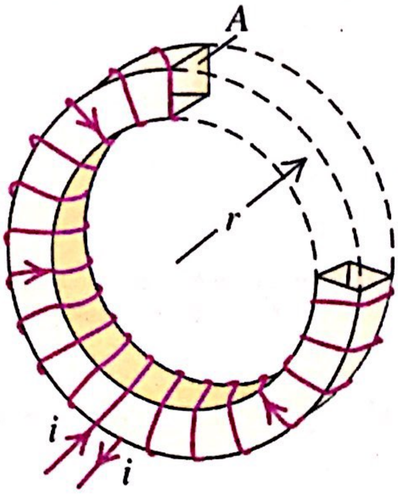
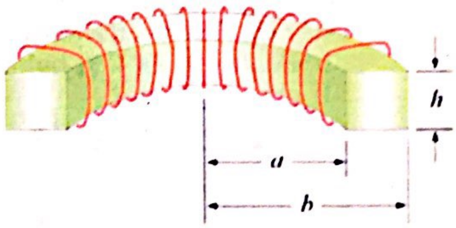
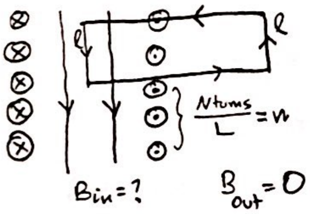
1. **Previous Midterm** *(by Prof. Simon)*. The figure below schematically illustrates a complete toroid with a rectangular cross-section. The wire is wound in N loops.





a) Find the magnetic field B(r) inside the toroid, $a \leq r \leq b$. **[5 points]**

Start w/ solenoid:

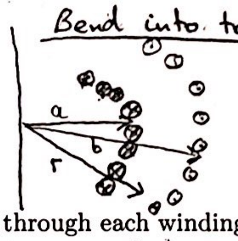


$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{encl.}$

$B\ell = \mu_0 \frac{N\ell}{L} \cdot I$

$B = \mu_0 n I$

Bend into toroid: N = same as solenoid, $n \neq$ same

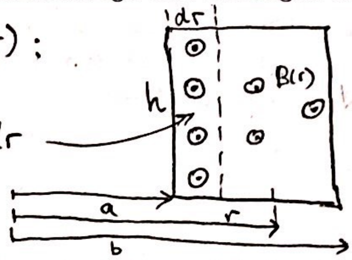


$n = \frac{N}{L}$; $n(a) = \frac{N}{2\pi a}$, $n(b) = \frac{N}{2\pi b}$,

$n(r) = \frac{N}{2\pi r}$ $\Rightarrow$ $\boxed{B(r) = \frac{\mu_0 I N}{2\pi r}}$

b) What is the magnetic flux through each winding in the toroid? **[5 points]**

Careful, $B(\underline{r})$:

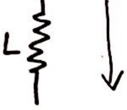


$\Phi_B = \int B(r)\, dA =$

$= \int_a^b B(r)\, h\, dr \Rightarrow$

$\Rightarrow \boxed{\Phi_B = \frac{\mu_0 I N h}{2\pi} \ln(b/a)}$

c) What is the self-inductance L of the toroid? **[5 points]**



Voltage drop, $V_L = L \frac{dI}{dt}$ (general)

Inductance, $\mathcal{E} = -N \frac{d\Phi_B}{dt}$

Self-inductance, $V_L = |\mathcal{E}|$:

$$N \frac{d\Phi_B}{dt} = L \frac{dI}{dt}$$

N, L = const., so $L_{self} = \frac{N \Phi_B}{I}$

$L_{self} = \frac{N\Phi}{I} \Rightarrow$

$$\boxed{L = \frac{\mu_0 N^2 h}{2\pi} \ln(b/a)}$$

d) What is the magnetic *energy density*, u_B, inside the toroid? **[5 points]**

$$u_B = \frac{B^2}{2\mu_0} \; ; \quad B = \frac{\mu_0 I N}{2\pi r}$$

$$\boxed{u_B = \frac{\mu_0 I^2 N^2}{8\pi^2 r^2}}$$

e) Integrate the magnetic energy density over the volume to get the magnetic energy U_B stored in the toroid. Express your answer in terms of the self-inductance L and the current I. **[5 points]**

$$U_B = \int u_B \, dV \; ; \quad dV = ? \rightarrow \text{use cylindrical coord's (have axis-symmetry)}$$

$\Rightarrow \int d\phi = 2\pi$

$\int dz = h$

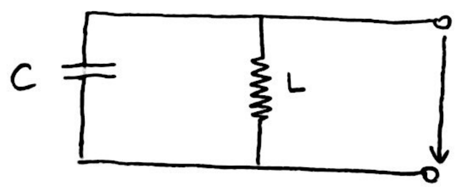
$\int u_B(r)\, r dr = ?$

$$= 2\pi h \int_a^b u_B(r)\, r dr =$$

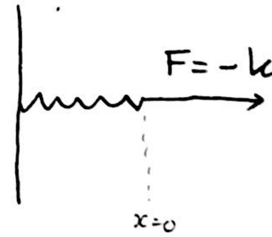
$$= \frac{2\pi h \, \mu_0 I^2 N^2}{8\pi^2} \int_a^b \frac{1}{r} dr = \frac{1}{2} \underbrace{\frac{h \mu_0 N^2 \ln(b/a)}{2\pi}}_{L} \cdot I^2 \Rightarrow$$

$$\Rightarrow \boxed{U_B = \frac{1}{2} L I^2 = \text{Magnetic energy}} \sim \text{looks like kinetic energy!}$$

2. Compare the equation for charge motion in an LC circuit with the equation of motion of a mass on a spring. Can you spot an analogy? Predict what role the circuit's resistance would play. **[8 points]**



$V_{tot} = 0 \Rightarrow V_L = -V_c$



$F = -kx = m\frac{d^2x}{dt^2}$

$$V_c = \frac{Q}{c} \; , \quad V_L = L\frac{dI}{dt}$$

$$\Rightarrow L\frac{dI}{dt} = -\frac{Q}{c}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx \Rightarrow$$

$$I = \frac{dQ}{dt} \quad \Rightarrow \quad \frac{d^2Q}{dt^2} = -\frac{1}{cL} Q \sim \text{simple harmonic motion, } \omega^2 = \frac{1}{cL} \Rightarrow \frac{d^2x}{dt^2} = -\underbrace{\frac{k}{m}}\, x$$

$$= -\omega^2 Q$$

$$= -\omega^2 x$$

$$\Rightarrow \frac{1}{c} \sim k \, , \; L \sim m \, , \; U = RI = R\frac{dQ}{dt} \sim F_{damping} = \gamma v = \gamma \frac{dx}{dt} \Rightarrow R \sim \text{damping constant.}$$