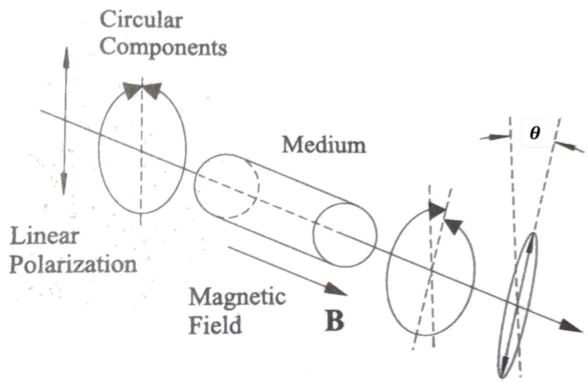
10. (Electromagnetism)



Consider the effect of a birefringent medium with a complex linear susceptibility χ on a linearly polarized electromagnetic wave, $\vec{\mathcal{E}} = \mathcal{E}_0 \hat{\epsilon} \cos(kz - \omega t)$, with $\mathcal{E}_0$ the electric field amplitude, $\hat{\epsilon}$ a unit vector in the direction of the polarization, k the wave number, and ω the angular frequency. Near a resonance ω_0 in the medium, the susceptibility has a complex Lorentzian lineshape given by

$$\chi(\Delta) = \chi_0 \left[\frac{2\Delta}{1 + 4\Delta^2} - i \frac{1}{1 + 4\Delta^2} \right]$$

where χ_0 is the amplitude of linear susceptibility and $\Delta = (\omega - \omega_0)/\gamma_0$ is the detuning from resonance in units of the resonance width γ_0. For $\chi_0 \ll 1$ the complex index of refraction is approximated by $n \approx 1 + \chi/2$.

(a) Discuss qualitatively what effects the real and imaginary parts of the susceptibility can have on the plane wave.

(b) On resonance ($\Delta = 0$) what is the absorption length L, defined as the distance where the electric field amplitude is $1/e$ smaller than the initial value the wave had when it entered the medium?

(c) In the presence of a magnetic field, the resonance in a birefringent medium splits, giving two different detunings

$$\Delta_\pm = \Delta \pm \Delta_B$$

where the +,- refer to the left- and right-handed circular polarizated waves. Consider a wave that initially has linear polarization. After a distance l in the birefringent medium and on resonance , what is the angle θ that the polarization has rotated? Sketch the angle of rotation θ versus Δ_B.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

At first glance, there seems to be a lot going on in this problem. The key is to take the parts one at a time, since each part relies only on a small portion of the given information.

(a) The problem tells us that the complex index of refraction is approximately $n \approx 1 + \chi/2$. (Note that we are meant to assume $\chi_0 \ll 1$.) The index of refraction is related to the wavenumber of a plane wave via the speed of the wave:

$$\omega = vk = \frac{ck}{n} \quad \text{where } v = \frac{c}{n} \text{ is the speed of the wave} \tag{336}$$

v is properly defined as real, but the relationship $\omega = \frac{ck}{n}$ is also true for complex k and n. Therefore, the complex wavenumber of the plane wave is

$$\begin{aligned} k &= \frac{\omega n}{c} \\ &\approx \frac{\omega}{c}\left(1 + \frac{\chi}{2}\right) \qquad (337) \\ &= \frac{\omega}{c}\left[\left(1 + \frac{1}{2}\operatorname{Re}(\chi)\right) + \frac{i}{2}\operatorname{Im}(\chi)\right] \qquad (338) \end{aligned}$$

So the real part of the susceptibility affects the real part of the wavenumber, and the imaginary part of the susceptibility affects the imaginary part of the wavenumber. The complex version of the electric field is

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, t) &= \varepsilon_0\, \hat{\epsilon}\, e^{i(kz - \omega t)} \\ &= \varepsilon_0\, \hat{\epsilon}\, e^{i((\operatorname{Re}(k) + i\operatorname{Im}(k))z - \omega t)} \\ &= \varepsilon_0 e^{-\operatorname{Im}(k) z}\, \hat{\epsilon}\, e^{i(\operatorname{Re}(k) z - \omega t)} \end{aligned} \tag{339}$$

so the real electric field is

$$\mathbf{E}(\mathbf{r}, t) = \operatorname{Re}\left(\tilde{\mathbf{E}}(\mathbf{r}, t)\right) = \varepsilon_0 e^{-\operatorname{Im}(k) z}\, \hat{\epsilon}\, \cos\left(\operatorname{Re}(k) z - \omega t\right) \tag{340}$$

From this equation, we can tell that

- Changing the real part of the electric susceptibility changes the real part of the wavenumber $\operatorname{Re}(k)$. This changes the speed of the wave, defined as $v \equiv \frac{k}{\omega}$. For constant z, this can also be thought of as a phase shift in the wave (but this phase shift is not spatially constant).
- Changing the imaginary part of the electric susceptibility changes the imaginary part of the wavenumber $\operatorname{Im}(k)$. This changes the spatial attenuation of the amplitude of the electric field, which is controlled by the prefactor $e^{-\operatorname{Im}(k) z}$. In other words, the imaginary part of the electric susceptibility describes absorption of electromagnetic waves in the medium.

(b) If $\Delta = 0$, the electric susceptibility takes a simpler form:

$$\chi(0) = -i\chi_0 \tag{341}$$

By (337), we can use this to write the complex wavenumber k:

$$k \approx \frac{\omega}{c}\left(1 + \frac{\chi}{2}\right) = \frac{\omega}{c}\left(1 - i\frac{\chi_0}{2}\right) \tag{342}$$

Looking at the prefactor $e^{-\operatorname{Im}(k) z}$ in (340), we can see that the electric field amplitude is attenuated by a factor $1/e$ when $z = 1/\operatorname{Im}(k)$. Therefore, the absorption length L is defined by $L = 1/|\mathrm{Im}(k)|$. (The absolute value is there because lengths are always nonnegative.) In this case, using (342), we find that it is equal to

$$L = \frac{1}{|\mathrm{Im}(k)|} = \frac{1}{\left|-\frac{\omega\chi_0}{2c}\right|}$$

$$\boxed{L = \frac{2c}{\omega\chi_0}} \tag{343}$$

(c) Most birefringence problems follow a specific pattern:

How to solve birefringence problems:
Step 1. Diagonalize the dielectric matrix to find the principal axes, i.e., the coordinate system in which the dielectric matrix is diagonal.
Step 2. Write the polarization of the incident electromagnetic wave in terms of the principal axes.
Step 3. For each principal axis, there is a (possibly different) dispersion relation that relates ω and k, typically involving the index of refraction n. Use this dispersion relation (and the fact that the frequency ω is spatially constant) to find the wavenumber for each principal axis.
Step 4. Use the (possibly different) wavenumbers for each principal axis to write the electromagnetic wave inside the medium.

Here, Step 1 is done for us already. We are told that the different electric susceptibilities $\chi(\Delta_\pm)$ correspond to left- and right-handed circular polarizations. Therefore, the principal axis are the left- and right-handed circular polarizations.

Moving on to Step 2, we need to write the polarization of the incident wave in terms of the left- and right-handed circular polarizations. We are told the incident electromagnetic wave is linearly polarized, so we can set our coordinate system so that the incident polarization is $\hat{\mathbf{x}}$. The left- and right-handed circular polarizations are

$$\hat{\epsilon}_\pm \equiv \frac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}} \tag{344}$$

Note the extra factor of $1/\sqrt{2}$, which ensures that the polarization remains a unit vector. We can write $\hat{\mathbf{x}}$ in terms of the circular polarization vectors:

$$\hat{\mathbf{x}} = \frac{\hat{\epsilon}_+ + \hat{\epsilon}_-}{\sqrt{2}} \tag{345}$$

Using this relation, the complex incident wave can be written in the principal axes of circular polarization:

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \varepsilon_0 \hat{\mathbf{x}} e^{i(kz-\omega t)} = \frac{\varepsilon_0}{\sqrt{2}} \left(\hat{\epsilon}_+ e^{i(kz-\omega t)} + \hat{\epsilon}_- e^{i(kz-\omega t)}\right) \tag{346}$$

Now for Step 3, which tells us that the dispersion relation between k and ω could be different for each polarization. In this case, defining $n_\pm \equiv n(\Delta_\pm)$, we can write this dispersion relation using (336):

$$k_\pm = \frac{\omega n_\pm}{c} \tag{347}$$

On to step 4, the effect of the birefringent medium is captured by writing $k_\pm$ along each principal axis in accordance with dispersion relation (347):

$$\begin{aligned} \tilde{\mathbf{E}}(\mathbf{r}, t) &= \frac{\varepsilon_0}{\sqrt{2}} \left[\hat{\epsilon}_+ \exp\left(i(k_+ z - \omega t)\right) + \hat{\epsilon}_- \exp\left(i(k_- z - \omega t)\right)\right] \\ &= \frac{\varepsilon_0}{\sqrt{2}} \left[\hat{\epsilon}_+ \exp\left(i\left(\frac{\omega n_+}{c} z - \omega t\right)\right) + \hat{\epsilon}_- \exp\left(i\left(\frac{\omega n_-}{c} z - \omega t\right)\right)\right] \end{aligned} \tag{348}$$

The problem asks us to find the polarization after a distance ℓ in the medium. In other words, we need to find the direction that the $\mathbf{E}$ field points after that distance. To find this, plug in ℓ for z and pull out the time-harmonic factor $e^{-i\omega t}$:

$$\begin{aligned}\tilde{\mathbf{E}}(z=\ell,t) &= \frac{\varepsilon_0}{\sqrt{2}}\left[\hat{\epsilon}_+ \exp\left(i\left(\frac{\omega n_+}{c}\ell - \omega t\right)\right) + \hat{\epsilon}_- \exp\left(i\left(\frac{\omega n_-}{c}\ell - \omega t\right)\right)\right] \\ &= \frac{\varepsilon_0}{\sqrt{2}}\left[\hat{\epsilon}_+ \exp\left(i\frac{\omega\ell}{c}n_+\right) + \hat{\epsilon}_- \exp\left(i\frac{\omega\ell}{c}n_-\right)\right] e^{-i\omega t} \end{aligned} \tag{349}$$

The goal is to rewrite the term in brackets in terms of a single real-valued vector. Now, define n_0 as the average between n_+ and n_-, and define Δn as the difference between n_+ and n_-:

$$n_0 \equiv \frac{n_+ + n_-}{2}; \qquad \Delta n \equiv n_+ - n_-; \qquad n_\pm = n_0 \pm \frac{\Delta n}{2} \tag{350}$$

We can require the electric field in terms of these quantities:

$$\begin{aligned}\tilde{\mathbf{E}}(z=\ell,t) &= \frac{\varepsilon_0}{\sqrt{2}}\left[\hat{\epsilon}_+ \exp\left(i\frac{\omega\ell}{c}\left(n_0 + \frac{\Delta n}{2}\right)\right) + \hat{\epsilon}_- \exp\left(i\frac{\omega\ell}{c}\left(n_0 - \frac{\Delta n}{2}\right)\right)\right] e^{-i\omega t} \\ &= \frac{\varepsilon_0}{\sqrt{2}}\left[\hat{\epsilon}_+ \exp\left(i\frac{\omega\ell\Delta n}{2c}\right) + \hat{\epsilon}_- \exp\left(-i\frac{\omega\ell\Delta n}{2c}\right)\right] e^{i\omega\ell n_0/c} e^{-i\omega t} \end{aligned} \tag{351}$$

Finally, we can rewrite the circular polarization vectors in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$:

$$\begin{aligned}\tilde{\mathbf{E}}(z=\ell,t) &= \frac{\varepsilon_0}{\sqrt{2}}\left[\left(\frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}}\right)\exp\left(i\frac{\omega\ell\Delta n}{2c}\right) + \left(\frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}}\right)\exp\left(-i\frac{\omega\ell\Delta n}{2c}\right)\right] e^{i\omega\ell n_0/c} e^{-i\omega t} \\ &= \varepsilon_0\left[\hat{\mathbf{x}}\left(\frac{\exp\left(i\frac{\omega\ell\Delta n}{2c}\right) + \exp\left(-i\frac{\omega\ell\Delta n}{2c}\right)}{2}\right) + \hat{\mathbf{y}}\left(\frac{i\left(\exp\left(i\frac{\omega\ell\Delta n}{2c}\right) - \exp\left(-i\frac{\omega\ell\Delta n}{2c}\right)\right)}{2}\right)\right] e^{i\omega\ell n_0/c} e^{-i\omega t} \\ &= \varepsilon_0\left[\hat{\mathbf{x}}\cos\left(\frac{\omega\ell\Delta n}{2c}\right) - \hat{\mathbf{y}}\sin\left(\frac{\omega\ell\Delta n}{2c}\right)\right] e^{i\omega\ell n_0/c} e^{-i\omega t} \end{aligned} \tag{352}$$

The physical electric field is the real part of this complex electric field. Assuming Δn is real, which will be justified in a moment, we have

$$\begin{aligned}\mathbf{E}(z=\ell,t) &= \mathrm{Re}\left(\tilde{\mathbf{E}}(z=\ell,t)\right) \\ &= \varepsilon_0 \underbrace{\left[\hat{\mathbf{x}}\cos\left(\frac{\omega\ell\Delta n}{2c}\right) - \hat{\mathbf{y}}\sin\left(\frac{\omega\ell\Delta n}{2c}\right)\right]}_{\text{new polarization vector}} \cos\left(\frac{\omega\ell n_0}{c} - \omega t\right) \end{aligned} \tag{353}$$

Recall that the original polarization vector was $\hat{\mathbf{x}}$. We have just found that the new polarization vector is $\hat{\mathbf{x}}\cos\left(\frac{\omega\ell\Delta n}{2c}\right) - \hat{\mathbf{y}}\sin\left(\frac{\omega\ell\Delta n}{2c}\right)$, so the polarization has rotated by an angle

$$\theta = \frac{\omega\ell\Delta n}{2c} \tag{354}$$

All we need to do now is write θ in terms of Δ_B. The problem tells us that $\Delta_\pm = \Delta \pm \Delta_B$, but it then tells us that we are working at resonance ($\Delta = 0$). Therefore, $\Delta_\pm = \pm\Delta_B$. From $\Delta_\pm$, we can find the electric susceptibilities $\chi_\pm$:

$$\begin{aligned}\chi_\pm &= \chi(\Delta_\pm) \\ &= \chi(\pm\Delta_B) \\ &= \chi_0\left[\frac{\pm 2\Delta_B}{1+4\Delta_B^2} - i\frac{1}{1+4\Delta_B^2}\right]\end{aligned}$$

From the electric susceptibilities $\chi_\pm$, we can find Δn using the formula $n \approx 1 + \chi/2$:

$$\begin{aligned}
\Delta n = n_+ - n_- &\approx \left(1 + \frac{1}{2}\chi_+\right) - \left(1 + \frac{1}{2}\chi_-\right) \\
&= \frac{1}{2}(\chi_+ - \chi_-) \\
&= \frac{1}{2}\chi_0 \left[\left(\frac{2\Delta_B}{1+4\Delta_B^2} - i\frac{1}{1+4\Delta_B^2}\right) - \left(\frac{-2\Delta_B}{1+4\Delta_B^2} - i\frac{1}{1+4\Delta_B^2}\right)\right] \\
&= \frac{1}{2}\chi_0 \, \frac{4\Delta_B}{1+4\Delta_B^2} \\
&= \chi_0 \, \frac{2\Delta_B}{1+(2\Delta_B)^2} \qquad (355)
\end{aligned}$$

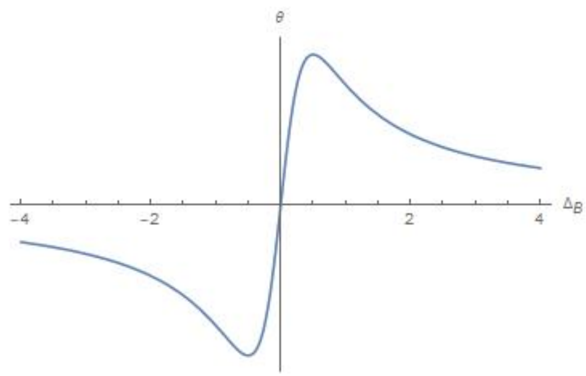
From the difference in the indices of refraction Δn, we can write the angle that the polarization has rotated (354):

$$\theta = \frac{\omega \ell \Delta n}{2c}$$

$$\boxed{\theta = \frac{\omega \ell \chi_0}{2c}\,\frac{2\Delta_B}{1+(2\Delta_B)^2} = \frac{\ell}{L}\,\frac{2\Delta_B}{1+(2\Delta_B)^2}} \qquad (356)$$

where $L = \frac{2c}{\omega\chi_0}$ is the absorption length we found in part (a).

Here is a sketch of the angle of rotation:



Note that the maximum angle occurs for $\Delta_B = 1/2$.

2014 Q11 is another birefringence problem.