1. *Quantum Mechanics* (Spring 2006)

An electron is at rest in a constant magnetic field pointing along the z-direction. The Hamiltonian is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = g\mu_0 \frac{\mathbf{s}}{\hbar} \cdot \mathbf{B}$$

where $\mathbf{B} = B_0 \hat{\mathbf{n}}_z$. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

(a) What are the eigenstates of the Hamiltonian in terms of $|\psi_\pm\rangle$, and what is the energy difference between them?

(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. What is $\big|\psi(0)\big\rangle$ in terms of $|\psi_\pm\rangle$? Calculate $\big|\psi(t)\big\rangle$ for any later time t in terms of these same two states.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

2. *Quantum Mechanics* (Spring 2006)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, ...$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0\,|\psi_0\rangle + c_1\,|\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $\big|\psi(0)\big\rangle = |\phi\rangle$. What is $\big|\psi(t)\big\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

3. *Quantum Mechanics* (Spring 2006)

A hydrogen atom is placed in a constant weak electric field of strength $\mathcal{E}$. Ignoring spin, what are the energies of the $n = 1$ and $n = 2$ levels including effects to first order in $\mathcal{E}$ (but ignoring second order effects)?

Note: You may want to use some of the following:

Radial Wave Functions $R_{nl}(r)$ (a is the Bohr radius):

$$\boxed{\begin{array}{c} R_{10}(r) = \frac{1}{a^{3/2}} 2e^{-r/a} \qquad R_{21}(r) = \frac{1}{a^{3/2}} \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a} \\ R_{20}(r) = \frac{1}{a^{3/2}} \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \end{array}}$$

Spherical Harmonics $Y_l^m(\theta, \phi)$:

$$\boxed{Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi}}$$

An integral:

$$\boxed{\int_0^\infty x^n e^{-x/a} dx = a^{n+1} n!}$$

4. *Quantum Mechanics* (Spring 2006)

Consider the scattering of a beam of non-relativistic spin 0 particles by a repulsive spherical potential of depth V_0 and radius a in three dimensions:

$$V(r) = \begin{cases} V_0, & \text{for } r < a; \\ 0, & \text{for } r > a. \end{cases}$$

Find the scattering cross section in the Born approximation.

5. *Quantum Mechanics* (Spring 2006)

A particle with mass m is confined to move on a circle of radius r. It is perturbed by a potential $V(\theta) = a(1+\cos(2\theta))$.

(a) What are the unperturbed energy levels?

(b) Find the shift in the energy levels to first order in a.

(c) Find the second order energy shift for all the states.

Hint: Beware of the special care needed for some of the states.

6. *Statistical Mechanics and Thermodynamics* (Spring 2006)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|\, c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

7. *Statistical Mechanics and Thermodynamics* (Spring 2006)

(a) In the case of a set of non-relativistic, noninteracting spin-1/2 fermions confined to two dimensions, what is the paramagnetic susceptibility at $T = 0$? Give your answer in terms of the mass m of the fermions, the gyromagnetic ratio γ, the number density σ, and whatever fundamental constants (e.g., $\hbar$, c, ...) are necessary to completely specify the system.

(b) Now, suppose that the external magnetic field is replaced by an effective field H_{eff}, where

$$H_{\text{eff}} = H + \Gamma \frac{M}{A}$$

where H is the external field and M/A is the induced magnetic moment per unit area. Above what threshold value of the parameter Γ is this two-dimensional system ferromagnetic at $T = 0$?

8. *Statistical Mechanics and Thermodynamics* (Spring 2006)

(a) A system consists of N particles, each of which can exist in two states, with energies ϵ_0 and ϵ_1, respectively. Given that the total energy of this system is U, what is its entropy?

(b) Obtain the expression for the entropy in the limit that N is large.

(c) Now, give an expression for the temperature of this system, as a function of U and the energies of the single particle states. Does this expression have any properties that require some discussion?

Stirling's formula: $n! \approx (\frac{n}{e})^n$, when n is large.

9. *Statistical Mechanics and Thermodynamics* (Spring 2006)

A researcher claims that a particular substance in thermal equilibrium exhibits the following total-number-of-states function

$$\Omega(E) = c(E - E_0)^\alpha V^\gamma \exp\left(-\frac{g}{V}\right)$$

where E_0, c, α, γ, and g are positive coefficients independent of the energy E, the volume V, and the temperature T.

(a) Find the equation of state for this substance.

(b) What is the relationship between the average energy and the temperature?

(c) Does this substance satisfy the third law of thermodynamics? Why?

(d) What values should E_0, c, α, γ, and g take for this substance to behave as an ideal gas?

10. *Electricity and Magnetism* (Spring 2006)

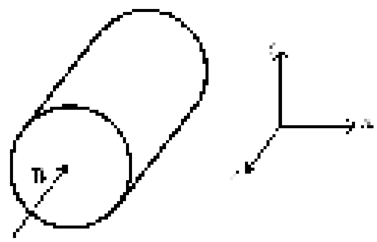
An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

(a) if the shell is uncharged;

(b) if the total charge on the shell is Q.

11. *Electricity and Magnetism* (Spring 2006)

Consider a long solid cylinder made of uniform resistive material. The cylinder is in a region in which there is an applied magnetic field that is uniform and is directed along the axis of the cylinder. The magnetic field is time-dependent and it is oscillating with angular frequency ω: $\mathbf{B}(t) = B_z \cos \omega t \hat{\mathbf{z}}$. The length of the cylinder is L and its radius is R ($R \ll L$). The resistivity of the cylinder material is ρ.



(a) Calculate the current density $\mathbf{j}(t)$ in the volume of the cylinder. Assume initially that you can ignore the self-inductance of the cylinder. Ignore end effects and the Hall effect.

(b) For large values of ω the effect of self-inductance cannot be ignored. Calculate the correction to the current density $\Delta\mathbf{j}(t)$ due to the self-inductance of the cylinder in next order of ω.

(c) Give the condition on ω such that the self-inductance of the cylinder can be ignored.

12. *Electricity and Magnetism* (Spring 2006)

Two point charges $+Q_0$ and $-Q_0$ are placed at opposite poles of a spherical balloon of initial radius R_0. The radius of the balloon is set to oscillate as follows: $R(t) = R_0 + \rho \sin \omega t$. Assume $\rho\omega \ll c$.

(a) Determine the total power radiated by the oscillating balloon, if any, in terms of Q_0, R_0, ρ, and ω. Show your work and explain your reasoning.

Note: If you are unable to write an expression for the total power radiated, explain how the total power radiated scales with each of the above variables.

(b) Suppose instead that charges are deposited on the balloon as described below. For each case, determine the ratio of the total power radiated by the oscillating balloon, if any, to the total power radiated in (a). Show your work and explain your reasoning.

(i) One point charge $+Q_0$ is placed at a given point on the balloon. The radius of the balloon is set to oscillate as above.

(ii) A total charge $+Q_0$ is deposited uniformly on the surface of the balloon. The radius of the balloon is set to oscillate as above.

13. *Electricity and Magnetism* (Spring 2006)

Consider an infinitely long filamentary current (i.e., a δ-function) carrying a total current I along the z-direction.

(a) Find the magnetic vector potential at a radial distance r from the current filament.

Now a non-relativistic particle of charge q and mass m is fired from a radial location d with velocity $\mathbf{v}$ pointing in the radial direction, away from the current filament.

(b) Evaluate the constants of the motion associated with the orbit of this particle.

(c) Deduce the maximum radial distance reached by the particle.

(d) What condition is required for the orbit size to be well-approximated by the usual Larmor radius expression?

14. *Electricity and Magnetism* (Spring 2006)

A plane, transverse electromagnetic wave of frequency ω propagates through a scalar medium whose complex dielectric coefficient is given by

$$\epsilon(\omega) = 1 - \frac{a}{\omega(\omega + ib)}$$

where a and b are positive real constants.

(a) What is the electrical conductivity of the medium?

(b) What is the ratio of the magnitude of the material current density to the displacement current density in the medium?

(c) Find the spatial damping coefficient of this wave (i.e., the imaginary part of k) in the limit of small b.

(d) Find the phase-shift between the electric and magnetic fields in the limit of small b.

(e) Does this $\epsilon(\omega)$ satisfy the required symmetry relation for general dielectric coefficients? Why?

1. *Quantum Mechanics* (Spring 2006)

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where $\mathbf{B} = B_0\hat{\boldsymbol{n}}_z$. Since the electron is at rest, you can treat this as a two-state system. Let $|\psi_\pm\rangle$ be the eigenstates of s_z with eigenvalues $\pm\frac{\hbar}{2}$ respectively.

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(b) At time $t = 0$ the electron is in an eigenstate of s_x with eigenvalue $+\hbar/2$. What is $|\psi(0)\rangle$ in terms of $|\psi_\pm\rangle$? Calculate $|\psi(t)\rangle$ for any later time t in terms of these same two states.

(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

a. $H = g\mu_0\frac{\vec{s}}{\hbar}\cdot\vec{B} = \frac{g}{2}\mu_0\vec{\sigma}\cdot\vec{B} = \frac{g}{2}\mu_0\sigma_z B_0 \cong \mu_0\sigma_z B_0$

The eigenvalues of σ_z are ± 1, so the eigenvalues of H are $\pm\mu_0 B_0 \Rightarrow \Delta E = \mu_0 B_0 - (-\mu_0 B_0) = 2\mu_0 B_0$

b. $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow |\psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ since $\sigma_x|\psi_{x+}\rangle = (+1)|\psi_{x+}\rangle$

So $|\psi(0)\rangle = |\psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}|\psi_{z+}\rangle + \frac{1}{\sqrt{2}}|\psi_{z-}\rangle$

$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\mu_0 B_0 t/\hbar}|\psi_{z+}\rangle + \frac{1}{\sqrt{2}}e^{i\mu_0 B_0 t/\hbar}|\psi_{z-}\rangle$

c. $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix} \Rightarrow a = \pm b$

$\Rightarrow |\psi_{x+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|\psi_{-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}-ib\\ia\end{pmatrix} \Rightarrow b = \pm ia$

$\Rightarrow |\psi_{y+}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ and $|\psi_{y-}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \Rightarrow \pm\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\-b\end{pmatrix} \Rightarrow a=0$ or $b=0$

$\Rightarrow |\psi_{z+}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\psi_{z-}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\langle s_x\rangle = \frac{\hbar}{2}|\langle\psi_{x+}|\psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\psi_{x-}|\psi(t)\rangle|^2$

$= \frac{\hbar}{2}|\cos(\mu_0 B_0 t/\hbar)|^2 + \left(-\frac{\hbar}{2}\right)|-\sin(\mu_0 B_0 t/\hbar)|^2$

$= \frac{\hbar}{2}\cos(2\mu_0 B_0 t/\hbar)$

$\langle s_y\rangle = \frac{\hbar}{2}|\langle\psi_{y+}|\psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\psi_{y-}|\psi(t)\rangle|^2$

$= \frac{\hbar}{2}\left|\frac{1}{2}\left(e^{-i\mu_0 B_0 t/\hbar} + ie^{i\mu_0 B_0 t/\hbar}\right)\right|^2 + \left(-\frac{\hbar}{2}\right)\left|\frac{1}{2}\left(e^{-i\mu_0 B_0 t/\hbar} - ie^{i\mu_0 B_0 t/\hbar}\right)\right|^2$

$= \frac{\hbar}{2}\frac{1}{4}\left[(\cos-\sin)^2 + (-\sin+\cos)^2\right] + \left(-\frac{\hbar}{2}\right)\frac{1}{4}\left[(\cos+\sin)^2 + (-\sin-\cos)^2\right]$

$= \frac{\hbar}{2}\frac{1}{2}(\cos-\sin)^2 - \frac{\hbar}{2}\frac{1}{2}(\cos+\sin)^2$

$= -\frac{\hbar}{2}\frac{1}{2}4\sin\cos = -\frac{\hbar}{2}\sin(2\mu_0 B_0 t/\hbar)$

$\langle s_z\rangle = \frac{\hbar}{2}|\langle\psi_{z+}|\psi(t)\rangle|^2 + \left(-\frac{\hbar}{2}\right)|\langle\psi_{z-}|\psi(t)\rangle|^2 = \frac{\hbar}{2}\left(\frac{1}{2}\right) - \frac{\hbar}{2}\left(\frac{1}{2}\right) = 0$

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(c) For the state you calculated in part (b), what are the expectation values of the three components of the spin at any time t?

$$H = -\vec{\mu}\cdot\vec{B} = +\frac{\mu_B}{\hbar}(\cancel{\vec{L}}^{\,0} + g\,\vec{s})\cdot\vec{B} = g\mu_B\frac{\vec{s}}{\hbar}\cdot\vec{B} = g\mu_B\frac{S_z}{\hbar}B_0 = \frac{g}{2}\mu_B\sigma_z B_0$$

$$\approx \mu_B B_0\sigma_z \qquad \text{where} \quad \mu_B = \frac{e\hbar}{2m}$$

a) $$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad |\psi_+\rangle \doteq \begin{pmatrix}1\\0\end{pmatrix} \qquad |\psi_-\rangle \doteq \begin{pmatrix}0\\1\end{pmatrix}$$

$\Rightarrow$ $|\psi_\pm\rangle$ are the eigenstates of H with eigenvalues

$E_\pm = \pm\mu_B B_0.$

$$\Delta E = E_+ - E_- = \mu_B B_0 + \mu_B B_0 = 2\mu_B B_0$$

b) $$\sigma_x|\psi(0)\rangle = |\psi(0)\rangle \doteq \begin{pmatrix}a\\b\end{pmatrix} \qquad |\psi(0)\rangle = a|\psi_+\rangle + b|\psi_-\rangle$$

$$\Rightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix} \quad\Rightarrow\quad a = b = \frac{1}{\sqrt{2}} \text{ for normalization } \left(\text{up to phase}\right)$$

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-iHt/\hbar}(|\psi_+\rangle + |\psi_-\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|\psi_+\rangle + e^{i\omega_0 t}|\psi_-\rangle\right) \qquad \text{where} \quad \omega_0 = \frac{\mu_B B_0}{\hbar} = \frac{eB_0}{2m}$$

c) $$\langle s_x\rangle = \langle\psi(t)|s_x|\psi(t)\rangle = \frac{\hbar}{2}\langle\psi(t)|\sigma_x|\psi(t)\rangle$$

$$= \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(e^{i\omega_0 t}\;\; e^{-i\omega_0 t}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t}\\e^{i\omega_0 t}\end{pmatrix} = \frac{\hbar}{4}\left(e^{i2\omega_0 t} + e^{-i2\omega_0 t}\right)$$

$$= \frac{\hbar}{2}\cos(2\omega_0 t)$$

$$\langle s_y\rangle = \frac{\hbar}{4}\left(e^{i\omega_0 t}\;\; e^{-i\omega_0 t}\right)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}e^{-i\omega_0 t}\\e^{i\omega_0 t}\end{pmatrix} = i\frac{\hbar}{4}\left(-e^{i2\omega_0 t} + e^{-i2\omega_0 t}\right)$$

$$= \frac{\hbar}{2}\sin(2\omega_0 t)$$

$$\langle s_z\rangle = \frac{\hbar}{4}\left(e^{i\omega_0 t}\;\; e^{-i\omega_0 t}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}e^{-i\omega_0 t}\\e^{i\omega_0 t}\end{pmatrix} = \frac{\hbar}{4}(1-1) = 0$$

2. *Quantum Mechanics* (Spring 2006)

The Hamiltonian for a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Let $|\psi_n\rangle$, $n = 0, 1, 2, \ldots$, be the usual energy eigenstates.

(a) Suppose the system is in a state $|\phi\rangle$ that is some linear combination of the two lowest states only:

$$|\phi\rangle = c_0 |\psi_0\rangle + c_1 |\psi_1\rangle$$

and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

$$\langle\phi|x|\phi\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}}$$

What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate?

a. For the simple harmonic oscillator, $H|\Psi_n\rangle = (n+\frac{1}{2})\hbar\omega$

$\hbar\omega = \langle\Phi|H|\Phi\rangle = \left(\langle\Psi_0|c_0^* + \langle\Psi_1|c_1^*\right)H\left(c_0|\Psi_0\rangle + c_1|\Psi_1\rangle\right)$

$= |c_0|^2\langle\Psi_0|H|\Psi_0\rangle + |c_1|^2\langle\Psi_1|H|\Psi_1\rangle$ by orthogonality

$= |c_0|^2(\frac{1}{2}\hbar\omega) + |c_1|^2(\frac{3}{2}\hbar\omega)$

$\Rightarrow 1 = \frac{1}{2}|c_0|^2 + \frac{3}{2}|c_1|^2$

Normalization of $|\Phi\rangle$ implies $\langle\Phi|\Phi\rangle = 1 \Rightarrow |c_0|^2 + |c_1|^2 = 1$

$\Rightarrow 1 = \frac{1}{2}|c_0|^2 + \frac{3}{2}(1 - |c_0|^2) = \frac{3}{2} - |c_0|^2$

$\Rightarrow |c_0|^2 = \frac{1}{2}$ and $|c_1|^2 = \frac{1}{2}$

Therefore $|c_0| = \frac{1}{\sqrt{2}}$ and $|c_1| = \frac{1}{\sqrt{2}}$

b. Recall $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)$

$\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} = \langle\Phi|x|\Phi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle\Psi_0|c_0^* c_1 a|\Psi_1\rangle + \langle\Psi_1|c_1^* c_0 a^+|\Psi_0\rangle\right]$

$= \sqrt{\frac{\hbar}{2m\omega}}\left[c_0^* c_1 + c_1^* c_0\right]$

$= \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{2m\omega}}\left[c_1 + c_1^*\right]$ Since c_0 real and pos $\Rightarrow c_0 = \frac{1}{\sqrt{2}}$

$\Rightarrow 1 = c_1 + c_1^* = \frac{1}{\sqrt{2}}e^{i\theta_1} + \frac{1}{\sqrt{2}}e^{-i\theta_1} = \frac{1}{\sqrt{2}}2\cos(\theta_1)$

$\Rightarrow \cos(\theta_1) = \frac{\sqrt{2}}{2} \Rightarrow \theta_1 = \pi/4$

c. $|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t/2}|\Psi_0\rangle + \frac{1}{\sqrt{2}}e^{-3i\omega t/2 + i\pi/4}|\Psi_1\rangle$

$\langle\Psi(t)|x|\Psi(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\frac{1}{2}e^{-i\omega t + i\pi/4} + \frac{1}{2}e^{i\omega t - i\pi/4}\right] = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \frac{\pi}{4})$

The angular frequency of oscillation is ω.

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and suppose it is known that the expectation value of the energy is $\hbar\omega$. What are $|c_0|$ and $|c_1|$?

(b) Choose c_0 to be real and positive, but let c_1 have any phase: $c_1 = |c_1|e^{i\theta_1}$. Suppose further that not only is the expectation value of H known to be $\hbar\omega$, but the expectation value of x is also known:

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What is θ_1?

(c) Now suppose the system is in the state $|\phi\rangle$ described above at time $t = 0$. That is, $|\psi(0)\rangle = |\phi\rangle$. What is $|\psi(t)\rangle$ at a later time t? Calculate the expectation value of x as a function of t. With what angular frequency does it oscillate? $\boxed{\omega}$

a) $\langle E\rangle = \langle\phi|H|\phi\rangle = \hbar\omega\langle\phi|(N+\tfrac{1}{2})|\phi\rangle = \hbar\omega\left(\tfrac{1}{2}|c_0|^2 + \tfrac{3}{2}|c_1|^2\right) = \hbar\omega$

and given that $|\phi\rangle$ is normalized, $\langle\phi|\phi\rangle = |c_0|^2 + |c_1|^2 = 1$

$$\Rightarrow \begin{array}{l} \tfrac{1}{2}a + \tfrac{3}{2}b = 1 \\ a + b = 1 \end{array} \qquad \begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \tfrac{1}{2} \\ 0 & 1 & \tfrac{1}{2} \end{pmatrix}$$

$$\Rightarrow |c_0|^2 = a = \tfrac{1}{2},\quad |c_1|^2 = b = \tfrac{1}{2} \qquad \Rightarrow \qquad |c_0| = |c_1| = \frac{1}{\sqrt{2}}$$

b) Let $c_0 = \frac{1}{\sqrt{2}}$ and $c_1 = \frac{1}{\sqrt{2}}e^{i\theta_1}$. Given $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$

$$\langle\phi|x|\phi\rangle = x_0\langle\phi|(a^\dagger + a)|\phi\rangle = x_0\langle\phi|\left(c_0\sqrt{1}\,|\psi_1\rangle + \cancel{c_1\sqrt{2}\,|\psi_2\rangle} + c_1\sqrt{1}\,|\psi_0\rangle\right)$$

$$= x_0\left(c_1^* c_0 + c_0^* c_1\right) = \frac{x_0}{2}\left(e^{-i\theta_1} + e^{i\theta_1}\right) = x_0\cos\theta_1$$

$$= \frac{1}{\sqrt{2}}\cos\theta_1\sqrt{\frac{\hbar}{m\omega}} = \frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} \qquad \Rightarrow \qquad \cos\theta_1 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta_1 = \pm\frac{\pi}{4}$$

c) $$|\psi(t)\rangle = e^{-iHt/\hbar}|\phi\rangle = c_0 e^{-iE_0 t/\hbar}|\psi_0\rangle + c_1 e^{-iE_1 t/\hbar}|\psi_1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|\psi_0\rangle + e^{i(\theta_1 - 3\omega_0 t)}|\psi_1\rangle\right) \qquad \text{where } \omega_0 = \frac{\omega}{2}$$

$$\langle x\rangle = \frac{x_0}{2}\left(e^{i\omega_0 t}\langle\psi_0| + e^{-i(\theta_1 - 3\omega_0 t)}\langle\psi_1|\right)\left(e^{-i\omega_0 t}\sqrt{1}\,|\psi_1\rangle + \cancel{\#|\psi_2\rangle} + e^{i(\theta_1 - 3\omega_0 t)}\sqrt{1}\,|\psi_0\rangle\right)$$

$$= \frac{x_0}{2}\left(e^{i(\theta_1 - 2\omega_0 t)} + e^{-i(\theta_1 - 2\omega_0 t)}\right) = x_0\cos(\theta_1 - \omega t) = \sqrt{\frac{\hbar}{2m\omega}}\cos(\theta_1 - \omega t)$$

3. *Quantum Mechanics* (Spring 2006)

A hydrogen atom is placed in a constant weak electric field of strength ε. Ignoring spin, what are the energies of the $n=1$ and $n=2$ levels including effects to first order in ε (but ignoring second order effects)?

Note: You may want to use some of the following:

Radial Wave Functions $R_{nl}(r)$ (a is the Bohr radius):

$$R_{10}(r) = \frac{1}{a^{3/2}} 2e^{-r/a} \qquad R_{21}(r) = \frac{1}{a^{3/2}} \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a}$$

$$R_{20}(r) = \frac{1}{a^{3/2}} \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

Spherical Harmonics $Y_l^m(\theta, \phi)$:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

An integral:

$$\int_0^\infty x^n e^{-x/a} dx = a^{n+1} n!$$

The unperturbed energy is $E_n = -\frac{mZ^2e^4}{2\hbar^2 n^2}$ where $Z=1$.

Assume $\vec{E} = \mathcal{E}\hat{z}$ so the perturbation is $H' = eV = -e\mathcal{E}z$

First order perturbation theory says $\Delta_n^{(1)} = H'_{nn} = \langle \Psi_n | H' | \Psi_n \rangle$

$$\Delta_{100}^{(1)} = -e\mathcal{E} \int R_{10}^*(r) Y_{00}^*(\theta,\phi)\, r\cos(\theta)\, R_{10}(r) Y_{00}(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= -e\mathcal{E} \int_0^\infty R_{10}^*(r) R_{10}(r) r^3 dr \int_0^\pi \frac{1}{4\pi} \cos(\theta)\sin(\theta)\, d\theta \int_0^{2\pi} d\phi$$

$$= -\frac{1}{2} e\mathcal{E} \int_0^\infty R_{10}^*(r) R_{10}(r) r^3 dr \overbrace{\int_0^\pi \frac{1}{2}\sin(2\theta)\, d\theta}^{0} = 0$$

The $n=2$ states are degenerate, so we use degenerate perturbation theory.

Let V be the 4 by 4 matrix for H' in the basis of the 4 degenerate states: $|200\rangle, |210\rangle, |211\rangle, |21{-1}\rangle$, so elements are of the form $\langle a | H' | b \rangle$ where a and b are from this set of 4.

The selection rules from the Wigner-Eckart Theorem help us evaluate the matrix V. The operator z is the 0^{th} spherical component of a rank 1 tensor $\vec{X}$. So $\Delta l = k = 1$ and $m' = m + q = m$.

$$V = -e\mathcal{E} \begin{pmatrix} \langle 200|z|200\rangle & \langle 200|z|210\rangle & \langle 200|z|211\rangle & \langle 200|z|21{-1}\rangle \\ \langle 210|z|200\rangle & \langle 210|z|210\rangle & \langle 210|z|211\rangle & \langle 210|z|21{-1}\rangle \\ \langle 211|z|211\rangle & \langle 211|z|210\rangle & \langle 211|z|211\rangle & \langle 211|z|21{-1}\rangle \\ \langle 21{-1}|z|21{-1}\rangle & \langle 21{-1}|z|210\rangle & \langle 21{-1}|z|211\rangle & \langle 21{-1}|z|21{-1}\rangle \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha = -e\mathcal{E}\langle 200|z|210\rangle = -e\mathcal{E} \int R_{20}^*(r) Y_{00}^*(\theta,\phi)\, r\cos(\theta)\, R_{21}(r) Y_{10}(\theta,\phi)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= -e\mathcal{E} \int a^{-3/2} \frac{1}{\sqrt{2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \frac{1}{\sqrt{4\pi}}\, r\cos(\theta)\, a^{-3/2} \frac{1}{2\sqrt{6}} \frac{r}{a} e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos(\theta)\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$= -\frac{e\mathcal{E}}{16\pi} a^{-4} \int_0^\infty r^4 \left(1 - \frac{r}{2a}\right) e^{-r/a} dr \int_0^\pi \cos^2(\theta)\sin(\theta)\, d\theta \int_0^{2\pi} d\phi$$

$$= -\frac{e\mathcal{E}}{8} a^{-4} \left(a^5 4! - \frac{1}{2a} a^6 5!\right)\left(\frac{2}{3}\right) = -\frac{e\mathcal{E}}{12} a\,(24 - 60) = 3e\mathcal{E}a$$

Now we diagonalize V to find the energy eigenvalues, but we only need the first quarter.

$$\begin{pmatrix} 0 & 3e\mathcal{E}a \\ 3e\mathcal{E}a & 0 \end{pmatrix} \begin{pmatrix} \langle 200|\Psi\rangle \\ \langle 210|\Psi\rangle \end{pmatrix} = 3e\mathcal{E}a \begin{pmatrix} \langle 210|\Psi\rangle \\ \langle 200|\Psi\rangle \end{pmatrix} = \lambda \begin{pmatrix} \langle 200|\Psi\rangle \\ \langle 210|\Psi\rangle \end{pmatrix} \Rightarrow \begin{matrix} |\Psi_+\rangle = \frac{1}{\sqrt{2}}(|200\rangle + |210\rangle) \\ |\Psi_-\rangle = \frac{1}{\sqrt{2}}(|200\rangle - |210\rangle) \end{matrix}$$

$$\begin{vmatrix} -\lambda & 3e\mathcal{E}a \\ 3e\mathcal{E}a & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (3e\mathcal{E}a)^2 = 0 \Rightarrow \lambda = \pm 3e\mathcal{E}a$$

Therefore, to first order $E_{n=1} = \frac{me^4}{2\hbar^2}$ and $E_{n=2} = \frac{me^4}{8\hbar^2} \pm 3e\mathcal{E}a$

3. *Quantum Mechanics* (Spring 2006)

A hydrogen atom is placed in a constant weak electric field of strength $\mathcal{E}$. Ignoring spin, what are the energies of the $n = 1$ and $n = 2$ levels including effects to first order in $\mathcal{E}$ (but ignoring second order effects)?

Note: You may want to use some of the following:

Radial Wave Functions $R_{nl}(r)$ (a is the Bohr radius):

$$R_{10}(r) = \frac{1}{a^{3/2}} 2e^{-r/a} \qquad R_{21}(r) = \frac{1}{a^{3/2}}\frac{1}{2\sqrt{6}}\frac{r}{a}e^{-r/2a}$$
$$R_{20}(r) = \frac{1}{a^{3/2}}\frac{1}{\sqrt{2}}\left(1-\frac{r}{2a}\right)e^{-r/2a}$$

Spherical Harmonics $Y_l^m(\theta,\phi)$:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}} \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

An integral:

$$\int_0^\infty x^n e^{-x/a}dx = a^{n+1}n!$$

Unperturbed energies: $E_n^{(0)} = -\frac{mK^2e^4}{2\hbar^2 n^2}$, degenerate (in l, m) for $n \geq 2$

Perturbation: $H' = -\vec{p}\cdot\vec{\mathcal{E}} = -q\vec{r}\cdot\vec{\mathcal{E}} = e\mathcal{E}z$ given $\vec{\mathcal{E}} = \mathcal{E}\hat{z}$

$n=1$ ($l=m=0$): (nondegenerate perturbation theory)

$(\Delta E)_1^{(1)} = \langle\Psi_{100}|H'|\Psi_{100}\rangle$, but H' (i.e., z) is an odd-parity operator, which only "connects" states of different parity

$\Rightarrow (\Delta E)_1^{(1)} = 0$

$n=2$ ($|200\rangle, |210\rangle, |21\pm1\rangle$): (degenerate perturbation theory, four-fold)
(rename $\hookrightarrow |0\rangle$ $\hookrightarrow |1\rangle$ $\hookrightarrow |\pm\rangle$)

$$H' \doteq \begin{pmatrix} \langle 0|H'|0\rangle & \langle 0|H'|1\rangle & \langle 0|H'|+\rangle & \langle 0|H'|-\rangle \\ \langle 1|H'|0\rangle & \langle 1|H'|1\rangle & \langle 1|H'|+\rangle & \langle 1|H'|-\rangle \\ \langle +|H'|0\rangle & \langle +|H'|1\rangle & \langle +|H'|+\rangle & \langle +|H'|-\rangle \\ \langle -|H'|0\rangle & \langle -|H'|1\rangle & \langle -|H'|+\rangle & \langle -|H'|-\rangle \end{pmatrix}$$

the Wigner-Eckart selection rules tell us, since z is the zero-component of a rank one spherical tensor, that the non-zero matrix elements have $\Delta m = 0$, $|\Delta l| \leq 1 \leq 2l$ (remember also, H' doesn't connect a state to itself)

$$= \begin{pmatrix} 0 & c & 0 & 0 \\ c^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow (\Delta E)_\pm^{(1)} = 0$$

$c = \langle 0|H'|1\rangle$ where $H' = e\mathcal{E}r\cos\theta = e\mathcal{E}\sqrt{\frac{4\pi}{3}}rY_1^0$ given $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$

$$= e\mathcal{E}\sqrt{\frac{4\pi}{3}}\int_0^\infty a^{-3/2}\frac{1}{\sqrt{2}}\left(1-\frac{r}{2a}\right)e^{-r/2a}\,a^{-3/2}\frac{1}{2\sqrt{6}}\frac{r}{a}e^{-r/2a}r^3dr\int\frac{1}{\sqrt{4\pi}}Y_1^0Y_1^0\,d\Omega$$

$$= e\mathcal{E}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\frac{1}{2\sqrt{6}}a^{-3}a^{-1}\int_0^\infty\left(r^4-\frac{r^5}{2a}\right)e^{-r/a}dr = \frac{1}{12}e\mathcal{E}a^{-4}\left[a^5 4! - \frac{1}{2a}a^6 5!\right]$$

$$= e\mathcal{E}a[2-5] = -3e\mathcal{E}a \quad \text{and} \quad c^* = c$$

3. Quantum Mechanics (Spring 2006)

(continued)

We want the remaining two eigenvalues of H', so we can simply examine the upper left quarter of the matrix:

$$\begin{pmatrix} 0 & c \\ c^* & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = c\,\sigma_x \quad \Rightarrow \quad |\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

where $\sigma_x|\psi_\pm\rangle = \pm|\psi_\pm\rangle$

$\Rightarrow$ eigenvalues $\pm c$ for $|\psi_\pm\rangle$

Summarizing: $E^{(1)}_{|\psi\rangle} = E^{(0)}_n + (\Delta E)^{(1)}_{|\psi\rangle}$, so

$n=1$ $\quad |100\rangle$ $\quad E^{(1)}_{100} = -\dfrac{mK^2e^4}{2\hbar^2}$

$n=2$ $\quad |\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|200\rangle + |210\rangle\right)$ $\quad E^{(1)}_{|\psi_+\rangle} = -\dfrac{mK^2e^4}{8\hbar^2} - 3e\mathcal{E}a$

$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|200\rangle - |210\rangle\right)$ $\quad E^{(1)}_{|\psi_-\rangle} = -\dfrac{mK^2e^4}{8\hbar^2} + 3e\mathcal{E}a$

$|21+1\rangle$ $\quad E^{(1)}_{21+1} = -\dfrac{mK^2e^4}{8\hbar^2}$

$|21-1\rangle$ $\quad E^{(1)}_{21-1} = -\dfrac{mK^2e^4}{8\hbar^2}$

4. *Quantum Mechanics* (Spring 2006)

Consider the scattering of a beam of non-relativistic spin 0 particles by a repulsive spherical potential of depth V_0 and radius a in three dimensions:

$$V(r) = \begin{cases} V_0, & \text{for } r < a; \\ 0, & \text{for } r > a. \end{cases}$$

Find the scattering cross section in the Born approximation.

The scattering cross section is $\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega$

where $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ and in the Born approximation

for spherically symmetric potentials, $f^{(1)}(\theta,\phi) = -\frac{2m}{q}\int_0^\infty r\sin(qr)V(r)dr$

where $q = 2K\sin(\theta_{sc}/2)$ and $\hbar K$ is the momentum transferred.

Our potential is $V(r) = V_0\Theta(a-r)$

$$\begin{aligned}
f^{(1)}(\theta,\phi) &= -\frac{2m}{q}\int_0^\infty r\sin(qr)V_0\Theta(a-r)\,dr \\
&= -\frac{2mV_0}{q}\int_0^a r\sin(qr)\,dr \\
&= -\frac{2mV_0}{q}\left[r\frac{-\cos(qr)}{q}\Big|_0^a - \int_0^a \frac{-\cos(qr)}{q}dr\right] \\
&= -\frac{2mV_0}{q}\left[-\frac{a\cos(qa)}{q} + \frac{\sin(qr)}{q^2}\Big|_0^a\right] \\
&= -\frac{2mV_0}{q}\left[-\frac{a\cos(qa)}{q} + \frac{\sin(qa)}{q^2}\right] \\
&= 2mV_0\left[\frac{a}{q^2}\cos(qa) - \frac{1}{q^3}\sin(qa)\right]
\end{aligned}$$

$$\frac{d\sigma}{d\Omega} = |f^{(1)}(\theta,\phi)|^2 = 4m^2V_0^2\left|\frac{a}{q^2}\cos(qa) - \frac{1}{q^3}\sin(qa)\right|^2$$

$$\sigma_{tot} = \int\frac{d\sigma}{d\Omega}d\Omega = 4m^2V_0^2\int\left|\frac{a}{q^2}\cos(qa) - \frac{1}{q^3}\sin(qa)\right|^2 d\Omega$$

6. *Statistical Mechanics and Thermodynamics* (Spring 2006)

Consider a gas of relativistic, conserved bosons. The relation between energy and momentum is

$$E = |\mathbf{p}|c$$

(a) Derive the condition for Bose-Einstein condensation in three dimensions.

(b) Does Bose-Einstein condensation occur in two dimensions? Justify your answer.

(c) What is the highest dimension for which Bose-Einstein condensation does not occur?

The simplest definition of T_c is the minimum temperature for which all particles in the system are expected to be in excited states. Our strategy:

1. Find the density of states
2. Integrate occupancy times density of states to get the total number of particles in excited states N_e (since $\epsilon=0$ for ground state, they aren't counted in this integral because $\rho(0)=0$).
3. Maximize N_e by setting $\mu=0$ so the minimum temperature comes out.
4. Set $N_e=N$ and solve for $T=T_c$.

a. $\epsilon = pc = \hbar k c = \left(\frac{\hbar\pi c}{L}\right)n \Rightarrow n = \left(\frac{L}{\hbar\pi c}\right)\epsilon \Rightarrow dn = \left(\frac{L}{\hbar\pi c}\right)d\epsilon$

$$\rho(\epsilon) = \frac{1}{8}4\pi n^2 dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^3\epsilon^2 d\epsilon = \frac{V}{2\pi^2}\frac{\epsilon^2}{(\hbar c)^3}d\epsilon$$

$$\begin{aligned}
N_e &= \int_0^\infty f(\epsilon)\rho(\epsilon)\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}}\sum_{\ell=0}^\infty e^{-\beta\ell(\epsilon-\mu)}\,d\epsilon \qquad (\text{must have } \epsilon>\mu \text{ for } \bar{n}>0) \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\int_0^\infty \epsilon^2 \sum_{\ell=1}^\infty e^{-\beta\ell(\epsilon-\mu)}\,d\epsilon \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{\ell=1}^\infty\left[e^{\beta\ell\mu}\int_0^\infty \epsilon^2 e^{-\beta\ell\epsilon}\,d\epsilon\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\sum_{\ell=1}^\infty\left[e^{\beta\ell\mu}\left(\frac{1}{\beta\ell}\right)^3\int_0^\infty x^2 e^{-x}\,dx\right] \\
&= \frac{V}{2\pi^2}\frac{1}{(\hbar c)^3}\frac{2!}{\beta^3}\sum_{\ell=1}^\infty \frac{e^{\beta\ell\mu}}{\ell^3} \\
&\xrightarrow{\mu\to 0} \frac{V}{\pi^2}\frac{1}{(\hbar c\beta)^3}\sum_{\ell=1}^\infty \frac{1}{\ell^3} \qquad \text{and } \sum_{\ell=1}^\infty \frac{1}{\ell^3} \equiv \zeta(3)
\end{aligned}$$

$$\begin{aligned}
N_e = N &\Rightarrow \frac{1}{\beta^3} = \frac{N}{V}\pi^2\frac{(\hbar c)^3}{\zeta(3)} \\
&\Rightarrow \frac{1}{\beta} = \hbar c\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3} \\
&\Rightarrow T_c = \frac{\hbar c}{k}\left(\pi^2\frac{N}{V}/\zeta(3)\right)^{1/3}
\end{aligned}$$

b. In 2D, $\rho(\epsilon)d\epsilon = \frac{1}{4}2\pi n\,dn = \frac{\pi}{2}\left(\frac{L}{\hbar\pi c}\right)^2\epsilon\,d\epsilon$ gives $\zeta(2)$ with a similar procedure and $\zeta(2)$ converges so everything is fine and condensation does occur.

c. In 1D, $\rho(\epsilon)d\epsilon = dn = \frac{L}{\hbar\pi c}d\epsilon$ gives $\zeta(1)$ with a similar procedure, but $\zeta(1)$ diverges, so the resulting T_c is $T_c=0$, so BEC does not occur in 1D, making 1 the highest dimension for which BEC does not occur.

7. *Statistical Mechanics and Thermodynamics* (Spring 2006)

(a) In the case of a set of non-relativistic, noninteracting spin-1/2 fermions confined to two dimensions, what is the paramagnetic susceptibility at $T = 0$? Give your answer in terms of the mass m of the fermions, the gyromagnetic ratio γ, the number density σ, and whatever fundamental constants (e.g., $\hbar$, c, ...) are necessary to completely specify the system.

(b) Now, suppose that the external magnetic field is replaced by an effective field H_{eff}, where

$$H_{\text{eff}} = H + \Gamma \frac{M}{A}$$

where H is the external field and M/A is the induced magnetic moment per unit area. Above what threshold value of the parameter Γ is this two-dimensional system ferromagnetic at $T = 0$?

a. We want to find χ, which is defined by $M_0 = \chi H$ where $M_0 = M/A$ and M is the total magnetization.

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 = \frac{\hbar^2\pi^2}{2mL^2}n^2$$

$$\Rightarrow n^2 = \frac{2mL^2}{\hbar^2\pi^2}\epsilon \quad \Rightarrow n = \sqrt{2m}\frac{L}{\hbar\pi}\epsilon^{1/2}$$

$$\Rightarrow dn = \frac{1}{2}\sqrt{2m}\frac{L}{\hbar\pi}\epsilon^{-1/2}d\epsilon$$

In 2D, $\rho(\epsilon)d\epsilon = \underbrace{2}_{\text{for spin}}\frac{1}{4}2\pi n\,dn = \cancel{2}\frac{\pi}{\cancel{2}}\frac{1}{\cancel{2}}\cancel{2}m\left(\frac{L}{\hbar\pi}\right)^2 d\epsilon$

$$= \frac{mA}{\pi\hbar^2}d\epsilon$$

$$\Rightarrow \rho(\epsilon) = \frac{mA}{\pi\hbar^2} \quad \text{so } \rho(\epsilon) \text{ is constant.}$$

$\bar{M} = \sum_r \mu_{H_r}\bar{n}_r = \sum_{\pm}\mu_{H\pm}\sum_p n_p$ where p runs over momentum states

$$= \mu_H\sum_p \bar{n}_p^{(+)} - \mu_H\sum_p \bar{n}_p^{(-)} \cong \mu_H\int_0^\infty \bar{n}(\epsilon_+)\rho(\epsilon_+)d\epsilon - \mu_H\int_0^\infty \bar{n}(\epsilon_-)\rho(\epsilon_-)d\epsilon$$

$= \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon_+)d\epsilon - \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon_-)d\epsilon$ since ρ is constant

$= \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon - \mu_H H)d\epsilon - \mu_H\rho(0)\int_0^\infty \bar{n}(\epsilon + \mu_H H)d\epsilon$ since $U = -\vec{\mu}_H\cdot\vec{H}$

$= \mu_H\rho(0)\int_0^\infty \theta(\mu - (\epsilon - \mu_H H))d\epsilon - \mu_H\rho(0)\int_0^\infty \theta(\mu - (\epsilon + \mu_H H))d\epsilon$ at $T=0$

$$= \mu_H\rho(0)(\mu + \mu_H H) - \mu_H\rho(0)(\mu - \mu_H H)$$

$$= \mu_H\rho(0)(2\mu_H H) = 2\mu_H^2 H\rho(0) = 2\mu_H^2 H\frac{mA}{\pi\hbar^2}$$

$$\chi = \frac{M_0}{H} = \frac{M}{AH} = \frac{2m\mu_H^2}{\pi\hbar^2}$$

b. A ferromagnetic substance has $\chi \gg 1$.

$$M = 2\mu_H^2\left(H + \Gamma\frac{M}{A}\right)\frac{mA}{\pi\hbar^2} \Rightarrow \frac{1}{2}\frac{\pi\hbar^2}{mA\mu_H^2}M = H + \Gamma\frac{M}{A}$$

$$\Rightarrow H = \left(\frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2} - \Gamma\right)\frac{M}{A} \Rightarrow \chi = \frac{M}{AH} = \left(\frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2} - \Gamma\right)^{-1}$$

$\Rightarrow \Gamma \cong \frac{1}{2}\frac{\pi\hbar^2}{m\mu_H^2}$ will produce ferromagnetism

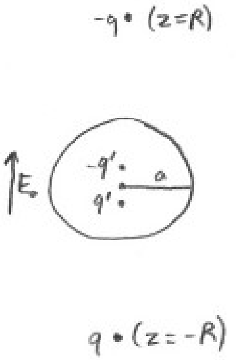
10. *Electricity and Magnetism* (Spring 2006)

An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres from separating

(a) if the shell is uncharged;

This is Jackson 2.9

(b) if the total charge on the shell is Q.



a. The strategy is:

1) Imagine the E_0 field is created by two point charges a large distance R away and use the method of images to find the potential outside the sphere

2) Find σ using $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}\Big|_{r=a}$

3) Find the force using $F_z = 2\int_{hemi} E_z dq$

$$= 2\int_{hemi} \left(\frac{\sigma}{2\epsilon_0}\cos(\theta)\right)\sigma\, dA$$

$$= 2\int_0^{2\pi}\int_0^{\pi/2} \frac{\sigma^2}{2\epsilon_0}\cos(\theta)\, a^2 \sin(\theta)\, d\theta\, d\phi$$

where the factor of 2 comes from having to push both hemispheres in order to hold them together.

Method of Images:

$$z' = \frac{a^2}{R} \qquad q' = -\frac{a}{R} q$$

1) $E_0 = 2\frac{1}{4\pi\epsilon_0}\frac{q}{R^2} \Rightarrow q = 2\pi\epsilon_0 R^2 E_0$

Recall $\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\, d^3r' \Rightarrow \vec{p}$ points toward positive charge, unlike fields

$$\vec{p} = |2z'|\cdot|q'|\hat{z} = 2\frac{a^3}{R^2}q\hat{z} = 4\pi\epsilon_0 a^3 E_0\hat{z}$$

Remember that $V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ so

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{4\pi\epsilon_0 a^3 E_0}{r^2}\cos(\theta) = \frac{a^3}{r^2}E_0\cos(\theta)$$

$$V_{field} = -\int \vec{E}\cdot d\vec{r} = -E_0 r\cos(\theta)$$

$$V = V_{dip} + V_{field} = -\left(r - \frac{a^3}{r^2}\right)E_0\cos(\theta)$$

2) $\sigma = -\epsilon_0\frac{\partial V}{\partial n}\Big|_{r=a} = -\epsilon_0\frac{\partial V}{\partial r}\Big|_{r=a} = \epsilon_0\left(1 + 2\frac{a^3}{r^3}\right)E_0\cos(\theta)\Big|_{r=a}$

$= 3\epsilon_0 E_0\cos(\theta)$ which is Jackson (2.15)

3) $F_z = 9\epsilon_0 E_0^2 a^2 (2\pi)\int_0^{\pi/2}\cos^3(\theta)\sin(\theta)\, d\theta$ Let $x = \cos(\theta)$

$$= 9\epsilon_0 E_0^2 a^2 (2\pi)\int_0^1 x^3 dx = \frac{9}{2}\pi\epsilon_0 E_0^2 a^2$$

b. In part a, the surface was an equipotential, so an additional charge Q will spread out uniformly: $\sigma \to 3\epsilon_0 E_0\cos(\theta) + Q/4\pi a^2$

$$F_z = \frac{2\pi a^2}{\epsilon_0}\int_0^{\pi/2}\left[9\epsilon_0 E_0\cos^2(\theta) + 2\frac{3\epsilon_0 E_0 Q}{4\pi a^2}\cos(\theta) + \frac{Q^2}{16\pi^2 a^4}\right]\cos(\theta)\sin(\theta)\, d\theta$$

$$= \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + 3E_0 Q\int_0^{\pi/2}\cos^2(\theta)\sin(\theta)\, d\theta + \frac{Q^2}{8\pi a^2\epsilon_0}\int_0^{\pi/2}\cos(\theta)\sin(\theta)\, d\theta$$

$$= \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + E_0 Q + \frac{Q^2}{16\pi a^2\epsilon_0}$$

But the middle term is the Q charge interacting with the field, which is in the same direction for both hemispheres, so

$$F = \frac{9}{2}\pi\epsilon_0 E_0^2 a^2 + \frac{Q^2}{16\pi a^2\epsilon_0}$$

8. *Statistical Mechanics and Thermodynamics* (Spring 2006)

(a) A system consists of N particles, each of which can exist in two states, with energies ϵ_0 and ϵ_1, respectively. Given that the total energy of this system is U, what is its entropy?

(b) Obtain the expression for the entropy in the limit that N is large.

(c) Now, give an expression for the temperature of this system, as a function of U and the energies of the single particle states. Does this expression have any properties that require some discussion?

Stirling's formula: $n! \approx (\frac{n}{e})^n$, when n is large.

a. The single particle partition function is

$$z = \sum_r e^{-\beta \epsilon_r} = e^{-\beta \epsilon_0} + e^{-\beta \epsilon_1}$$

$$\Rightarrow Z = \frac{z^N}{N!} = \frac{1}{N!}\left(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}\right)^N$$

$$S = k(\ln(Z) + \beta U) = k\left[N\ln(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}) - \ln(N!) + \beta U\right]$$

b. Using Stirling's Formula, $\ln(N!) \cong N\ln(\frac{N}{e}) = N\ln(N) - N$

$$\Rightarrow S \cong k\left[N\ln(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}) - N\ln(N) + N + \beta U\right]$$

c. $U = -\frac{\partial \ln(Z)}{\partial \beta}$ where $\ln(Z) = N\ln(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}) - \ln(N!)$

$$U = N\,\frac{\epsilon_0 e^{-\beta\epsilon_0} + \epsilon_1 e^{-\beta\epsilon_1}}{e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}}$$

Now group like terms to solve for β

$$U(e^{-\beta\epsilon_0} + e^{-\beta\epsilon_1}) = N(\epsilon_0 e^{-\beta\epsilon_0} + \epsilon_1 e^{-\beta\epsilon_1})$$

$$(U - N\epsilon_0)e^{-\beta\epsilon_0} = (N\epsilon_1 - U)e^{-\beta\epsilon_1}$$

$$e^{\beta(\epsilon_1 - \epsilon_0)} = \frac{N\epsilon_1 - U}{U - N\epsilon_0}$$

$$\beta(\epsilon_1 - \epsilon_0) = \ln\left(\frac{N\epsilon_1 - U}{U - N\epsilon_0}\right)$$

$$T = \frac{\epsilon_1 - \epsilon_0}{k}\left[\ln\left(\frac{N\epsilon_1 - U}{U - N\epsilon_0}\right)\right]^{-1}$$

This expression has the property that it is negative if $N\epsilon_1 - U < U - N\epsilon_0 \Leftrightarrow U > \frac{1}{2}N(\epsilon_0 + \epsilon_1)$, which is a characteristic of systems with an upper limit to their total energy (see Reif page 105).

9. *Statistical Mechanics and Thermodynamics* (Spring 2006)

A researcher claims that a particular substance in thermal equilibrium exhibits the following total-number-of-states function

$$\Omega(E) = c(E - E_0)^\alpha V^\gamma \exp\left(-\frac{g}{V}\right)$$

where E_0, c, α, γ, and g are positive coefficients independent of the energy E, the volume V, and the temperature T.

(a) Find the equation of state for this substance.

(b) What is the relationship between the average energy and the temperature?

(c) Does this substance satisfy the third law of thermodynamics? Why?

(d) What values should E_0, c, α, γ, and g take for this substance to behave as an ideal gas?

a. $dE = TdS - pdV \Rightarrow dS = \frac{1}{T}dE + \frac{p}{T}dV$

$dS = \left(\frac{\partial S}{\partial E}\right)_V dE + \left(\frac{\partial S}{\partial V}\right)_E dV \Rightarrow \left(\frac{\partial S}{\partial E}\right)_V = \frac{1}{T}$ and $\left(\frac{\partial S}{\partial V}\right)_E = \frac{p}{T}$

The most typical equation of state here is

$p = T\left(\frac{\partial S}{\partial V}\right)_E$ and $S = k\ln(\Omega(E)) = k\ln\left[c(E-E_0)^\alpha V^\gamma \exp(-\frac{g}{V})\right]$

$= T\left(\frac{\partial}{\partial V}\right)_E \left(k\ln[c(E-E_0)^\alpha] + k\ln[V^\gamma \exp(-\frac{g}{V})]\right)$

$= kT \frac{1}{V^\gamma \exp(-g/V)}\left(\gamma V^{\gamma-1}\exp(-\frac{g}{V}) + V^\gamma(-g)(-\frac{1}{V^2})\exp(-\frac{g}{V})\right)$

$= kT\left(\frac{\gamma}{V} + \frac{g}{V^2}\right)$

$\Rightarrow pV = kT(\gamma + \frac{g}{V})$

b. $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V = \left(\frac{\partial}{\partial E}\right)_V \left(k\ln[(E-E_0)^\alpha] + k\ln[cV^\gamma \exp(-\frac{g}{V})]\right)$

$= \left(\frac{\partial}{\partial E}\right)_V \left(\alpha k\ln(E-E_0)\right)$

$= \frac{\alpha k}{E-E_0}$

$\Rightarrow T = \frac{E-E_0}{\alpha k}$

c. The 3rd Law states $S \xrightarrow{T\to 0} S_0$ and as $T\to 0$ we have $E \to E_0$ and $S \to k\ln(0) = -\infty$, which is not some constant S_0, so the answer is no.

Note: The ideal gas also does not satisfy the 3rd Law because it is not a valid approximation for low temperatures.

d. For an ideal gas, $E = \frac{3}{2}NkT$ and $pV = NkT$

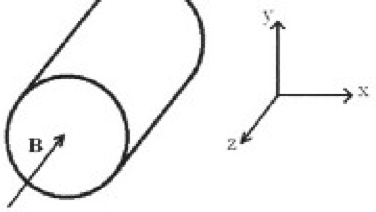
$E \to 0$ as $T \to 0 \Rightarrow \underline{E_0 = 0}$, $E = \frac{3}{2}NkT = \alpha kT \Rightarrow \underline{\alpha = \frac{3}{2}N}$,

$pV = NkT = kT(\gamma + \frac{g}{V}) \Rightarrow N = \gamma + \frac{g}{V} \Rightarrow \underline{g = 0}$ and $\underline{\gamma = N}$

Now $\Omega(E) = c(VE^{3/2})^N \Rightarrow \underline{c = 1}$ since $S(N=0) \to 0$.

11. *Electricity and Magnetism* (Spring 2006)

Consider a long solid cylinder made of uniform resistive material. The cylinder is in a region in which there is an applied magnetic field that is uniform and is directed along the axis of the cylinder. The magnetic field is time-dependent and it is oscillating with angular frequency ω: $\mathbf{B}(t) = B_z \cos\omega t\hat{\mathbf{z}}$. The length of the cylinder is L and its radius is R ($R \ll L$). The resistivity of the cylinder material is ρ.



(a) Calculate the current density $\mathbf{j}(t)$ in the volume of the cylinder. Assume initially that you can ignore the self-inductance of the cylinder. Ignore end effects and the Hall effect.

(b) For large values of ω the effect of self-inductance cannot be ignored. Calculate the correction to the current density $\Delta\mathbf{j}(t)$ due to the self-inductance of the cylinder in next order of ω.

(c) Give the condition on ω such that the self-inductance of the cylinder can be ignored.

a. $\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \Rightarrow \int_c \vec E\cdot d\vec l = -\int_s \frac{\partial \vec B}{\partial t}\cdot d\vec a = -\int_s -\omega B_z \sin(\omega t)\hat z\cdot d\vec a$

$\Rightarrow 2\pi r E = \omega B_z \sin(\omega t)\pi r^2 \Rightarrow \vec E = \frac12 \omega B_z \sin(\omega t) r\,\hat\phi$

$\vec J = \sigma \vec E = \frac1\rho \vec E \Rightarrow \vec J = \frac{1}{2\rho}\omega B_z\sin(\omega t) r\hat\phi$

b. First we find the correction to the magnetic field due to all the solenoids outside radius r. $\vec B_{sol}(r,t) = \mu_0 \vec K(r,t)$

$d\vec K(r) = \vec J(r)dr$ and $d^2 \vec I(r) = d\vec K(r) dz = \vec J(r)\, dr\, dz \Rightarrow d\vec B_{sol}(r,t) = \mu_0 \vec J(r,t) dr$

$\Delta\vec B(r,t) = \int_r^R d\vec B_{sol}(r',t) = \frac{\mu_0}{2\rho}\omega B_z \sin(\omega t)\hat\phi\int_r^R r'dr'$

$= \frac{\mu_0}{2\rho}\omega B_z\sin(\omega t)\hat\phi\left(\frac12(R^2-r^2)\right) = \frac{\mu_0}{4\rho}\omega B_z \sin(\omega t)(R^2-r^2)\hat\phi$

$\vec\nabla\times\vec E = -\frac{\partial \vec B}{\partial t} \Rightarrow \int_c \vec E\cdot d\vec l = -\int_s \frac{\partial\vec B}{\partial t} = -\int_0^{2\pi}\int_0^r \frac{\partial\vec B}{\partial t}\, r'dr'd\theta'$

$\Rightarrow 2\pi r\Delta E = -2\pi\int_0^r \frac{\partial}{\partial t}(\Delta B)\, r'dr'$

$\Rightarrow \Delta E = -\frac1r\int_0^r \frac{\mu_0}{4\rho}\omega^2 B_z\cos(\omega t)(R^2-r'^2)r'dr'$

$= -\frac1r\frac{\mu_0}{4\rho}\omega^2 B_z\cos(\omega t)\left(\frac12 r^2R^2 - \frac14 r^4\right)$

$= -\frac{\mu_0}{4\rho}\omega^2 B_z \cos(\omega t)\left(\frac12 rR^2 - \frac14 r^3\right)$

$\Rightarrow \Delta\vec J = \frac1\rho\Delta\vec E = -\frac{\mu_0}{4\rho^2}\omega^2 B_z\cos(\omega t)\left(\frac12 rR^2-\frac14 r^3\right)\hat\phi$

c. $\left|\frac{\Delta \vec J}{\vec J}\right| \ll 1 \Leftrightarrow \frac{\frac{\mu_0}{4\rho^2}\omega^2 B_z}{\frac{1}{2\rho}\omega B_z} \ll 1 \Leftrightarrow \frac{\mu_0}{\rho}\omega \ll 1 \Leftrightarrow \omega \ll \frac{\rho}{\mu_0}$