1. Quantum Mechanics

Two distinguishable spin-1/2 particles interact via the Hamiltonian $H_0 = g\vec{S}_1 \cdot \vec{S}_2$.

a) What are the energy eigenstates and eigenvalues? Express the states in terms of eigenstates of $S_{1,z}$ and $S_{2,z}$.

We now add a time dependent perturbation: $H = H_0 + \epsilon \exp\left(-\frac{t^2}{2\alpha^2}\right) S_{1,z}$.

b) Assume the system is in an eigenstate of H_0 at $t = -\infty$. Compute the probabilities for the system to be in a given eigenstate of H_0 at $t = +\infty$, working to lowest nontrivial order in ϵ. In particular, you are being asked to consider transitions between all possible initial and final eigenstates of H_0.

1. Quantum Mechanics (solution)

a) We employ the standard trick of writing

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(\vec{S}_1 + \vec{S}_2)^2 - \frac{1}{2}\vec{S}_1^2 - \frac{1}{2}\vec{S}_2^2 = \frac{1}{2}(\vec{S}_1 + \vec{S}_2)^2 - \frac{3\hbar^2}{4}$$

The total spin squared, $(\vec{S}_1 + \vec{S}_2)^2$ has eigenvalues $j(j+1)\hbar^2$ where either $j = 0$ (antisymmetric singlet state) or $j = 1$ (symmetric triplet states. We thus have

$$E_0 = -\frac{3\hbar^2 g}{4}\ , \quad |0,0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$
$$E_0 = \frac{g\hbar^2}{4}\ , \quad |1,1\rangle = |\uparrow\uparrow\rangle\ , \quad |1,0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]\ , \quad |1,-1\rangle = |\downarrow\downarrow\rangle \tag{1}$$

b) Using the standard formula of time dependent perturbation theory, the probability for a system in state $|i\rangle$ in the far past to end up in state $|f\rangle$ (where $f \neq i$) in the far future is $|A|^2$ where

$$A = -\frac{i}{\hbar}\int_{-\infty}^{\infty} \langle f|\delta H(t)|i\rangle e^{\frac{i(E_f - E_i)t}{\hbar}} dt$$

The states $|1,1\rangle$ and $|1,-1\rangle$ are eigenstates of $S_{1,z}$ and hence of δH. To order ϵ^2 the probability for one of these states to transition to a different one is therefore zero. Similarly, the probability for one of the other two states to transition to either $|1,1\rangle$ or $|1,-1\rangle$ is zero

Thus just leaves the transition probability between $|0,0\rangle$ and $|1,0\rangle$. We have

$$\langle 1,0|S_{1,z}|0,0\rangle = \langle 0,0|S_{1,z}|1,0\rangle = \frac{\hbar}{2} \tag{2}$$

The energy difference between these states is $E_f - E_i = \pm\hbar^2 g^2$. We then have

$$\begin{aligned} |A|^2 &= \left|\frac{\epsilon}{2}\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\alpha^2}\right) e^{i\hbar g^2 t}\right|^2 \\ &= \frac{\pi\alpha^2\epsilon^2}{2} e^{-\hbar^2\alpha^2 g^4} \end{aligned} \tag{3}$$

This gives the transition probability for $|0,0\rangle \to |1,0\rangle$ and the reverse process $|1,0\rangle \to |0,0\rangle$.

2. Quantum Mechanics

A spin-1/2 particle of mass m is restricted to move in the x-direction only. It moves in a potential whose x dependence is that of an infinite square well of width $2L$

$$V(x) = \begin{cases} 0 & -L \leq x \leq L \\ \infty & \text{otherwise} \end{cases}$$

The particle also couples to a magnetic field via a term in the Hamiltonian $H_B = \mu_0 \vec{\sigma} \cdot \vec{B}$, where σ_i are the Pauli matrices. The magnetic field takes the form

$$\vec{B} = \begin{cases} B\hat{z} & -L \leq x \leq 0 \\ B\hat{x} & 0 < x \leq L \end{cases}$$

What are the energy levels of the particle to first order in B?

We use first order perturbation theory. For $B = 0$ we have

$$E_n = \frac{\hbar^2\pi^2 n^2}{8mL^2}\ , \quad \psi_n^{\pm}(x) = \frac{1}{\sqrt{L}}\begin{cases}\cos(n\pi x/2L) & n = 1,3,5,\ldots\\ \sin(n\pi x/2L) & n = 2,4,6,\ldots\end{cases}$$

where $\pm$ denotes spin up or down in the z-direction.

We need to use degenerate perturbation theory because spin-up and down states have the same energy at $B = 0$. In degenerate perturbation theory we compute the matrix elements of the perturbation in each degenerate subspace. The matrix elements are

$$\begin{pmatrix} \langle\psi_n^+|H_B|\psi_n^+\rangle & \langle\psi_n^+|H_B|\psi_n^-\rangle \\ \langle\psi_n^-|H_B|\psi_n^+\rangle & \langle\psi_n^-|H_B|\psi_n^-\rangle \end{pmatrix}$$

Using that $\sigma_z(\sigma_x)$ only has nonzero matrix elements between states of the same(opposite) spin we have

$$\begin{aligned}\langle\psi_n^+|H_B|\psi_n^+\rangle = -\langle\psi_n^-|H_B|\psi_n^-\rangle = \mu_0 B\int_{-L}^{0}\left(\psi_n^+(x)\right)^2 dx = \tfrac{\mu_0 B}{2}\\ \langle\psi_n^+|H_B|\psi_n^-\rangle = \langle\psi_n^+|H_B|\psi_n^-\rangle = \mu_0 B\int_0^L\left(\psi_n^+(x)\right)^2 dx = \tfrac{\mu_0 B}{2}\end{aligned} \qquad (1)$$

The matrix of perturbations is therefore

$$\frac{\mu_0 B}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The shifts in energy are given by the eigenvalues of this matrix, which are determined by the equation

$$\left(\frac{\mu_0 B}{2} - \lambda\right)\left(-\frac{\mu_0 B}{2} - \lambda\right) - \left(\frac{\mu_0 B}{2}\right)^2 = 0$$

with solutions $\lambda = \pm\frac{\mu_0 B}{\sqrt{2}}$. The energies to first order in B are therefore

$$E_n = \frac{\hbar^2\pi^2 n^2}{8mL^2} \pm \frac{\mu_0 B}{\sqrt{2}}$$

3. Quantum Mechanics

The purpose of this problem is to show that a spin zero particle with electric charge e in the presence of a certain radial magnetic field $\mathbf{B}$ effectively behaves as a particle with spin $\frac{1}{2}$. The classical Lagrangian for the spin 0 particle is given by (here $\mathbf{v} = \dot{\mathbf{r}}$ and $r = |\mathbf{r}|$),

$$L = \frac{1}{2} m \mathbf{v}^2 + e\mathbf{A} \cdot \mathbf{v} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A} = g\,\frac{\mathbf{r}}{r^3}$$

where m is the mass of the particle and g is a real parameter.

1. Compute the canonical momenta $\mathbf{p}$ conjugate to the position variables $\mathbf{r}$.
2. Write down the Euler-Lagrange equation for the system in terms of $\mathbf{r}$ and $\mathbf{v}$.
3. Using the results of 2. above, show that the combination $\mathbf{L} = (L_x, L_y, L_z)$ defined by,

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} - eg\,\frac{\mathbf{r}}{r}$$

   is time-independent.
4. Compute the commutators $[L_i, r_j]$ (i.e. the commutators of the components of the vectors $\mathbf{L}$ and $\mathbf{r}$). An analogous result – which you are not asked to derive – for $[L_i, p_j]$ establishes that $\mathbf{L}$ represents angular momentum.
5. Compute the quantum operator L_z in spherical coordinates r, θ, ϕ using the result of 1.
6. Show that the eigenvalues of L_z are half-odd-integer multiples of $\hbar$ when the electric charge e and the parameter g are related by $eg = \frac{\hbar}{2}$.

[Hint: In a convenient gauge, the vector potential $\mathbf{A}$ for the field $\mathbf{B}$ is given by $\mathbf{A} = g\mathbf{n}_\phi(1 - \cos\theta)/(r\sin\theta)$ where $\mathbf{n}_\phi$ is the unit vector given by $\mathbf{n}_\phi = (-\sin\phi, \cos\phi, 0)$ in spherical coordinates where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.]

3. Quantum Mechanics (solution)

1. The canonical momentum conjugate to $\mathbf{v} = \dot{\mathbf{r}}$ is given by $\mathbf{p} = \partial L/\partial \mathbf{v} = m\mathbf{v} + e\mathbf{A}$.

2. The Euler-Lagrange equations are the equations of motion for a charged particle in the presence of a static magnetic field, given by the Lorentz force equation,

$$m\dot{\mathbf{v}} = e\mathbf{v} \times \mathbf{B}$$

3. Using the result of item 2 and the expression for $\mathbf{B}$ we compute,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \mathbf{r} \times m\dot{\mathbf{v}} - eg\frac{\mathbf{v}}{r} + eg(\mathbf{r}\cdot\mathbf{v})\frac{\mathbf{r}}{r^3} \\ &= \frac{eg}{r^3}\Big(\mathbf{r} \times (\mathbf{v} \times \mathbf{r}) - r^2\mathbf{v} + (\mathbf{r}\cdot\mathbf{v})\mathbf{r}\Big) = 0\end{aligned}$$

4. Using Cartesian coordinates $\mathbf{r} = (x, y, z)$ and spherical coordinates (r, θ, ϕ), we may express L_z by $L_z = x\,mv_y - y\,mv_x - eg\cos\theta$, and eliminate velocities in favor of momenta using the result of item 1,

$$L_z = xp_y - yp_y - exA_y + eyA_x - eg\cos\theta \tag{0.1}$$

Using the coordinate representation of canonical momenta,

$$p_x = -i\hbar\frac{\partial}{\partial x} \qquad\qquad p_y = -i\hbar\frac{\partial}{\partial y} \tag{0.2}$$

and converting $xp_y - yp_x$ to spherical coordinates, we obtain,

$$xp_y - yp_x = -i\hbar\frac{\partial}{\partial \phi} \tag{0.3}$$

Finally, using the form of $\mathbf{A}$ given in the hint, we compute,

$$xA_y - yA_x = g(1 - \cos\theta) \tag{0.4}$$

Putting all together, we obtain,

$$L_z = -i\hbar\frac{\partial}{\partial \phi} - eg \tag{0.5}$$

5. Given that the angle ϕ has range 2π, wave functions must be periodic with period 2π, so that the eigenvalues of the term $-i\partial_\phi$ must be integers. Since for the minimal monopole charge consistent with the Dirac charge quantization condition we have $eg = \hbar/2$, we see that the eigenvalues of L_z are half-odd-integers so that the spinless particle in the presence of thsi monopole in fact behaves as a fermion.

4. Quantum Mechanics

In the standard approximation, the hydrogen atom consists of a heavy, essentially motionless proton of charge e, together with a lighter electron (charge $-e$) and mass m that orbits around it. We treat both the proton and the electron as point charges, and the hydrogen wave functions are given by $\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi)$, with the quantum numbers n, l and m. We usually use $1s$, $2s$, $2p$, etc to represent the energy levels, which combine the quantum numbers nl. For example, we have $n = 1$ and $l = 0$ for the $1s$ state, while $n = 2$ and $l = 1$ for the $2p$ state.

In this problem, we instead take the proton to be a uniformly charged ball of radius R, with $R/a \ll 1$, where $a = 4\pi\epsilon_0\hbar^2/me^2$ is the "Bohr radius" for hydrogen.

(a) Compute the change in energies of the $2s$ and $2p$ states due to the fact that the proton is not point-like. Work to order $(R/a)^2$. In other words, compute $\Delta E_{2s} = E_{2s}(R) - E_{2s}(R = 0)$ to order $(R/a)^2$ and do the same for the $2p$ state. Here, $E_{2s}(R)$ is the energy of the $2s$ state for the proton with finite size.

Compare the energies of the states $2s$ and $2p$: E_{2s} and E_{2p}. Are they equal to each other?

(b) Show that the difference in energy of the $2s$ and $2p$ states, $E_{2s}(R) - E_{2p}(R)$, gives a measure of the proton radius R.

Useful formulas: for the hydrogen wave functions $\psi_{nlm} = R_{nl}(r)Y_{lm}(\theta, \phi)$, the expressions for the first few $R_{nl}(r)$ are given by

$$R_{10}(r) = 2a^{-3/2} \exp(-r/a) , \tag{1}$$

$$R_{20}(r) = \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{1}{2}\frac{r}{a}\right) \exp(-r/2a) , \tag{2}$$

$$R_{21}(r) = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp(-r/2a) . \tag{3}$$

4. Quantum Mechanics (solution)

(a) When we treat the proton as a point charge, the potential energy is given by

$$V_0(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}. \tag{4}$$

When the proton is treated as a ball of radius R, we can compute the potential energy as follows. First, we use Gauss's law for electric field

$$\oint \vec{E}\cdot d\vec{A} = E\,(4\pi r^2) = \begin{cases} \frac{1}{\epsilon_0}e & r \geq R\,, \\ \frac{1}{\epsilon_0}\frac{e}{\frac{4\pi R^3}{3}}\frac{4\pi r^3}{3} = \frac{1}{\epsilon_0}e\left(\frac{r}{R}\right)^3 & r < R\,. \end{cases} \implies E = \begin{cases} \frac{e}{4\pi\epsilon_0}\frac{1}{r^2} & r \geq R\,, \\ \frac{e}{4\pi\epsilon_0}\frac{r}{R^3} & r < R\,. \end{cases} \tag{5}$$

Then the potential energy of the electron is given by

$$V(r) = -\int_r^\infty e\,E\,dr = \begin{cases} -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} & r \geq R\,, \\ -\frac{e^2}{4\pi\epsilon_0}\frac{1}{2R}\left[3-\left(\frac{r}{R}\right)^2\right] & r < R\,. \end{cases} \tag{6}$$

We treat the change from $V_0(r)$ to $V(r)$ as a perturbation H' in the Hamiltonian, with H' given by

$$H'(r) = V(r) - V_0(r) = \begin{cases} 0 & r \geq R\,, \\ \frac{e^2}{4\pi\epsilon_0}\left\{\frac{1}{r} - \frac{1}{2R}\left[3-\left(\frac{r}{R}\right)^2\right]\right\} & r < R\,. \end{cases} \tag{7}$$

Using the perturbation theory, we obtain the energy shift to the $2s$ state

$$\begin{aligned} \Delta E_{2s} &= \langle 2s|H'|2s\rangle \\ &= \int d^3\vec{r}\,H'(r)|\psi_{200}|^2 = \int_0^\infty dr\,r^2\,H'(r)R_{20}^2(r)\int d\Omega\,|Y_{00}(\theta,\phi)|^2 \\ &= \int_0^R dr\,r^2\,\frac{e^2}{4\pi\epsilon_0}\left\{\frac{1}{r}-\frac{1}{2R}\left[3-\left(\frac{r}{R}\right)^2\right]\right\}\left[\frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)\exp\left(-r/2a\right)\right]^2 \\ &\approx \frac{e^2}{4\pi\epsilon_0\,a}\frac{1}{20}\left(\frac{R}{a}\right)^2, \end{aligned} \tag{8}$$

where we have used the fact that the angular part is normalized as $\int d\Omega\,|Y_{00}(\theta,\phi)|^2 = \int \sin(\theta)\,d\theta\,d\phi|Y_{00}(\theta,\phi)|^2 = 1$, and thus we only need to compute the integral over the radial part. In the final step, we take the approximation $R/a \gg 1$ and keep only the leading term. Likewise, the energy shift to the $2p$ state

$$\begin{aligned} \Delta E_{2p} &= \langle 2p|H'|2p\rangle \\ &= \int_0^\infty dr\,r^2\,H'(r)R_{21}^2(r) \\ &= \int_0^R dr\,r^2\,\frac{e^2}{4\pi\epsilon_0}\left\{\frac{1}{r}-\frac{1}{2R}\left[3-\left(\frac{r}{R}\right)^2\right]\right\}\left[\frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,\exp\left(-r/2a\right)\right]^2 \\ &\approx 0\,, \end{aligned} \tag{9}$$

at the order of $(R/a)^2$.

(b) If one treats the proton as a point charge, since $E_n = E_1/n^2$, i.e., only depends on the principal quantum number, we would have the energy levels $E_{2s}(R=0) = E_{2p}(R=0)$. However, now for a finite-size proton, the energy levels of $2s$ and $2p$ are no longer equal to each other. Instead, they differ by

$$E_{2s}(R) - E_{2p}(R) = \Delta E_{2s} - \Delta E_{2p} = \frac{e^2}{4\pi\epsilon_0\, a}\frac{1}{20}\left(\frac{R}{a}\right)^2 . \tag{10}$$

If one measures the energy difference between $2s$ and $2p$, $E_{2s} - E_{2p}$, one can then determine the size of the proton.

5. Classical Mechanics

An electron (charge e = -|e| and rest mass m) with mechanical momentum

$$\boldsymbol{p} = p_0(\hat{\boldsymbol{x}} \sin\theta + \hat{\boldsymbol{y}} \cos\theta)$$

enters into a static magnetic field region ($x>0$) from a region of free space (zero magnetic field and zero vector potential) at $x<0$.

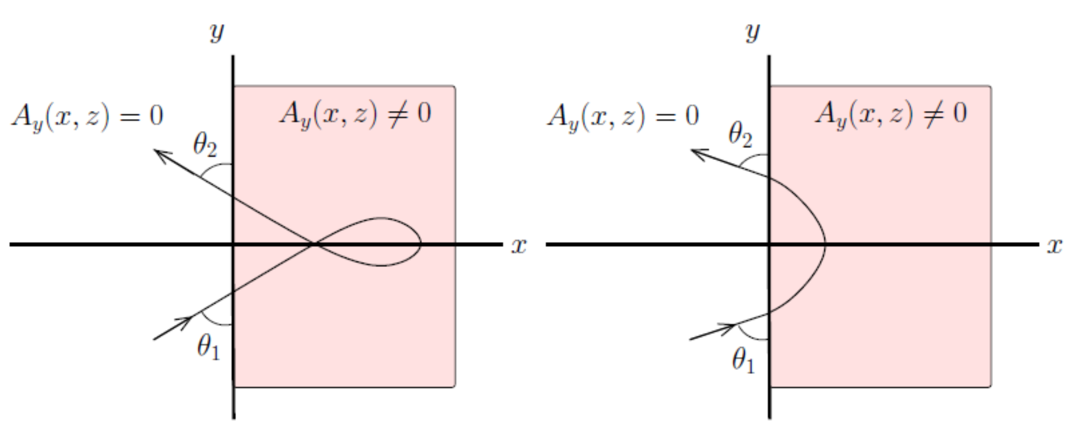
The magnetic field has no y-component. It is due to a vector potential which has only a y-component A_y with (x,z) dependence, i.e.

$$\boldsymbol{A} = \hat{\boldsymbol{y}} A_y(x, z)$$

In addition, at z=0 the magnetic field is perpendicular to the x-y plane.

Note: for this problem, you can assume the electron to be non-relativistic or fully relativistic, just make that clear in your answers.

a) Starting from the Lagrangian of a charged particle in external electromagnetic fields, construct the relativistic Hamiltonian of the system and the canonical momentum of the particle.
b) Show that a trajectory of an electron located at z = 0 with its momentum in the x-y plane will stay in the x-y plane.
c) Obtain two conserved quantities for the problem above and show, assuming that the electron eventually leaves the static magnetic field region, that this system is indeed a mirror for trajectories in the x-y plane, namely an electrons with initial momentum **p** is reflected such that the angles that the incoming and outgoing trajectories make with the y-axis are equal in magnitude and opposite in sign (i.e. $\theta_1 = \theta_2$ in the picture below).
d) Find an equation for the depth the penetration (the furthest the electron reaches into the magnetic field region) and solve the resulting equation for the particular case of field **B** = G (($\hat{\boldsymbol{x}}z - \hat{\boldsymbol{z}}x$). Which sign of G corresponds to the trajectories shown in each figure of the figures below?



a) For an electrons in x-y plane ($z=0$)

$$F_x = q(\vec{v}\times\vec{B})_z = -e(v_x B_y - v_y B_x) = 0$$

if $B_x = B_y = 0$ as specified

b) $$L = -mc^2\frac{1}{\sqrt{1-\frac{\dot{r}^2}{c^2}}} + q\vec{A}\cdot\dot{\vec{r}} = -mc^2\gamma + q\vec{A}\cdot\dot{\vec{r}}$$ with $\vec{A} = A_y(x,z)\hat{y}$

canonical momentum $$\vec{P} = \frac{\partial L}{\partial\dot{\vec{r}}} = \gamma m\dot{\vec{r}} + q\vec{A} = \vec{p} + q\vec{A}$$

$$H = \vec{P}\cdot\dot{\vec{r}} - L = \gamma mc^2 = \sqrt{p^2c^2 + m^2c^4}$$

Two integrals of motion. Lagrangian is Time independent and cyclic in y $\Rightarrow$ H = energy is conserved

and $P_y = p_0\cos\vartheta_1 = p_y + qA_y$

$$(p_x^2 + p_y^2)c^2 + m^2c^4 = p_0^2c^2 + m^2c^4$$

c) for $x<0$ $\quad p_x = \pm\sqrt{p_0^2 - p_y^2}$

and $p_y = p_0\cos\vartheta_1$ $\quad\Rightarrow\quad \vartheta_1 = \vartheta_2$

d) Turning point for $p_y = \pm p_0 = p_0\cos\vartheta_1 - qA_y(x_{min},0)$ $\qquad A_y(x_{min},0) = \frac{p_0(\cos\vartheta_1 \mp}{q}$

$$\vec{A} = -\frac{G}{2}(x^2+z^2)\hat{y}$$

$$\Rightarrow x_{min}^2 = \frac{2p_0}{qG}(\pm 1 - \cos\vartheta_1)$$

$G<0$ $\quad x_{min}^2 = \frac{2p_0}{qG}(1-\cos\vartheta_1)$

$G>0$ $\quad x_{min}^2 = \frac{2p_0}{e|G|}(1+\cos\vartheta_1)$

6. Classical Mechanics

Consider a walled rectangular enclosure of length L that is open at the end $z = L$. The cross section is $a \times b$ running from 0 to a along the x-axis and 0 to b along the y-axis. The walls at $z = 0$, $x = 0$, $y = 0$, $x = a$, and $y = b$ are sealed (so that there is no flow across them). The cavity is filled with a fluid that has a density ρ_0 and speed of sound c. The open end is exposed to the atmosphere with ambient pressure p_0. (Gravity acts to hold the fluid in the cavity but otherwise can be neglected for this problem.) Now an external force causes the pressure at the open end $z = L$ to oscillate (around p_0) by an amount $p' e^{i\omega t}$. Find the most general solution for the pressure in the fluid in the small amplitude approximation. This is the limit in which the physical equations can be linearized, so terms higher than first order in p' can be neglected. Viscosity and damping (such as thermal conduction) are zero and the motion is irrotational. Note that the solution includes a driven response as well as a homogeneous response.

6. Classical Mechanics (solution)

Best to solve in terms of the velocity potential ϕ where $v = \nabla\phi$ and

$p = p_0 - \rho_0[\partial\phi / \partial t]$ and $\partial^2\phi / \partial t^2 - c^2\nabla^2\phi = 0$. For this problem the homogeneous term can be written down by inspection noting that at the solid walls $v_\perp = 0$ and at x=0; $\phi = 0$.

$$\phi_h = \sum A(n_x, n_y, n_z)\cos(n_x\pi x / a)\cos(n_y\pi y / b)\cos([n_z + (1/2)]\pi z / L)$$
$$x\exp[i\omega(n_x, n_y, n_z)t]$$

$$\omega^2(n_x, n_y, n_z) = c^2\pi^2[(n_z + 1/2)^2 / L^2 + n_x^2 / a^2 + n_y^2 / b^2]$$

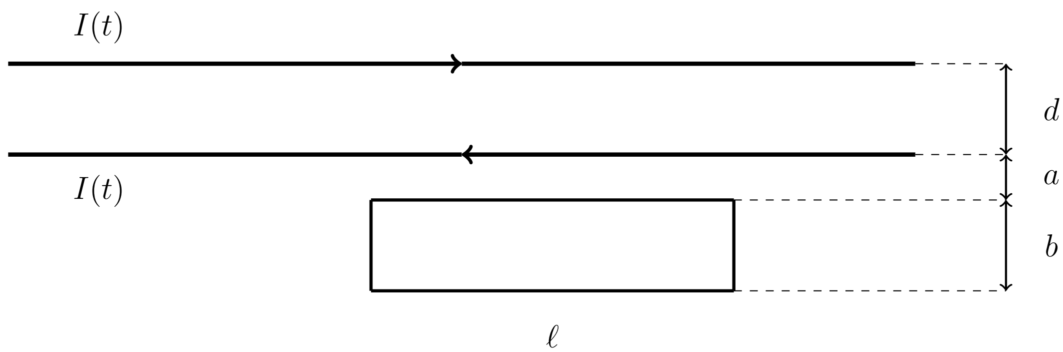
The particular solution depends only on 'z' and $-i\omega\rho_0\phi_p(z = L) = p'$

And $\partial\phi / \partial z]_{z=0} = 0$. Take $\phi_p = B\cos[(\omega / c)z]\exp(i\omega t)$

$B\cos[(\omega / c)L] = ip' / \omega\rho_0$ yields B.

7. Electromagnetism

Electro-magnetic induction makes it possible to carry out a wire-tap on a landline phone without cutting any wires. Here is how it works. The telephone wires are represented by two infinitely long straight parallel wires separated by a distance d, carrying time-dependent currents $\pm I(t)$ in the two wires of equal magnitude but opposite direction, as indicated by the arrows in the figure. Parallel to and in the same plane as the wires, we install a small rectangular closed circuit of length ℓ and width b representing the wire-tapping device, separated from the closest telephone wire by a distance a, as shown in the figure below.



1. Write down the magnetic field produced by a single wire traversed by a current $I(t)$.
2. Compute the magnetic flux $\Phi(t)$ through the rectangular loop as a function of the geometrical data specified in the figure and the current $I(t)$.
3. Compute the electromotive force $\varepsilon(t)$ generated by this flux.
4. Is the current induced in the rectangular loop clockwise or counterclockwise ?
   Justify your answer in terms of the sign of $I(t)$ and its derivative at any given time t.

7. Electromagnetism (solution)

1. The magnetic field produced by an infinitely long straight wire traversed by a current $I(t)$ is given by Ampère's law,

$$B(t) = \frac{\mu_0 I(t)}{2\pi r} \tag{0.1}$$

where r is the distance from the wire, and the field is oriented in the plane perpendicular to the wire and the radial direction away from the wire. If the current moves to the left on the page, then the field points perpendicularly out of the page.

2. The magnetic flux Φ_1 across the loop due to the first wire, in the convention that it points out of the page, is given by the integral of the magnetic field through the loop,

$$\Phi_1(t) = -\int_{d+a}^{d+a+b} dr\, \frac{\mu_0 \ell I(t)}{2\pi r} = -\frac{\mu_0 \ell I(t)}{2\pi} \ln \frac{d+a+b}{d+a} \tag{0.2}$$

while the flux $\Phi_2(t)$ due to the second wire, in the same convention, is given by,

$$\Phi_2(t) = \int_{a}^{a+b} dr\, \frac{\mu_0 \ell I(t)}{2\pi r} = \frac{\mu_0 \ell I(t)}{2\pi} \ln \frac{a+b}{a} \tag{0.3}$$

whose sum gives,

$$\Phi(t) = \Phi_1(t) + \Phi_2(t) = -\frac{\mu_0 \ell I(t)}{2\pi} \ln \frac{(d+a+b)a}{(d+a)(a+b)} \tag{0.4}$$

3. The emf induced in the rectangular loop is then given by,

$$\varepsilon(t) = -\frac{d\Phi(t)}{dt} = \frac{\mu_0 \ell}{2\pi} \frac{dI(t)}{dt} \ln \frac{(d+a+b)a}{(d+a)(a+b)} \tag{0.5}$$

4. The current induced in the rectangular loop must oppose the change in the magnetic flux by Lenz's law, so that the corresponding magnetic field must be into the page and the orientation of the current induced in the rectangular loop is clockwise.

8. Electromagnetism

In this problem we consider charged objects in nonrelativistic circular motion in the plane transverse to a uniform magnetic field $\boldsymbol{B}$.

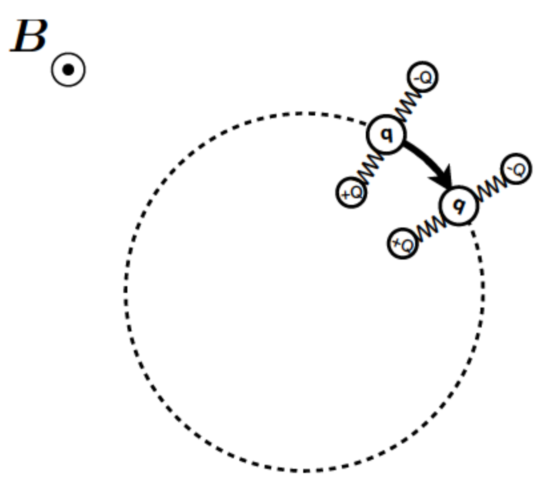
(A) For a point particle with mass m and charge q, what is the angular velocity ω?

We now consider the case that the point particle is replaced by a finite size polarizable molecular ion. As a simple model for this, along with the point charge q we introduce two more point charges $+Q$ and $-Q$, of negligible masses, which are attracted to q by a harmonic potential, $V = \frac{1}{2}k\left[|\vec{x}_Q - \vec{x}_q|^2 + |\vec{x}_{-Q} - \vec{x}_q|^2\right]$. Since this is meant to capture all the interactions between the particles you should not include any additional Coulomb interactions between the particles.

(B) We now look for a solution in which the molecule undergoes circular motion at some angular velocity ω'. Take the orbital radius of charge q to be R, and assume that the spring constant k is large so that the molecular size is small compared to R, i.e. $|\vec{x}_{\pm Q} - \vec{x}_q| \ll R$. Compute the angular velocity ω' to lowest nontrivial order in the small parameter $1/k$. As shown in the figure, you can assume that the distance between the charges is constant and that the axis of the molecule always points toward the center of the orbit. You can also set the masses of charges Q and $-Q$ to zero.

(C) We want to express the result for ω' in terms of the electric polarizability of the molecule, α, which you now compute. Place the molecule in a constant electric field $\boldsymbol{E}$ (assume the charge q is held in place by some other force). Compute the induced electric dipole moment $\boldsymbol{d}$, taking the location of charge q to be the origin of coordinates. Compute the polarizability α via $\boldsymbol{d} = \alpha \boldsymbol{E}$.

(D) Using your results above, express the fractional angular velocity shift, $\Delta\omega/\omega = (\omega' - \omega)/\omega$ in terms of α, working to first nontrivial order in small α. The resulting expression for the angular velocity shift holds for more general molecules, and has been observed experimentally.



8. Electromagnetism (solution)

We work in units where $c = 1$. We take Q, q and $B = |\boldsymbol{B}|$ to all be positive, in which case the molecule orbits clockwise.

(A) Let R be the orbital radius. Force balance gives $m\omega^2 R = q\omega RB$, and thus $\omega = qB/m$.

(B) We write out $\boldsymbol{F} = m\boldsymbol{a}$ for each of the three particles. For charges $\pm Q$ we are instructed to neglect the masses, so we just need to balance the Lorentz force against that of the harmonic potential. Since Q, q, B are all positive we have $R_{-Q} > R > R_Q$. The force balance equation for charge $-Q$ is then $k(R_{-Q} - R) = Q\omega' R_{-Q} B$; the force balance equation for charge $+Q$ is $k(R - R_Q) = Q\omega' R_Q B$. We solve these to get

$$R_{-Q} = \frac{kR}{k - Q\omega' B}, \quad R_Q = \frac{kR}{k + Q\omega' B}.$$

For use in the next step we expand the sum $R_{-Q} + R_Q$ for large k to get

$$R_{-Q} + R_Q \approx 2R + 2\left(\frac{Q\omega' B}{k}\right)^2 R + \dots$$

The force balance equation for charge q is

$$m\omega'^2 R = q\omega' RB + k(R - R_Q) - k(R_{-Q} - R)$$

Using the result above this becomes

$$m\omega'^2 R \approx q\omega' RB - 2\frac{Q^2\omega'^2 B^2 R}{k}$$

We solve this by writing $\omega' = \frac{qB}{m} + \Delta\omega$ and solve to first order in $\Delta\omega$ as

$$\frac{\Delta\omega}{\omega} \approx -\frac{2Q^2 B^2}{km}$$

(C) In an electric field the $\pm Q$ charges are displaced by $\delta x = \pm\frac{EQ}{k}$, giving an electric dipole moment $d = 2\frac{Q^2}{k}E$, and so the polarizability is $\alpha = \frac{2Q^2}{k}$.

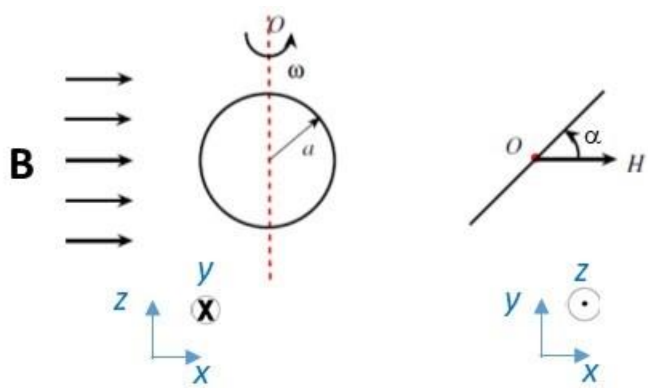
(D) Combining the results above, we have

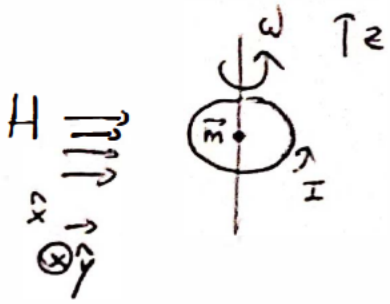
$$\frac{\Delta\omega}{\omega} \approx -\frac{\alpha B^2}{m}$$

9. Electromagnetism

Consider a closed circuit formed into a circular coil of N turns with radius a, resistance R. You can neglect the self-inductance of the circuit. The coil rotates around the **z**-axis in a uniform magnetic field **B** directed along the **x**-axis (see below)

a) Find the current in the coil as a function of α for rotation at a constant angular velocity ω. Here $\alpha(t) = \omega t$ is the angle between the plane of the coil and **B** (the **x**-axis).
b) Find the externally applied torque that is needed to maintain the coil's uniform rotation.
c) Due to the time-dependent currents induced in the coil, electromagnetic waves are radiated. What is the frequency of the radiation?
d) What is the polarization of the radiated waves propagating along the positive z-axis?
e) Compute the total power radiated by the rotating coil of the wire.





$$\mathcal{E} = -\frac{d\phi_B}{dt} = IR$$

$$\phi_B = \pi a^2 N B \sin\omega t$$

a) $\Rightarrow I = -\frac{\pi a^2 N B\omega \cos\omega t}{R}$

b) $\vec{m} = I\vec{A} = -(\pi a^2 N)^2 \frac{B\omega}{R} \cos\omega t \left(\sin\omega t\,\hat{x} - \cos\omega t\,\hat{y}\right)$

$$\vec{\tau} = \vec{m} \times \vec{H} = (\pi a^2 N)^2 \frac{B\omega}{R}\left(\cos^2\omega t\,\hat{z}\right)$$

c) $\vec{m}(t) = \frac{m_0}{2}\left(\sin 2\omega t\,\hat{x} + \cos 2\omega t\,\hat{y}\right)$ + constant

$\rightarrow 2\omega$ radiation

d) rotating magnetic dipole generates circularly polarized radiation along $\hat{z}$ axis

formula for magnetic dipoles

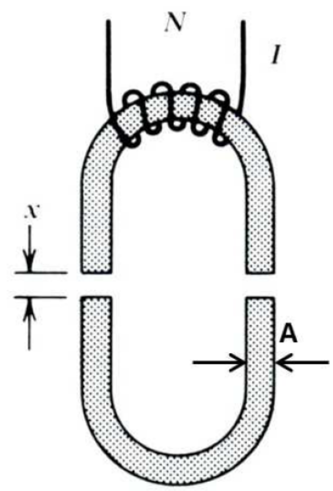
e) $P = \frac{\mu_0 \omega^4}{12\pi c^3} \vec{m} \cdot \vec{m}^*$. In our case $\omega \to 2\omega$ and $\vec{m} \to m_0(-\hat{x} + i\hat{y})$

$$= \frac{2\mu_0 m_0^2 \omega^4}{3\pi c^3}$$

10. Electromagnetism

Consider an electromagnet with an iron core. Each segment of iron has length *L*, constant cross-sectional area *A* and permeability $\mu >> \mu_0$ where μ_0 is the permeability of free space. The two halves of the magnet are separated by a small distance $x << L$. The magnet is powered by a coil of *N* turns carrying a constant current *I*.

a) Determine the magnetic fields in the iron when x=0 (the gap is closed).
   Hint: since $\mu >> \mu_0$, the magnetic field lines follow the shape of the iron (no magnetic flux leakage), and you can assume that the magnetic field strength is constant inside the iron core.
b) Determine the fields H and B in the gap when x is non-zero, but very small, so that you can still assume that the magnetic field vanishes outside of the iron core and the small gap region.
c) Determine the total magnetic field energy as a function of x<<L
d) Calculate the force (magnitude and direction) between the two halves for vanishing small gap x.



a) $\hat{n} \times \vec{H}_1 = \hat{n} \times \vec{H}_2$

$\hat{n} \cdot \vec{B}_1 = \hat{n} \cdot \vec{B}_2$

b) $NI = HL$ neglecting flux leakage

c) $NI = LH_{core} + H_{gap}\, x$

$$= B\left(\frac{L}{\mu} + \frac{x}{\mu_0}\right) \overset{\mu \gg \mu_0}{\approx} \frac{Bx}{\mu_0}$$

$$\Rightarrow B = \frac{\mu_0 NI}{x}$$

d) $$U = \int \frac{1}{2} \mu H^2 dV = \frac{1}{2} \frac{B^2}{\mu} AL + \frac{1}{2} \frac{B^2}{\mu_0} Ax$$

$$\approx \frac{1}{2} \mu_0 \frac{(NI)^2 A}{x}$$

e) $$F_x = + \frac{dU}{dx} = -\frac{1}{2} \mu_0 \frac{(NI)^2}{x^2} A\, \hat{x}$$ attractive

(+: taking into account power supply)

11. Statistical Mechanics

When a certain molecule A absorbs a photon it decays into molecules B and C according to the reaction $\gamma + A \to B + 3C$. The time reversed process is also possible. The masses of the molecules are m_A, m_B and m_C. Let E_b be the binding energy (i.e E_b is the minimum photon energy needed to produce the reaction).

Assume the validity of non-relativistic statistical mechanics and Maxwell-Boltzmann statistics for the molecules. Further, ignore the internal degrees of freedom of the molecules.

a) Now suppose some A, B and C molecules are placed in a box, whose walls can absorb and emit photons, and allowed to come to thermal equilibrium at temperature T. What is the density per unit volume, n_A, of A molecules in terms of the densities $n_{B,C}$ of B and C molecules? Your answer should involve $(n_B, n_C, m_A, m_B, m_C, E_b)$ as well as fundamental constants.

b) Now suppose that the photon γ is replaced by another particle γ^* which has mass m^* (and is relativistic), spin-1 (and so obeys Bose-Einstein statistics), and which is not emitted/absorbed by the the walls of the box. We start with some number of γ^* particles and molecules in the box and let the system come to thermal equilibrium at temperature T. Derive an expression for the density n_{γ^*} in the box in terms of the densities of molecules n_A, n_B, n_C. Your answer can be given in the form of an unevaluated integral.

11. Statistical Mechanics (solution)

a) At equilibrium the Gibbs free energy should be stationary with respect to the reaction. This condition is expressed in terms of the chemical potentials of the objects involved

$$\mu_\gamma + \mu_A = \mu_B + 3\mu_C$$

Since photons are created and absorbed by the box we have $\mu_\gamma = 0$. For the molecules, under the stated assumptions we can use the formula for a particle of mass m,

$$\mu = kT\log(n/n_Q)\ , \quad n_Q = \left(\frac{mkT}{2\pi\hbar}\right)^{3/2}$$

We use this formula for B and C, while for A we take into account the binding energy,

$$\mu_A = kT\log(n_A/n_Q^A) - E_b$$

The equilibrium condition then reads

$$kT\log(n_A/n_Q^A) - E_b = kT\log(n_B/n_Q^B) + 3kT\log(n_C/n_Q^C)$$

Solving for n_A gives

$$n_A = \left(\frac{m_A}{m_B m_C^3}\right)^{3/2}\left(\frac{2\pi\hbar}{kT}\right)^{9/2} e^{E_b/kT} n_B n_C^3$$

b) Under the assumptions, we now assign a chemical potential μ_γ to the "photons" and at equilibrium require

$$\mu_{\gamma^*} + \mu_A = \mu_B + 3\mu_C$$

The density of "photons" is given by the Bose-Einstein distribution (with a factor of three included for polarization states)

$$n_\gamma = 3\int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{(E_k-\mu_\gamma)/kT} - 1}$$

with $E_k = \sqrt{p^2c^2 + m_{\gamma^*}^2 c^4}$, $p^2 = \hbar^2|\vec{k}|^2$. We have $\mu_{\gamma^*} = -\mu_A + \mu_B + 3\mu_C$. The chemical potentials of the molecules are given in terms of their densities by

$$\mu_A = kT\log(n_A/n_Q^A) - E_b \ , \quad \mu_B = kT\log(n_B/n_Q^B) \ , \quad \mu_C = kT\log(n_C/n_Q^C)$$

This gives

$$n_{\gamma^*} = 3\int \frac{d^3k}{(2\pi)^3}\frac{1}{\frac{n_Q^B(n_Q^C)^3 n_A}{n_B(n_C)^3 n_Q^A}e^{(E_k - E_b)/kT} - 1}$$

where the n_Q's were given above.

12. Statistical Mechanics

Consider a volume V containing N electrons of mass m which have no mutual interactions, other than those associated with the Pauli exclusion principle. The spectrum of states available to each electron contains a continuous part with energy $E = \mathbf{p}^2/2m$ for $E > 0$, and a bound state part with energy $E = -\varepsilon$ and $\varepsilon > 0$. The total number of bound states $\mathcal{M}$ available to the N electrons is assumed to be larger than N. The parameters $m, \varepsilon, \mathcal{M}$ are considered fixed throughout.

1. Give the expressions for the number N_b of electrons in the bound states, and the number N_c of electrons in the continuum, as a function of T, V and the chemical potential μ. Justify your answers.

2. Obtain the relation between N and μ for given T, V.

3. At low temperature $k_B T \ll \varepsilon$, most of the electrons occupy the bound states, and therefore few occupy the continuum states. Using the approximation of the classical distribution for the occupation number of states in the continuum spectrum, and in the limit $k_B T \ll \varepsilon$, evaluate N_c and μ as a function of T, V.

4.* The system is placed in a uniform external magnetic field B that splits each energy level into two levels whose energies are shifted by $+\kappa B$ and $-\kappa B$ respectively, where κ is the magnetic moment of the electron. Work with the thermodynamic potential suitable for the independent variables T, V, μ, B. Define the magnetic susceptibility as the induced magnetic dipole moment per unit volume per unit applied magnetic field B. Expressing the magnetic susceptibility at $B = 0$ in terms of the thermodynamic potential in the absence of a magnetic field, obtain the magnetic susceptibility as a function of T, V, N, using the results obtained in part 3.

$\star$ Note that part 4 is more difficult than the first three parts of this problem.

1. The numbers N_b and N_c of electrons is given by (we use the notation $k_B T = 1/\beta$),

$$\begin{aligned} N_b &= \frac{\mathcal{M}}{e^{\beta(-\varepsilon-\mu)}+1} \\ N_c &= 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(p^2/2m-\mu)}+1} \end{aligned}$$

where we have used the Fermi-Dirac occupation numbers at each energy level and included a factor of 2 for the continuum part to account for the spin 1/2 degeneracy of the electron.
2. The total number of electrons is $N = N_b + N_c$, so that,

$$N = \frac{\mathcal{M}}{e^{-\beta(\varepsilon+\mu)}+1} + 2V \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(p^2/2m-\mu)}+1}$$

3. In the approximation $\beta\varepsilon \gg 1$ we have $N_c \ll N$, since we have $N < \mathcal{M}$ and all electrons have the options of filling all bound states. Thus, we may use Boltzmann statistics for the contribution from the continuum part of the spectrum,

$$N_c \approx 2V \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\beta(p^2/2m-\mu)} = N_0\, e^{\beta\mu}$$

where N_0 is given by the Gaussian integral,

$$N_0 = 2V \int \frac{d^3p}{(2\pi\hbar)^3} e^{-\beta p^2/2m} = 2V \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{3}{2}}$$

The exact relation between N and μ now simplifies as follows,

$$N = \frac{\mathcal{M}}{e^{-\beta(\varepsilon+\mu)}+1} + N_0\, e^{\beta\mu} \tag{0.1}$$

In the approximation where $\beta\varepsilon \gg 1$ and $N \gg N_c = N_0 e^{\beta\mu}$, the above equation may be solved approximately by dropping its last term on the right side, and we find,

$$N_c \approx \frac{NN_0}{\mathcal{M}-N} e^{-\beta\varepsilon} \qquad\qquad \mu \approx -\varepsilon - k_BT \ln\frac{\mathcal{M}-N}{N} \tag{0.2}$$

4. We work with the grand canonical partition function $Z_G(T,V,\mu\,\beta) = \sum e^{-\beta(E+\mu N)}$, and the associated potential is $\Omega = -T\ln Z_G$. Ω is sometimes called the "grand potential".

Denoting the grand potential of the electrons in the absence of the magnetic field by $2\Omega_0(T,V,\mu)$, the grand potential $\Omega(T,V,\mu,B)$ for $B \neq 0$ is obtained as follows,

$$\Omega(T,V,\mu,B) = \Omega_0(T,V,\mu+\kappa B) + \Omega_0(T,V,\mu-\kappa B)$$

where,

$$\Omega_0(T,V,\mu) = -\frac{\mathcal{M}}{2\beta}\ln(1+e^{-\beta(-\varepsilon-\mu)}) - \frac{V}{\beta}\int\frac{d^3p}{(2\pi\hbar)^3}\ln(1+e^{-\beta(p^2/2m-\mu)})$$

Within the approximations used, the second term becomes,

$$-\frac{V}{\beta}\int\frac{d^3p}{(2\pi\hbar)^3}\ln(1+e^{-\beta(p^2/2m-\mu)}) \approx -\frac{V}{\beta}\int\frac{d^3p}{(2\pi\hbar)^3}e^{-\beta(p^2/2m-\mu)} = -\frac{N_0}{2\beta}e^{\beta\mu}$$

so that we have,

$$\Omega_0(T,V,\mu) \approx -\frac{\mathcal{M}}{2\beta}\ln(1+e^{-\beta(-\varepsilon-\mu)}) - \frac{N_0}{2\beta}e^{\beta\mu}$$

Combining these results, the exact formula for the magnetic susceptibility χ_{magn} at $B=0$ is given by,

$$\chi_{\text{magn}} = -\frac{1}{V}\frac{\partial^2\Omega}{\partial B^2}\bigg|_{B=0} = -2\frac{\kappa^2}{V}\frac{\partial^2\Omega_0}{\partial\mu^2}\bigg|_{T,V} = \frac{2\kappa^2}{V}\frac{\partial N}{\partial\mu}\bigg|_{T,V}$$

where we have used the standard thermodynamic relation $N = -\partial\Omega_0/\partial\mu$ at fixed T,V to obtain the last formula. The formula (0.1) gives the number of electrons N in terms of μ so we use this formula to compute the derivative,

$$\frac{\partial N}{\partial\mu}\bigg|_{T,V} = \frac{\beta\mathcal{M}\,e^{-\beta(\varepsilon+\mu)}}{(e^{-\beta(\varepsilon+\mu)}+1)^2} + \beta N_0\,e^{\beta\mu}$$

Eliminating μ using (0.2) we find after some simplifications,

$$\chi_{\text{magn}} = \frac{2\kappa^2 N}{Vk_BT}\left[\frac{\mathcal{M}-N}{\mathcal{M}} + \frac{N_0}{\mathcal{M}-N}\,e^{-\beta\varepsilon}\right]$$

In view of our approximations, the second term inside the brackets is negligible compared to the first term.