10. (Electromagnetism)

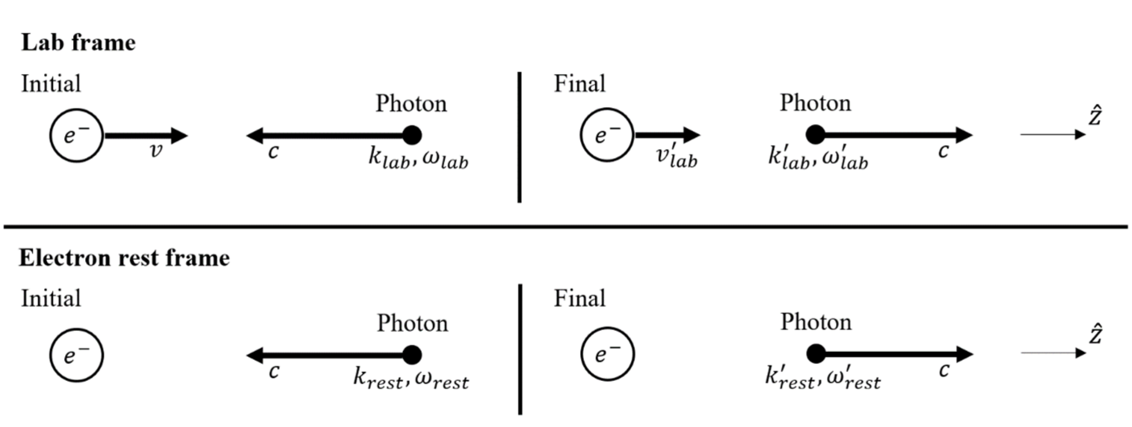
Consider the backscattering of laser photons from a counter-propagating relativistic electron. The electron is taken to be traveling in the z-direction with speed v giving a Lorentz factor $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$. The laser photons propagate in the opposite $(-z)$ direction and the scattered photons propagate in the positive z-direction. The laser photons have a free-space wavelength of 800 nm ($\hbar\omega = 1.55$ eV), and the electron's total energy is 100 MeV.

(a) Assuming the *Thomson* approximation, the frequency ω' of the scattered radiation in the *electron rest frame* obeys $\hbar\omega' \ll m_e c^2$, and the backscattered radiation has nearly the same frequency but opposite wavenumber k' (reversed propagation direction) as the laser in this frame. Write expressions for k' and ω' and evaluate the adequacy of the Thomson approximation.

(b) What is the energy of the scattered photons in the laboratory frame?

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This is a special relativistic collision problem, almost a classical mechanics problem. To start, let's draw a diagram of the setup before and after the collision in both the lab frame and the electron rest frame:



(a) First, note the relationship between frequency and wavenumber of photons in a vacuum, which is just the dispersion relation for a plane wave:

$$\omega = c|k| \tag{65}$$

(We will take the wavenumber k to be positive if the photon is moving in the $+\hat{\mathbf{z}}$ direction and negative if the photon is moving in the $-\hat{\mathbf{z}}$ direction. The angular frequency ω is always positive.) The problem tells us that in the electron rest frame (assuming the Thomson approximation), the initial and final frequencies of the photon are the same, and the initial and final wavenumbers of the photon are opposite. In other words,

$$\omega'_{\text{rest}} = \omega_{\text{rest}} \quad \text{and} \quad k'_{\text{rest}} = -k_{\text{rest}} \quad \text{in the Thomson approximation} \tag{66}$$

The problem gives us a value for k_{lab} when it tells us that the wavelength of the laser photons is 800 nm (recall that the wavenumber k and the wavelength λ are related by $k = \frac{2\pi}{\lambda}$). The problem also gives us a value for ω_{lab} when it tells us that $\hbar\omega = 1.55$ eV for the laser photons. Therefore, all we need to do is find ω_{rest} and k_{rest} in terms of ω_{lab} and k_{lab}, and (66) will do the rest.

To relate the wavenumber and frequency in the two frames, recall that for a photon, the energy is given by

$$E_{\text{photon}} = \hbar\omega \tag{67}$$

Since the photon is massless, the energy-momentum relation for relativistic particles, $E = \sqrt{(pc)^2 + (mc^2)^2}$ becomes $E_{\text{photon}} = |p_{\text{photon}}|c$, so

$$|p_{\text{photon}}| = \frac{\hbar\omega}{c} = \hbar|k| \quad \text{by (65)} \tag{68}$$

Therefore, the relevant components of the four-momentum of a photon are

$$p^{\mu}_{\text{photon}} = \begin{pmatrix} E_{\text{photon}}/c \\ p_{\text{photon}} \end{pmatrix} = \begin{pmatrix} \hbar\omega/c \\ \pm\hbar\omega/c \end{pmatrix} \tag{69}$$

where the $\pm$ is chosen according to the direction of motion of the photon. For us, the photons initially move in the $-\hat{\mathbf{z}}$ direction, so

$$p^\mu_{\text{photon, lab}} = \begin{pmatrix} \hbar\omega_{\text{lab}}/c \\ -\hbar\omega_{\text{lab}}/c \end{pmatrix} \quad \text{and} \quad p^\mu_{\text{photon, lab}} = \begin{pmatrix} \hbar\omega_{\text{rest}}/c \\ -\hbar\omega_{\text{rest}}/c \end{pmatrix} \tag{70}$$

Therefore, to transform ω_{lab} and k_{lab} into ω_{rest} and k_{rest}, perform a Lorentz boost of the four-momentum $p^\mu_{\text{photon, lab}}$ along the velocity $+v\,\hat{\mathbf{z}}$ of the electron. This gets us the four-momentum of the photon in the electron rest frame:

$$\begin{aligned} p^\mu_{\text{photon, rest}} &= \underbrace{\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}}_{\substack{\text{Lorentz} \\ \text{transformation}}} \underbrace{\begin{pmatrix} \hbar\omega_{\text{lab}}/c \\ -\hbar\omega_{\text{lab}}/c \end{pmatrix}}_{p^\mu_{\text{photon, lab}}} \quad \text{where } \beta \equiv \frac{v}{c} \text{ and } \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \\ &= \begin{pmatrix} \hbar\gamma\omega_{\text{lab}}\left(1+\beta\right)/c \\ -\hbar\gamma\omega_{\text{lab}}\left(1+\beta\right)/c \end{pmatrix} \end{aligned} \tag{71}$$

Comparing this to the expression for $p^\mu_{\text{photon, rest}}$ given in (70), we can extract an expression for ω_{rest}:

$$\begin{pmatrix} \hbar\gamma\omega_{\text{lab}}\left(1+\beta\right)/c \\ -\hbar\gamma\omega_{\text{lab}}\left(1+\beta\right)/c \end{pmatrix} = p^\mu_{\text{photon, rest}} = \begin{pmatrix} \hbar\omega_{\text{rest}}/c \\ -\hbar\omega_{\text{rest}}/c \end{pmatrix}$$

$$\Longrightarrow \omega_{\text{rest}} = \gamma\omega_{\text{lab}}(1+\beta) \tag{72}$$

$$\begin{aligned} &= \omega_{\text{lab}}\frac{1+\beta}{\sqrt{1-\beta^2}} \quad \text{since } \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \\ &= \omega_{\text{lab}}\frac{1+\beta}{\sqrt{(1+\beta)(1-\beta)}} \\ &= \omega_{\text{lab}}\sqrt{\frac{1+\beta}{1-\beta}} \end{aligned}$$

$$\omega_{\text{rest}} = \omega_{\text{lab}}\sqrt{\frac{1+(v/c)}{1-(v/c)}} \tag{73}$$

Using the dispersion relation (65), we get

$$|k_{\text{rest}}| = \frac{\omega_{\text{rest}}}{c} = \frac{\omega_{\text{lab}}}{c}\sqrt{\frac{1+(v/c)}{1-(v/c)}} \tag{74}$$

Since the photon is initially moving in the $-\hat{\mathbf{z}}$ direction, this implies

$$k_{\text{rest}} = -\frac{\omega_{\text{lab}}}{c}\sqrt{\frac{1+(v/c)}{1-(v/c)}} \tag{75}$$

Using (66), we can now write ω'_{rest} and k'_{rest} in the Thomson approximation:

$$\boxed{\omega'_{\text{rest}} = \omega_{\text{rest}} = \omega_{\text{lab}}\sqrt{\frac{1+(v/c)}{1-(v/c)}}} \tag{76}$$

$$\boxed{k'_{\text{rest}} = -k_{\text{rest}} = \frac{\omega_{\text{lab}}}{c}\sqrt{\frac{1+(v/c)}{1-(v/c)}}} \tag{77}$$

For approximate, back-of-the-envelope calculations, it is often easier to write our answers in terms of γ. If we wanted the answer in terms of γ, we could start from (72) and write

$$\begin{aligned}\omega_{\text{rest}} &= \gamma\omega_{\text{lab}}(1+\beta) \\ &= \gamma\omega_{\text{lab}}\left(1+\sqrt{1-\frac{1}{\gamma^2}}\right) \quad \text{since } \frac{1}{\gamma^2} = 1-\beta^2\end{aligned}$$

Then we could write equivalent answers

$$\boxed{\omega'_{\text{rest}} = \omega_{\text{rest}} = \omega_{\text{lab}}\gamma\left(1+\sqrt{1-\frac{1}{\gamma^2}}\right)} \tag{78}$$

$$\boxed{k'_{\text{rest}} = -k_{\text{rest}} = \frac{\omega_{\text{lab}}}{c}\gamma\left(1+\sqrt{1-\frac{1}{\gamma^2}}\right)} \tag{79}$$

We might also be interested in getting approximate answers in the limit that the electron is highly relativistic ($v \approx c$). In this limit, $\gamma \to \infty$, so we can write the approximation

$$\boxed{\omega'_{\text{rest}} \approx 2\omega_{\text{lab}}\gamma \quad \text{and} \quad k'_{\text{rest}} \approx \frac{2\omega_{\text{lab}}\gamma}{c} \quad \text{if the electron is highly relativistic}} \tag{80}$$

We will see in just a moment that this approximation is justified in this case.

In order to evaluate the adequacy of the Thomson approximation, as stated in the problem, we need to establish whether or not $\hbar\omega'_{\text{rest}} \ll m_e c^2$. The first step is to determine γ for the electron in the lab frame.

The electron's total energy in the lab frame is given to be 100 MeV, while the rest energy of an electron is approximately 0.5 MeV (a fact you should memorize or write on your formula sheet). This means that in the lab frame

$$E = \gamma m_e c^2 = 100 \text{ MeV} \quad \text{and} \quad E_{\text{rest}} = m_e c^2 \approx 0.5 \text{ MeV} \tag{81}$$

We can divide these expressions by one another to solve for γ:

$$\gamma = \frac{\gamma m_e c^2}{m_e c^2} = \frac{E}{E_{\text{rest}}} \approx \frac{100 \text{ MeV}}{0.5 \text{ MeV}} = 200 \tag{82}$$

Since γ is large, the electron is highly relativistic. Therefore, to evaluate the left-hand side of the equation $\hbar\omega'_{\text{rest}} \ll m_e c^2$, we can use our approximation for ω'_{rest} for highly relativistic electrons (80), along with the fact that $\hbar\omega = 1.55$ eV in the lab frame:

$$\begin{aligned}\hbar\omega'_{\text{rest}} &\approx 2\hbar\omega_{\text{lab}}\gamma \\ &\approx 2(1.55 \text{ eV})(200) \\ &\approx 600 \text{ eV}\end{aligned} \tag{83}$$

The right-hand side of the equation $\hbar\omega'_{\text{rest}} \ll m_e c^2$ is just the rest energy of the electron, which we know to be about 0.5 MeV:

$$m_e c^2 \approx 0.5 \text{ MeV} = 5 \cdot 10^5 \text{ eV} \tag{84}$$

Since 600 eV is indeed much less that $5 \cdot 10^5$ eV, we have $\hbar\omega'_{\text{rest}} \ll m_e c^2$ and the Thomson approximation is more than adequate.

(b) To find the energy of the scattered photons in the laboratory frame, we should first find the four-momentum of the scattered photons in the electron rest frame, which we'll call $p'^{\mu}_{\text{photon, rest}}$. Then, we will convert that four-momentum back into the lab frame.

We already know ω'_{rest} from part (a), and we know that the scattered photons propagate in the $+\hat{\mathbf{z}}$-direction. Therefore, by (69), the four-momentum of the scattered photons in the electron rest frame is

$$p'^{\mu}_{\text{photon, rest}} = \begin{pmatrix} \hbar\omega'_{\text{rest}}/c \\ \hbar\omega'_{\text{rest}}/c \end{pmatrix} \quad \text{where } \omega'_{\text{rest}} \text{ is our answer from part (a)} \tag{85}$$

To transform this answer back to the lab frame, perform a Lorentz boost back to the lab frame. This is the opposite of the Lorentz boost from part (a), so it is a boost of $-v\,\hat{\mathbf{z}}$. This gets us the four-momentum of the photon in the lab frame:

$$\begin{aligned} p'^{\mu}_{\text{photon, lab}} &= \underbrace{\begin{pmatrix} \gamma & +\beta\gamma \\ +\beta\gamma & \gamma \end{pmatrix}}_{\substack{\text{Lorentz} \\ \text{transformation}}} \underbrace{\begin{pmatrix} \hbar\omega'_{\text{rest}}/c \\ \hbar\omega'_{\text{rest}}/c \end{pmatrix}}_{p^{\mu}_{\text{photon, rest}}} \quad \text{where } \beta \equiv \frac{v}{c} \text{ and } \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \\ &= \begin{pmatrix} \hbar\gamma\omega'_{\text{rest}}\,(1+\beta)\,/c \\ \hbar\gamma\omega'_{\text{rest}}\,(1+\beta)\,/c \end{pmatrix} \end{aligned} \tag{86}$$

Comparing once again to (69), we can extract the angular frequency of the scattered photon in the lab frame:

$$\omega'_{\text{lab}} = \gamma\omega'_{\text{rest}}(1+\beta) \tag{87}$$

Therefore, the energy of the scattered photon in the lab frame is

$$\begin{aligned} E'_{\text{photon, lab}} &= \hbar\omega'_{\text{lab}} \\ &= \gamma(\hbar\omega'_{\text{rest}})(1+\beta) \quad \text{by (87)} \end{aligned}$$

It is helpful to write this in terms of γ and to impose the approximation that the electron is highly relativistic ($\gamma \to \infty$):

$$\begin{aligned} E'_{\text{photon, lab}} &= \gamma(\hbar\omega'_{\text{rest}})\left(1+\sqrt{1-\frac{1}{\gamma^2}}\right) \quad \text{since } \frac{1}{\gamma^2} = 1-\beta^2 \\ &\approx 2\gamma(\hbar\omega'_{\text{rest}}) \quad \text{as } \gamma \to \infty \end{aligned} \tag{88}$$

In part (a), we found that (82) $\gamma \approx 200$ and (83) $\hbar\omega'_{\text{rest}} \approx 600$ eV. With these numerical results, we can calculate $E'_{\text{photon, lab}}$ to be

$$\begin{aligned} E'_{\text{photon, lab}} &\approx 2(200)(600 \text{ eV}) \\ &= 240{,}000 \text{ eV} \end{aligned}$$

$$\boxed{E'_{\text{photon, lab}} \approx 240 \text{ keV}} \tag{89}$$