpha \\ (\omega+\Delta)/\alpha \end{pmatrix} = (-\lambda/\alpha)\,|0\rangle + ((\omega+\Delta)/\alpha)\,|1\rangle$$

(b) The time evolution of state $|0\rangle$ can be expressed as:

$$|0(t)\rangle = e^{-iE_+t/\hbar}((\omega+\Delta)/\alpha)\,|\mu_+\rangle - e^{-iE_-t/\hbar}(\lambda/\alpha)\,|\mu_-\rangle$$

State $|1\rangle$ can be expressed in the diagonal basis as:

$$|1\rangle = (\lambda/\alpha)\,|\mu_+\rangle + ((\omega+\Delta)/\alpha)\,|\mu_-\rangle$$

The probability is $P_{0\to 1} = |\langle 1|0(t)\rangle|^2 = (\lambda^2(\omega+\Delta)^2/\alpha^4)(2-2\cos((E_+-E_-)t/2\hbar)) = (\lambda^2/\Delta^2)\sin^2(\Delta t)$.

(c) Using time dependent perturbation theory, the probability amplitude for the transition is:

$$A_{0\to 1} = \frac{1}{i\hbar}\int_0^t (0\ 1)\begin{pmatrix} 0 & \hbar\lambda \\ \hbar\lambda & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{i(E_f-E_i)t_1/\hbar}dt_1 = \frac{1}{i}\int_0^t \lambda e^{-2i\omega t_1}dt_1$$

The transition probability is $P_{0\to 1} = |A_{0\to 1}|^2 = \lambda^2/\omega^2 \sin^2(\omega t)$.

(d) In the limit that $\omega \gg \lambda$, we get: $\Delta = \sqrt{\omega^2+\lambda^2} = \omega\sqrt{1+\lambda^2/\omega^2} = \omega(1+\lambda^2/2\omega^2+...)$. Therefore, to first order, $\Delta \approx \omega$ and (b) then resembles (c).

Question 3: Quantum Mechanics

Consider a system of two spin 1 particles: $|m_1\rangle$, $|m_2\rangle$ with $m_1, m_2 \in \{-1, 0, 1\}$. The system is governed by the Hamiltonian:

$$H = -\alpha \mathbf{S}_1 \cdot \mathbf{S}_2 + \beta (S_{1z} + S_{2z})^2$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the spin operators of the two particles and α, β are positive constants with $\beta > 2\alpha$.

By using the ladder operator $S_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\hbar |j, m-1\rangle$ or otherwise, find the energies and wavefunctions of the lowest two energy eigenstates. Express these energy eigenstates in the product basis of the two spin 1 particles.

Solution:

Let's represent the eigenstates of $(\mathbf{S}_1 + \mathbf{S}_2)^2, (S_{1z} + S_{2z})$ by $|j, m\rangle$, i.e.,

$$\begin{aligned}
(\mathbf{S}_1 + \mathbf{S}_2)^2 |j, m\rangle &= j(j+1)\hbar^2 |j, m\rangle \\
(S_{1z} + S_{2z}) |j, m\rangle &= m\hbar |j, m\rangle
\end{aligned}$$

The Hamiltonian can be recast as:

$$\begin{aligned}
H &= -(\alpha/2)\{(\mathbf{S}_1 + \mathbf{S}_2)^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\} + \beta(S_{1z} + S_{2z})^2 \\
H |j, m\rangle &= \{-(\alpha/2)\hbar^2\{(j(j+1) - 2 - 2\} + \beta(m)^2\hbar^2\} |j, m\rangle
\end{aligned}$$

The lowest energy eigenstate corresponds to largest j and smallest m and is thus $|j = 2, m = 0\rangle$ and the energy eigenvalue is $-\alpha\hbar^2$.

The corresponding eigenstate can be obtained by acting the ladder operator $S_- = S_{1-} + S_{2-}$ on

$|j = 2, m = 2\rangle = |1, 1\rangle \otimes |1, 1\rangle$

$$\begin{aligned}
S_- |j, m\rangle &= \sqrt{(j+m)(j-m+1)}\hbar |j, m-1\rangle \\
S_- |2, 2\rangle &= \sqrt{4}\hbar |2, 1\rangle \\
&= \sqrt{2}\hbar(|1, 0\rangle \otimes |1, 1\rangle + |1, 1\rangle \otimes |1, 0\rangle) \\
(S_-)^2 |2, 2\rangle &= \sqrt{4}\hbar S_- |2, 1\rangle \\
&= \sqrt{4}\sqrt{3}\sqrt{2}\hbar^2 |2, 0\rangle \\
&= \hbar^2\sqrt{2}(\sqrt{2} |1, -1\rangle |1, 1\rangle + \sqrt{2} |1, 0\rangle |1, 0\rangle + \sqrt{2} |1, 0\rangle |1, 0\rangle + \sqrt{2} |1, 1\rangle |1, -1\rangle) \\
&= 2\hbar^2(|1, -1\rangle |1, 1\rangle + 2 |1, 0\rangle |1, 0\rangle + |1, 1\rangle |1, -1\rangle) \\
\implies |2, 0\rangle &= \sqrt{\frac{1}{6}}(|1, -1\rangle |1, 1\rangle + 2 |1, 0\rangle |1, 0\rangle + |1, 1\rangle |1, -1\rangle)
\end{aligned}$$

The states with $j = 2$ have energies $-\alpha\hbar^2 + \beta\hbar^2 m^2$, the states with $j = 1$ have energies $\alpha\hbar^2 + \beta\hbar^2 m^2$

For $\beta > 2\alpha$, the first excited state is the $|j = 1, m = 0\rangle$ state.

To find this state, we first note that the state $|j = 1, m = 1\rangle$ is the unique state with $m = 1$ which is orthogonal to $|2, 1\rangle$. Thus, $|1, 1\rangle = \frac{1}{\sqrt{2}}(|1, 0\rangle \otimes |1, 1\rangle - |1, 1\rangle \otimes |1, 0\rangle)$

Then,

$$
\begin{aligned}
S_- \left| j=1, m=1 \right\rangle &= \sqrt{2}\hbar \left|1,0\right\rangle \\
&= S_- \frac{1}{\sqrt{2}} (\left|1,0\right\rangle \left|1,1\right\rangle - \left|1,1\right\rangle \left|1,0\right\rangle) \\
&= \frac{\hbar}{\sqrt{2}} (\sqrt{2} \left|1,-1\right\rangle \left|1,1\right\rangle + \sqrt{2} \left|1,0\right\rangle \left|1,0\right\rangle - \sqrt{2} \left|1,0\right\rangle \left|1,0\right\rangle - \sqrt{2} \left|1,1\right\rangle \left|1,-1\right\rangle) \\
\implies \sqrt{2} \left|1,0\right\rangle &= (\left|1,-1\right\rangle \left|1,1\right\rangle - \left|1,1\right\rangle \left|1,-1\right\rangle) \\
\implies \left|1,0\right\rangle &= \frac{1}{\sqrt{2}} (\left|1,-1\right\rangle \left|1,1\right\rangle - \left|1,1\right\rangle \left|1,-1\right\rangle)
\end{aligned}
$$

Question 4: Quantum Mechanics

A particle of mass m and charge q is confined to a circular ring of radius R lying in the x-y plane. There are also constant electric and magnetic fields: $\vec{E} = \mathcal{E}\hat{y}$, $\vec{B} = B\hat{z}$.

a) Write a Schrodinger equation for the energy levels of this system

b) Compute the energy spectrum in the regime where $\mathcal{E}$ is negligible compared to B. Hint: this will be simpler with the right choice of gauge

c) By making a suitable approximation in the Hamiltonian, compute the energy spectrum in the regime where $\mathcal{E}$ is large and B is negligible. Hint: think about where the wavefunction is concentrated in this limit.

Solution:

a) Set $c = 1$. We will start with the Lagrangian, although one could also directly write down the Hamiltonian.

The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - qV + q\vec{A} \cdot \dot{\vec{x}}\ , \quad x^2 + y^2 = R^2$$

We have the potentials

$$V = -\mathcal{E}y\ , \quad A_x = -\frac{1}{2}By\ , \quad A_y = \frac{1}{2}Bx$$

Polar coordinates

$$x = Rsin\theta\ , \quad y = R\cos\theta$$

This gives

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{qBR^2}{2}\dot{\theta} + q\mathcal{E}R\cos\theta$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} - \frac{qBR^2}{2}$$

The Hamiltonian is

$$H = p\dot{\theta} - L = \frac{1}{2mR^2}p^2 + \frac{qB}{2m}p - q\mathcal{E}R\cos\theta + \frac{1}{2m}\left(\frac{qBR}{2}\right)^2$$

The Schrodinger equation is

$$\left[-\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2} - \frac{i\hbar qB}{2m}\frac{d}{d\theta} - q\mathcal{E}R\cos\theta + \frac{1}{2m}\left(\frac{qBR}{2}\right)^2\right]\psi(\theta) = E\psi(\theta)$$

The wavefunction should be single valued on the ring: $\psi(\theta + 2\pi) = \psi(\theta)$.

b) If we ignore the $\mathcal{E}$ term, then we can take $\psi_n(\theta) = A_n e^{in\theta}$, where $n = 0, \pm1, \pm2, \ldots$. This gives

$$E_n = \frac{\hbar^2}{2mR^2}\left(n^2 + \frac{qBR^2n}{\hbar} + \left(\frac{qBR^2}{2\hbar}\right)^2\right) = -\frac{\hbar^2}{2mR^2}\left(n + \frac{qBR^2}{2\hbar}\right)^2$$

c) Now ignore the B terms,

$$\left[-\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2} - q\mathcal{E}R\cos\theta\right]\psi(\theta) = E\psi(\theta)$$

At large $\mathcal{E}$ the wavefunction will be localized near the minimum of the potential at $\theta = 0$, so we approximate $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. This gives

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{d\theta^2} + \frac{1}{2}q\mathcal{E}R^3\theta^2 \right]\psi(\theta) = R^2(E + q\mathcal{E}R)\psi(\theta)$$

The left hand side is a harmonic oscillator Hamiltonian with frequency $\omega^2 = q\mathcal{E}R^3/m$. Therefore the energy levels are

$$R^2(E_n + q\mathcal{E}R) = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{q\mathcal{E}R^3}{m}}$$

i.e.

$$E_n = \left(n + \frac{1}{2}\right)\hbar\sqrt{\frac{q\mathcal{E}}{mR}} - q\mathcal{E}R$$

Question 5: Classical Mechanics

a) Find a canonical transformation $P = P(p,q), Q = Q(p,q)$ that turns the Hamiltonian:

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$$

into

$H_0 = i\omega_0 PQ$ where P,Q are the new momentum and position.

b) Next, consider the driven harmonic oscillator:

$$m\ddot{q} + kq = F\cos\Omega t$$

Using the transformation $p,q \to P,Q$ that you derived in a), compute the transformed Hamiltonian and write down Hamiltons's equations of motion for P,Q.

Solution

a)-These are all quadratic forms so $P = ap + bq, Q = cp + dq$

$F_2(P,q) = \alpha P^2 + \beta Pq + \delta q^2$ so match terms.

$$P = -ip/\sqrt{2m\omega_0} - q\sqrt{m\omega_0/2},\ Q = p/\sqrt{2m\omega_0} + iq\sqrt{m\omega_0/2}$$

$$F_2(P,q) = (i/2)P^2 + iPq\sqrt{2m\omega_0} + (i/2)m\omega_0 q^2;$$

b)- With the driving term included $H = H_0 - Fq\cos\Omega t$;

in transformed coordinates:

$$q = -[P + iQ]/\sqrt{2m\omega_0}\ :$$

$$H = i\omega_0 PQ + F[P + iQ]/\sqrt{2m\omega_0}]\cos\Omega t$$

and by basic eqtns:

$$\dot{Q} = \partial H/\partial P = i\omega_0 Q + [F/\sqrt{2m\omega_0}]\cos\Omega t$$

$$\dot{P} = -\partial H/\partial Q = -i\omega_0 P - i[F/\sqrt{2m\omega_0}]\cos\Omega t$$

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$$q = -[P + iQ]/\sqrt{2m\omega_0}\ :$$

$$H = i\omega_0 PQ + F[P + iQ]/\sqrt{2m\omega_0}]\cos\Omega t$$

and by basic eqtns:

$$\dot{Q} = \partial H/\partial P = i\omega_0 Q + [F/\sqrt{2m\omega_0}]\cos\Omega t$$

$$\dot{P} = -\partial H/\partial Q = -i\omega_0 P - i[F/\sqrt{2m\omega_0}]\cos\Omega t$$

Question 7: Electromagnetism

A linearly polarized electromagnetic plane wave of wavenumber k and frequency ω is propagating in the z-direction. The electric field is in the y direction. The wave is scattered by two small dielectric spheres of radius a separated by a distance b with $b >> a$. The first sphere is centered at the origin while the second sphere is located on the z-axis with $z = b$. The two spheres have dielectric constant $\varepsilon/\varepsilon_0 = 1 + \chi$, with $\chi \ll 1$.

a) Consider the case $kb << 1$. Determine the differential and total cross section for the two spheres to leading order in kb.
b) Consider the case $kb \sim 1$, but still $ka << 1$. Determine the differential cross-section of the two spheres for light scattered at an angle θ in the x,z plane, measured with respect to the z-axis
c) Can you find values of kb for which the total radiated power from part a) and b) are the same?

Electromagnetism: scattering from 2 spheres

a) To leading order the two spheres are polarized in the same way

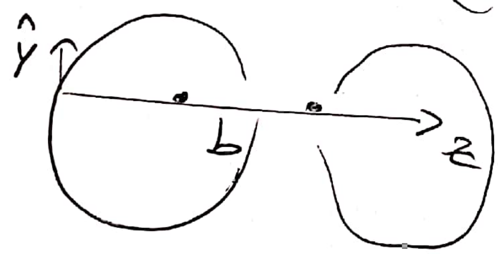
$$\vec{P}_1 = \varepsilon_0 \chi V E_0 e^{-i\omega t} \hat{y} \qquad \vec{P}_2 = \varepsilon_0 \chi V E_0 e^{-i\omega t} \hat{y}$$

$$V = \frac{4}{3}\pi a^3$$

Radiated power $\quad \dfrac{dP}{d\Omega} = \dfrac{p_0^2 \omega^4}{32\pi^2 \varepsilon_0 c^3} \left| \hat{r} \times \hat{y} \right|^2$

with $\quad p_0 = 2\varepsilon_0 \chi V E_0$

$$\frac{d\sigma}{d\Omega} = \frac{4\chi^2 V^2 k^4}{16\pi^2}\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)$$



radiation pattern.

$$\iint \left(\sin^2\theta\cos^2\varphi + \cos^2\theta \, d\cos\theta\right) d\varphi = \frac{8\pi}{3}$$

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{4\chi^2 V^2 \omega^4}{6\pi c^4}$$

b) The Two dipoles are driven out of phase

$$\vec{P}_1 = \varepsilon_0 \chi V E_0 e^{-i\omega t} \hat{y} \qquad \vec{P}_2 = \varepsilon_0 \chi V E_0 e^{-i\omega t + ikb} \hat{y}$$

$$\frac{dP}{d\Omega} = \frac{p_0^2 \omega^4}{32\pi^2 \varepsilon_0 c^3} \left| \hat{r}\times\hat{y} + \hat{r}\times\hat{y}\, e^{ikb(1-\cos\theta)} \right|^2$$

dipole located at $z'=b$

for $\hat{r}$ in x,z plane $\hat{r}\times\hat{y} = 1$ and

$$\frac{d\sigma}{d\Omega} = \frac{\chi^2 V^2 k^4}{16\pi^2} \left| 1 + e^{ikb(1-\cos\theta)} \right|^2 = \frac{\chi^2 V^2 k^4}{16\pi^2} \, 2\left(1 + \cos kb(1-\cos\theta)\right)$$

c) in the general case $\left|\hat{r}\times\hat{y}\right|^2 = \sin^2\theta \cos^2\varphi + \cos^2\theta$

$$\sigma = \frac{4\chi^2 V^2 k^4}{16\pi^2} \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \; \frac{1+\cos kb(1-\cos\theta)}{2} \left(\sin^2\theta \cos^2\varphi + \cos^2\theta\right)$$

$$= \frac{4\chi^2 V^2 k^4}{16\pi^2} \, 2\pi \left[\int_{-1}^{1} dx \, \frac{1+\cos kb(1-x)}{2} \, \frac{1+x^2}{2} \right]$$

$$\parallel \; \frac{2}{3} + \int_{-1}^{1} \frac{\cos kb(1-x)(1+x^2)}{4} \, dx$$

so a) = b) for all kb such that

$$\int_{-1}^{1} \cos kb(1-x) \cdot (1+x^2)\, dx = 0$$

for example $\cos kb = 0$

$$- \; kb = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Question 8: Electromagnetism

The purpose of this problem is to determine the current density and the magnetic field created by two spheres immersed in a medium with homogeneous and isotropic conductivity σ. Consider two spheres of equal radii a whose centers are separated by a distance $2d \gg a$, and which are held at constant potentials $+V$ and $-V$ respectively with $V > 0$. In the midplane between the two spheres, consider points at an equal distance $R \gg d$ from the two centers,

1. compute the current density vector $\mathbf{J}$ in terms of σ, V, a, d, R;
2. compute the magnetic field vector $\mathbf{B}$ in terms of σ, V, a, d, R.

Hint: place the center of the spheres at $(0, 0, \pm d)$, so the midplane is the xy-plane.

Fall 2020 Comprehensive Exam
Solution to EM8

We choose an oriented orthonormal frame with unit vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and place the centers of the spheres with potentials $\pm V$ on the z-axis at coordinates $(0, 0, \pm d)$ respectively. Since $a \ll d \ll R$ the potentials on the spheres may be generated by point-like charges $\pm Q$ located at the centers of the spheres whose magnitude is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 a}$$

1. To compute the current $\mathbf{J}$ we use Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, and compute the electric field at an arbitrary point $\mathbf{r}$ in space,

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0} \left(\frac{\mathbf{r} - d\mathbf{e}_z}{|\mathbf{r} - d\mathbf{e}_z|^3} - \frac{\mathbf{r} + d\mathbf{e}_z}{|\mathbf{r} + d\mathbf{e}_z|^3} \right)$$

Points at equal distance R from the two charges lie on a circle in the x, y plane given by,

$$R = |\mathbf{r} - d\mathbf{e}_z| = |\mathbf{r} + d\mathbf{e}_z|$$

the above formula simplifies and, eliminating Q in favor of V, we find,

$$\mathbf{E}(\mathbf{r}) = -2\frac{Vad}{R^3}\,\mathbf{e}_z, \qquad \mathbf{J} = -2\frac{\sigma V ad}{R^3}\,\mathbf{e}_z, \qquad \text{for } \mathbf{r} \cdot \mathbf{e}_z = 0$$

2. The magnetic field $\mathbf{B}$ at points in the x, y plane is constant on the circle $R^2 = \mathbf{r}^2 + d^2$ and points in the direction $\mathbf{e}_\phi$ tangent to the circle with $\mathbf{e}_\phi = \mathbf{e}_y$ for a point on the positive x-axis. By Ampère's law, we have,

$$\oint d\mathbf{l} \cdot \mathbf{B} = \mu_0 \int_{\mathbf{r}^2 \leq R^2 - d^2} d\mathbf{S} \cdot \mathbf{J}$$

The line integral on the left is over the circle in the x, y plane with radius $|\mathbf{r}| = \sqrt{R^2 - d^2}$. With $\mathbf{B} = B\mathbf{e}_\phi$, and using $RdR = \mathbf{r} \cdot d\mathbf{r}$ we have,

$$2\pi|\mathbf{r}|B = -2\sigma\mu_0 V ad \times 2\pi \int_d^R R' dR' \frac{1}{(R')^3}$$

Carrying out the integral, using the approximation $|\mathbf{r}| \sim R$ on the left side in view of $d \ll R$, and simplifying the result, we find,

$$B = 2\frac{\sigma\mu_0 V ad}{R} \left(\frac{1}{R} - \frac{1}{d} \right) \approx -2\frac{\sigma\mu_0 V a}{R}$$

This means that the orientation of the magnetic field at a point on the positive x-axis is in the direction $-\mathbf{e}_y = -\mathbf{e}_\phi$, which is consistent with the cork-screw mnemonic.

Question 9: Electromagnetism

A conductor is often modeled with a simple Ohm's law:

$$\vec{j} = \sigma_0 \vec{E}$$

in both real and Fourier space where σ_0 is the conductivity. However, a conductor is more accurately modeled with a modified frequency dependent conductivity $\sigma = \frac{\sigma_0}{1 - i\omega \frac{\sigma_0}{\epsilon_0 \omega_p^2}}$ (this frequency dependent conductivity assumes quantities vary as $e^{-i\omega t}$), which is equivalent to the modified Ohm's law,

$$\frac{\sigma_0}{\epsilon_0 \omega_p^2} \partial \vec{j} / \partial t + \vec{j} = \sigma_0 \vec{E}$$

where ω_p is the plasma frequency of the conduction electrons, $\omega_p^2 \equiv \frac{e^2 n_0}{\epsilon_0 m}$, n_0 is the density of conduction electrons, and ϵ_0 is the permittivity of free space. Finally, at t=0, a small amount of excess charge, Q, is uniformly distributed throughout a sphere of radius r_0 with conductivity σ_0 at t=0.

a) Using combinations of the relevant Maxwells equations and the continuity equation, derive the equation for the charge density inside the sphere and then obtain the solution for it for the correct initial conditions. Your answer should depend on r_0, ϵ_0, σ_0, ω_p, and t.

b) For copper the density of conduction electrons, $n_0 = .85 \times 10^{29} m^{-3}$ and the conductivity is $6 \times 10^7 S/m$. Show that for these parameters, $4 >> \epsilon_0^2 \omega_p^2 / \sigma_0^2$. Under this condition, what is the formula and the time in seconds that it takes for the charge density to decrease to $1/e$ of its initial value at any location in the sphere?

Solutions:

S1.a There are several ways to obtain a single equation for the charge density. The quickest is to take the divergence of Ohm's law to get

$$\frac{\sigma}{\epsilon_0 \omega_p^2} \partial \vec{\nabla} \cdot \vec{j} / \partial t + \vec{\nabla} \cdot \vec{j} = \sigma \vec{\nabla} \cdot \vec{E}$$

From the continuity equation substitute $\vec{\nabla} \cdot \vec{j} = -\partial \rho / \partial t$ and from Gauss's law substitute $\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ to get:

$$\partial^2 \rho / \partial t^2 + \frac{\epsilon_0 \omega_p^2}{\sigma_0} \partial \rho / \partial t + \omega_p^2 \rho = 0$$

This is a simple harmonic oscillator equation with a damping term. There are several ways to get a general solution which is of the form

$$\rho = A e^{-i\omega_+ t} + B e^{-i\omega_- t}$$

where

$$\omega_\pm = \frac{1}{2}(-i\epsilon_0 \omega_p^2 / \sigma_0 \pm \sqrt{4\omega_p^2 - \epsilon_0^2 \omega_p^4 / \sigma_0^2})$$

and

$$A = \rho_0 (1 - i\frac{\omega_+}{\omega_-} / (1 + (\frac{\omega_+}{\omega_-})^2)$$

$$B = \rho_0 (1 - i\frac{\omega_-}{\omega_+} / (1 + (\frac{\omega_-}{\omega_+})^2)$$

where we used the fact $\partial \rho / \partial t = 0$ at t=0. In addition, $\rho = Q / [(4/3)\pi r_0^3]$. This solution is valid for all space because there are no partial derivatives in space.

S1.b First, if you plug in the numbers you can see that $\frac{\epsilon_0^2 \omega_p^2}{4\sigma_0^2} \sim 10^{-5}$. Under this condition,

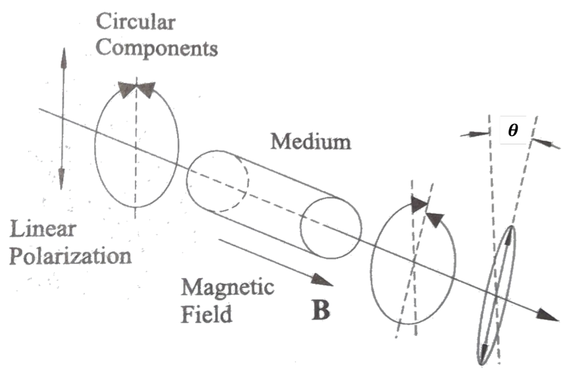
$$\omega_\pm \approx (-i\epsilon_0 \omega_p^2 / (2\sigma_0) \pm \omega_p,$$

therefore the charge decays as $e^{-\epsilon_0 \omega_p^2 / (2\sigma_0) t}$ and the charge will decay (towards the surface) in a time

$$\tau = 2\sigma_0 / (\epsilon_0 \omega_p^2) = 2.1 \times 10^{-14} s$$

while it oscillates at the plasma frequency.

Question 10: Electromagnetism



Consider the effect of a birefrigent medium with a complex linear susceptibility, χ, on a linearly polarized electromagnetic wave, $\vec{\mathcal{E}} = \mathcal{E}_0 \hat{\epsilon} \cos(kz - \omega t)$, with $\mathcal{E}_0$ the electric field amplitude, $\hat{\epsilon}$ a unit vector in the direction of the polarization, k the wavenumber, and ω the angular frequency. Near a resonance ω_0 in the medium, the susceptibility has a complex Lorentzian lineshape given by

$$\chi(\Delta) = \chi_0 \left[\frac{2\Delta}{1 + 4\Delta^2} - i \frac{1}{1 + 4\Delta^2} \right]$$

where χ_0 is the amplitude of linear susceptibilty and $\Delta = (\omega - \omega_0)/\gamma_0$ is the detuning from resonance in units of the resonance width γ_0. For $\chi_0 \ll 1$ the complex index of refraction is approximated by $n \approx 1 + \chi/2$.

(a) Discuss qualitatively what effects the real and imaginary parts of the susceptibility can have on the plane wave.

(b) On resonance ($\Delta = 0$) what is the absorption length, L, defined as the distance where the electric field amplitude is $1/e$ smaller than the initial value the wave had when it entered the medium?

(c) In the presence of a magnetic field, the resonance in a birefringent medium splits, giving twp different detunings

$$\Delta_\pm = \Delta \pm \Delta_B$$

where the $+, -$ refer to left- and right-handed circular polarizations, respectively. This leads to a difference in the indices of refraction

$$\Delta n = n(\Delta_+) - n(\Delta_-)$$

for the right- and left-handed circularly polarized waves. Consider a wave that initially has linear polarization. After a distance l in the birefringent medium and on resonance ($\Delta = 0$), what is the angle, θ, that the polarization has rotated? Sketch the angle of rotation, θ versus Δ_B.

Solution

(a) The index of refraction affects the wavenumber $k_\pm = n_\pm \omega / c$. Looking at the spatial part of our plane wave

$$\vec{\mathcal{E}} \propto \hat{\epsilon} \exp(ikz) = \hat{\epsilon} \exp(in\omega z/c)$$

it is clear that the real part of n leads to a phase shift while the imaginary part leads to exponential attenuation. Thus the real part of the susceptibility leads to a phase shift of an electromagnetic wave while the imaginary part leads to absorption.

(b) The field amplitude depends on the imaginary part of the index of refraction:

$$\mathcal{E}(l) = \mathcal{E}_0 \exp(-\mathrm{Im}(n)\omega l/c).$$

The amplitude is down by a factor of e when

$$\frac{\mathrm{Im}(n)\omega L}{c} = 1$$

or

$$L = \frac{2c}{\chi_0 \omega}.$$

(c) Circular polarizations

$$\hat{\epsilon}_\pm = \frac{\pm \hat{x} + i\hat{y}}{\sqrt{2}}$$

are related to linear polarizations by

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{\epsilon}_+ - \hat{\epsilon}_-)$$
$$\hat{y} = \frac{-i}{\sqrt{2}}(\hat{\epsilon}_+ + \hat{\epsilon}_-).$$

It is clear that a relative phase shift of $\Delta\phi = \pi$ between the two circular polarizations changes $\hat{x}$-polarized light to $\hat{y}$-polarized light, or a rotation of $\theta = \pi/2 = \Delta\phi/2$. The rotation angle is then given by

$$\theta = \Delta\phi/2 = \mathrm{Re}[n(\Delta_+) - n(\Delta_-)]\frac{\omega l}{2c}$$

Substituting in the new detunings gives

$$n(\Delta_\pm) \approx 1 + \frac{\chi_0}{2}\left[\frac{2(\Delta \pm \Delta_B)}{1 + 4(\Delta \pm \Delta_B)^2} - i\frac{1}{1 + 4(\Delta \pm \Delta_B)^2}\right].$$

The real part of the difference in the indices of refraction is then

$$\begin{aligned}\Delta n = \mathrm{Re}[n(\Delta_+) - n(\Delta_-)] &\approx \frac{\chi_0}{2}\left[\frac{2(\Delta+\Delta_B)}{1+4(\Delta+\Delta_B)^2} - \frac{2(\Delta-\Delta_B)}{1+4(\Delta-\Delta_B)^2}\right] \\ &= \chi_0\left\{\frac{(\Delta+\Delta_B)[1+4(\Delta-\Delta_B)^2] - (\Delta-\Delta_B)[1+4(\Delta+\Delta_B)^2]}{[1+4(\Delta+\Delta_B)^2][1+4(\Delta-\Delta_B)^2]}\right\} \\ &= 2\chi_0\left[\frac{\Delta_B(1+4\Delta_B^2-4\Delta^2)}{16\Delta^2+(1+4\Delta_B^2-4\Delta^2)^2}\right].\end{aligned}$$

Therefore the rotation angle is

$$\theta = \frac{\chi_0\omega l}{c}\frac{\Delta_B(1+4\Delta_B^2-4\Delta^2)}{16\Delta^2+(1+4\Delta_B^2-4\Delta^2)^2}.$$

On resonance, $\Delta = 0$, we have

$$\theta = \frac{\chi_0\omega l}{c}\frac{\Delta_B}{1+4\Delta_B^2} = 2\frac{l}{L}\frac{\Delta_B}{1+4\Delta_B^2}.$$

Question 11: Statistical Mechanics

The only degrees of freedom we retain in an approximate description of an assembly of N weakly interacting particles are their internal energy levels, which we assume to be non-degenerate and taking the three possible values $-\varepsilon$, 0, ε with $\varepsilon > 0$. The system is in equilibrium contact with a heat bath at temperature T . Assume Boltzmann statistics. Evaluate the following quantities for the system,

1. the entropy at zero temperature;
2. the maximal possible entropy;
3. the minimal possible energy;
4. the partition function;
5. the average energy;
6. the value of $\int_0^\infty dT \frac{C(T)}{T}$ where $C(T)$ is the specific heat function.

Solution:

1. $S(T=0)=0$ by the third law of thermodynamics;
2. $S_{\text{max}} = k\ln\Omega_{\text{max}} = k\ln 3^N = kN\ln 3$;
3. the minimum energy is $-N\varepsilon$
4. The partition function is,

$$Z = \left(e^{\beta\varepsilon} + 1 + e^{-\beta\varepsilon}\right)^N \qquad\qquad \beta = \frac{1}{kT}$$

5. The average energy is the internal energy,

$$E = -\frac{\partial}{\partial\beta}\ln Z = -N\varepsilon\frac{e^{\beta\varepsilon} - e^{-\beta\varepsilon}}{1 + e^{\beta\varepsilon} + e^{-\beta\varepsilon}}$$

6. The integrand satisfies $C(T)dT/T = dS$, so it follows,

$$\int_0^\infty dT\frac{C(T)}{T} = \int_{T=0}^{T=\infty} dS = kN\ln 3$$

Question 12: Statistical Mechanics

Consider a d-dimensional gas of spin-$1/2$ electrons (two spin states per electron). The gas is enclosed in a rectangular box whose sides have equal length L. Assume that the box is large enough such that the spectrum may be approximated by a continuum.

Define the surface area of the d-dimensional hypersphere of radius r as $S_d r^{d-1}$ (e.g. $S_2 = 2\pi$ and $S_3 = 4\pi$).

(a) Given electron density $\rho = N/L^d$, where N is the total number of electrons, calculate the Fermi wavevector k_F. Express your answer in terms of d, ρ, and S_d.

(b) Using the definition $E_F = \hbar^2 k_F^2/2m$, calculate the density of states per unit volume, $D(E)$, as a function of energy E. Express your answer in terms of ρ, d and E_F.

Solution

(a) The total number of electrons, N can be summed in the following way:

$$N = 2\int d^d n = 2\frac{L^d}{(2\pi)^d}\int_0^{k_F} d^d k = 2\frac{L^d S_d}{(2\pi)^d}\int_0^{k_F} dk k^{d-1} = 2\frac{L^d S_d k_F^d}{d(2\pi)^d} \quad (1)$$

We can then solve for k_F in terms of the density, $\rho = N/L^d$:

$$k_F = \left(\frac{d\rho(2\pi)^d}{2S_d}\right)^{1/d} \quad (2)$$

(b) The density of states, $D(E)$ can be obtained using $N = L^d \int D(E)dE$. Using $E = \hbar^2 k^2/2m$, we can write:

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad \text{and} \qquad dk = \frac{\sqrt{m}}{\hbar\sqrt{2E}}dE \quad (3)$$

Using the expression from (a), we can plug the appropriate values in:

$$N = 2\frac{L^d S_d}{(2\pi)^d}\int_0^{k_F} dk k^{d-1} = \frac{L^d S_d}{(2\pi)^d}\int_0^{E_F} dE \left(\frac{2m}{\hbar^2}\right)^{d/2} E^{\frac{d-2}{2}} \quad (4)$$

Therefore, the density of states per unit volume is therefore:

$$D(E) = \frac{S_d}{(2\pi)^d}\left(\frac{2m}{\hbar^2}\right)^{d/2} E^{\frac{d-2}{2}} \quad (5)$$

To express this in terms of the necessary quantities, we use that $E_F = \hbar^2 k_F^2/2m$ and the answer from part (a) to write:

$$D(E) = \frac{d\rho}{2E_F}\left(\frac{E}{E_F}\right)^{\frac{d-2}{2}} \quad (6)$$