WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Thursday, September 19, and Friday, 20, 2002

PART I – THURSDAY, September 19,2002

Important – please read carefully.

The exam (8 hours) is in two parts:

PART 1	Quantum Mechanics, Thermodynamics, Statistical Mechanics
Sept 19	7 Problems – **DO ALL PROBLEMS.**
9:00–1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.
PART 2	Electromagnetic Theory,Thermodynamics,Statistical Mechanics
Sept 20	7 Problems – **DO ALL PROBLEMS.**
9:00–1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.

Instructions

1) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. **Use one side only.**

3) Print **your name and problem number** on EACH AND EVERY page. (Note: Pages without names may not be counted.)

4) Return the problem page as the first page of your answers.

5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

1. *Quantum Mechanics.*

A Stern-Gerlach apparatus is adjusted so that the z-component of the spin of an electron (spin-1/2) transmitted through it is $-\hbar/2$. A uniform magnetic field in the x-direction is then switched on at time $t = 0$.

(a) What are the probabilities associated with finding the different allowed values of the z-component of the spin after time T?

(b) What are the probabilities associated with finding the different allowed values of the x-component of the spin after time T?

2. *Quantum Mechanics.*

The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2,$$

where the magnetic field $\mathbf{B}$ is related to the vector potential $\mathbf{A}$ by $\mathbf{B} = \nabla \times \mathbf{A}$. Here, $e$is the charge of the particle, m the mass, c the velocity of light and $\mathbf{p} = (p_x, p_y, p_z)$ is the momentum of the particle. Let $\mathbf{A} = -B_0 y\ \hat{x}$, corresponding to the magnetic field $\mathbf{B} = B_0\ \hat{z}$.

(a) Find the energy levels of the particle.

(b) Would the energy levels change if we chose $\mathbf{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$? Give reasons for your answer.

3. *Quantum Mechanics.*

A charged particle of charge, q, and mass, m, is bound in a one-dimensional harmonic oscillator potential $V = \frac{1}{2}m\omega^2x^2$, where ω is the frequency of the oscillator. The system is then placed in an electric field E that is constant in space and time.

(a) Calculate the shift of the ground state energy to order E^2.

(b) What are the third and the higher order (in E) shifts in the ground state energy? Give reasons for your answer.

Hint: If n labels the eigenstates of the unperturbed harmonic oscillator, then $\langle n'|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n'}\delta_{n,n'-1} + \sqrt{n'+1}\delta_{n,n'+1}\right]$.

4. *Quantum Mechanics.*

Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 - \frac{Ze^2}{r} \tag{1}$$

(a) What is the ground state energy of this Hamiltonian?

(b) What is the expectation value of the potential energy $\left\langle -\frac{Ze^2}{r} \right\rangle$ in the ground state?

(c) What is the expectation value of the kinetic energy $\left\langle -\frac{\hbar^2}{2m}\vec{\nabla}^2 \right\rangle$ in the ground state?

5. *Quantum Mechanics.*

In the Born-Oppenheimer approximation, the electrons are treated quantum mechanically, while the atomic nuclei are treated classically. The electronic energy is calculated as a function of the spacing between the nuclei. The sum of the electronic energy and the potential energy due to nuclei-nuclei interactions is minimized. The nuclear kinetic energy is neglected. As a toy model, we will consider the formation of a diatomic molecule in one dimension. Let us suppose that the electron is at x and the nuclei are at X_1, X_2. We assume that the interaction between an electron and a nucleus is $V(x - X_i) = -V_0\delta(x - X_i)$ for $i = 1, 2$, $V_0 > 0$. The interaction between nuclei is $U(X_1 - X_2) = \frac{Z^2e^2}{|X_1 - X_2|}$.

(a) Suppose that the nuclei are a distance a apart. What is the ground state energy of an electron to order a if $\frac{mV_0a}{\hbar^2} \ll 1$ (m is the electron mass).

(b) Consider a diatomic molecule composed of an electron and two nuclei. Using the Born-Oppenheimer approximation, find the separation between the two nuclei if $V_0 \gg Z^2e^2$. You need only compute the separation to lowest order in $Ze/V_0^{1/2}$.

6. *Statistical Mechanics and Thermodynamics*

A gas of N highly relativistic, and non-interacting, spin 1/2 Fermions occupies a volume V at a temperature that is effectively equal to zero.

(a) Find the pressure on this gas.

(b) Based on the calculation you have just done, show what (extreme) inequality must be satisfied in order that the assumption of a temperature that is "effectively equal to zero" is justified.

(c) Suppose that energy of the system due to gravitational self-attraction goes as $-\mathcal{A}N^2V^{-1/3}$, where $\mathcal{A}$ is a constant. What does this and your result for the pressure imply about the stability of this system, assuming that gravitational attraction is what keeps it together?

A chain consists of N links that can freely rotate in two dimensions. The links are joined end-to end, as shown below.

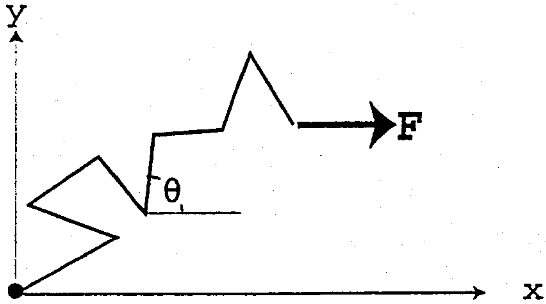


Figure 1: The freely-jointed chain. The angle, θ, between one link and the x-axis is shown.

The chain is subjected to a tension, F, in the x-direction, as indicated. The tension is applied at the end of the chain, so that the total energy of the chain is given by

$$E = -Fl \sum_{i=1}^{N} \cos\theta_i$$

where θ_i is the angle that the i^{th} link makes with the x-axis, and l is the length of each link in the chain.

(a) Calculate the partition function of this chain.

(b) From the partition function, find the relationship between the extension of the chain in the x-direction and the tension, F, assuming that the temperature is T.

(c) When the tension, F, is small, the extension-versus-tension expression implies a spring constant for the freely jointed chain. What is this effective spring constant?

If integrals do not evaluate to elementary functions in parts a and b, it is not necessary to attempt to reduce them. Leave them as integrals. However, in part c, it is necessary to come up with something explicit.

WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Thursday, September 19, and Friday, 20, 2002

PART II – FRIDAY, September 20, 2002

Important – please read carefully.

The exam (8 hours) is in two parts:

PART 1	Quantum Mechanics, Thermodynamics, Statistical Mechanics
Sept 19	7 Problems – DO ALL PROBLEMS.
9:00–1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.
PART 2	Electromagnetic Theory, Thermodynamics, Statistical Mechanics
Sept 20	7 Problems – **DO ALL PROBLEMS.**
9:00–1:00	This part will be collected at the end of four hours. Each problem counts for 20 points; the total is 140.

Instructions

2) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. **Use one side only.**

6) Print your **name and problem number** on **EACH AND EVERY** page. (Note: Pages without names may not be counted.)

7) Return the problem page as the first page of your answers.

8) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

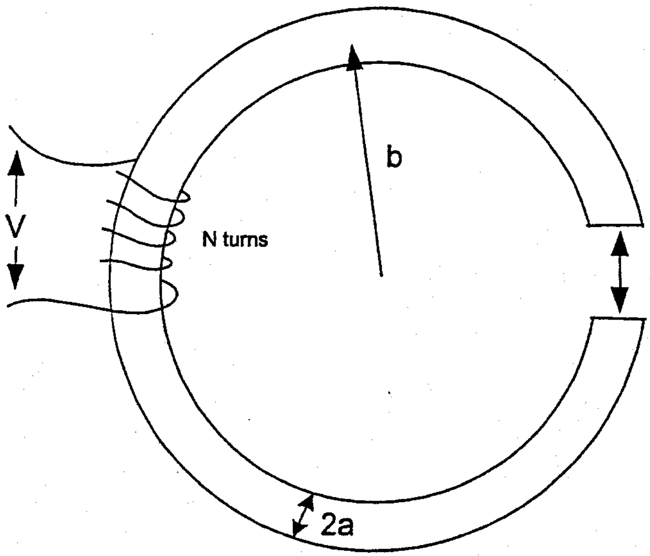
Radiating charges

(a) A point charge q under acceleration $\mathbf{a}(t)$ emits electromagnetic radiation. Giveq ualitative physical arguments why the radiated power, P, should beof the form $P = Bq^2a^2$, where B is a proportionality constant. Determine by dimensional analysis the dependence of B on fundamental physical constants. Explain how and why the exact expression for B differs from this estimate.

(b) A point charge $q$has mass m and is attached to a spring (of spring constant κ) hanging from a fixed support above an infinite horizontal **conducting** plane. The charge is set in motion with amplitude $A < h$, the equilibrium height of the charge abovethe conducting plane. Calculate its instantaneous radiating power.

9. *Electricity and Magnetism*

A D.C. electromagnet is constructed from a cylindrical soft-iron bar with radius a. The relative magnetic permeability of the iron is μ. The bar is bent into a C-shape as shown below with radius b. The width of the small gap is w. The magnet is energized by winding a coil of copper wire N turns tightly around the bar and connecting the coil to a D.C. power supply with voltage V. The copper wire has a resistivity ρ, and radius r_{wire}. Assume $r_{\text{wire}} \ll a \ll b$ and ignore fringe-field effects.



(a) What is the steady-state value of the magnetic field B in the gap?

(b) What is the time constant governing the response of the current in the coil when the voltage is turned on? (Assume μ is constant.)

10. *Electricity and Magnetism*

A point charge q is inside a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

(a) the potential inside the sphere;

(b) the induced surface-charge density at the point on the sphere nearest to q;

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surface?

11. *Electricity and Magnetism*

Describe how you would *measure* the following physical quantities:

(a) An electrostatic field $\mathbf{E}$.

(b) A vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}$ in the gauge $\nabla \cdot \mathbf{A} = 0$.

(c) The charge of an electron assuming its mass is known.

(d) The speed of light or electromagnetic waves.

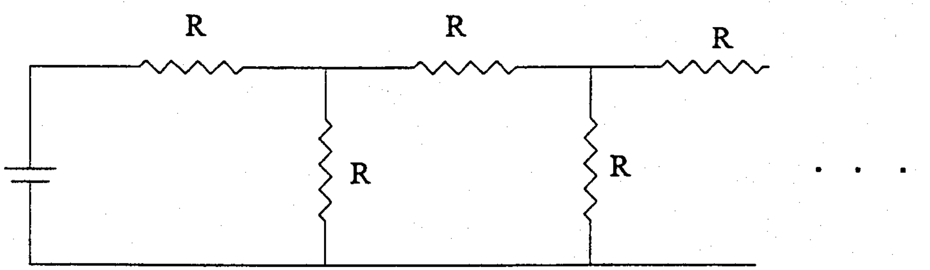
(e) The electrical conductivity of a flame

(f) The direction of wave propagation of a plane electromagnetic wave.

Please describe the approach and method as realistically as possible.

12. *Electricity and Magnetism*

A voltage is applied to the infinitely long resistor network shown below. Each resistor has the same resistance R. Calculate the power dissipated in each resistor.



13. *Statistical Mechanics and Thermodynamics*

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperature T.

Find the a) chemical potential, b) free energy, c) entropy, d) pressure and e) heat capacity at constant pressure.

14. *Statistical Mechanics and Thermodynamics*

In this problem, you will study the q-state Potts model using mean-field theory. The Hamiltonian is

$$H_{\text{Potts}} = -J \sum_{<i,j>} \delta_{\sigma_i \sigma_j} \tag{2}$$

where the 'spins' σ take values $\sigma_i = 0, 1, 2, \ldots, q-1$. For $q = 2$, this is the Ising model.

(a) Show that H_{Potts} can be rewritten in the form

$$H_{\text{Potts}} = -\frac{J}{q} \sum_{<i,j>} \left[(q-1)\mathbf{s}_i \cdot \mathbf{s}_j + 1\right] \tag{3}$$

where the vectors $\mathbf{s}_i$ are constrained to take values in a set of q vectors in $(q-1)$-dimensional space, $\{\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_q\}$ satisfying $\mathbf{S}_a \cdot \mathbf{S}_b = 1$ if $a = b$ and $\mathbf{S}_a \cdot \mathbf{S}_b = -\frac{1}{q-1}$ if $a \neq b$.

(b) In mean-field theory, we approximate the Hamiltonian H_{Potts} by a mean-field Hamiltonian H_{MF},

$$H_{\text{MF}} = \sum_i \left[\mathbf{h} \cdot \mathbf{s}_i - \frac{J}{q}\right] \tag{4}$$

in which there is no interaction between the different spins, but each spin is coupled to an effective magnetic field, $\mathbf{h}$. Calculate the partition function of H_{MF}.

(c) Using H_{MF}, compute $\langle \mathbf{s} \rangle$. Impose the self-consistency condition that the effective magnetic field is generated by the average spin $\langle \mathbf{s} \rangle$ so that H_{MF} approximates H_{Potts}. Requiring self-consistency, derive(bu t do not solve) the mean-field equation for $\langle \mathbf{s} \rangle$. (For simplicity, you may assume that the tetrahedron is oriented so that one of the allowed values of $\mathbf{s}_i$ is $(0, 0, \ldots, 0, 1)$. Assume that $\langle \mathbf{s} \rangle = s\,(0, 0, \ldots, 0, 1)$ and find s.)

Selected Answers

Fall 2002

1) (a) $P_{z\uparrow} = \sin^2 \omega t$ $P_{z\downarrow} = \cos^2 \omega t$ (b) $P_{x\uparrow} = \frac{1}{2} = P_{x\downarrow}$

2) (a) $E = \dfrac{\hbar^2 k^2}{2m} + (n+\frac{1}{2})\hbar \dfrac{e|B|}{mc}$, $n = 0,1,2,\ldots$ (b) no, b/c gauge invariance

3) (a) $\Delta E_0^{(2)} = \dfrac{-q^2 E^2}{2m\omega^2}$ (b) all higher order terms are zero

4) (a) $E = \dfrac{-Z^2 e^2}{a_0}$ (b) $\dfrac{-Z^2 e^2}{a_0}$ (c) $\dfrac{-Z^2 e^2}{2a_0}$

5) (a) $E_0 = \dfrac{-2m}{\hbar^2} V_0^2 + \left(\dfrac{2mV_0}{\hbar^2}\right)^2 V_0 a + \mathcal{O}(a^2)$ (b) $a = \dfrac{\hbar^2}{2mV_0}\left(\dfrac{Ze}{V_0^{1/2}}\right)$

6) $P = \dfrac{1}{3}\dfrac{\varepsilon}{V} = \dfrac{1}{4}(3\pi^2)^{1/3}\hbar c\left(\dfrac{N}{V}\right)^{4/3}$

7) (a) $Z = \left[\int_0^{2\pi} e^{\beta F \ell \cos\theta} d\theta\right]^N$ (b) $L = N\ell \dfrac{\int_0^{2\pi} e^{\beta F\ell\cos\theta}\cos\theta\, d\theta}{\int_0^{2\pi} e^{\beta F \ell \cos\theta}} =$

(c) $K = \dfrac{2k_B T}{N\ell^2}$

8) (a) (b) $P = 2B(qa)^2 = 2Bq^2\left[A\omega^2 \cos(\omega t)\right]^2$

9) (a) $B = \dfrac{\mu_0 \mu V r^2}{2A\rho(2\pi b + w\mu)}$ (from $\oint H \cdot d\vec{\ell} = NI$) (b) $\tau = \dfrac{L}{R} = \dfrac{N\mu_0 \mu a \pi r^2}{2\rho(\mu_0 2\pi b + \mu w)}$

12) $P_i = \dfrac{V^2}{R}\dfrac{4(1+\sqrt{5})^{i-3}}{(3+\sqrt{5})^{i-1}}$ series ; $P_i = \dfrac{V^2}{R}\left(\dfrac{1+\sqrt{5}}{3+\sqrt{5}}\right)^i$ parallel

13) (a) $\mu = \tau\left(\ln \dfrac{n}{n_q} - \ln\left[1+e^{-\Delta/\tau}\right]\right)$ (d) $p_{int} = -\left(\dfrac{\partial F_{int}}{\partial V}\right)_{\tau, V} = 0$

$\tau = k_B T$ (b) $F = N\tau\left[\ln\dfrac{n}{n_q} - 1 - \ln\left[1+e^{-\Delta/\tau}\right]\right]$ (e) $U = \frac{3}{2}N\tau + U_{int}$

(c) $\sigma = \sigma_{Jackson} - \sigma_{electrode} + \sigma_{int}$ $\dfrac{U_{int}}{N} = \dfrac{-\partial}{\partial\beta}\ln Z_{int}$

Fall 2002 #1.

$T=0 \quad \binom{0}{1} \qquad \hbar=1 \qquad \vec{B}=B_0\hat{x}$

$$H = \frac{g\,\mu_0\,\vec{\sigma}\cdot\vec{B}}{2} = \mu_0 B_0\,\sigma_x \qquad H=\mu_0 B_0\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$\lambda = \pm\,\mu_0 B_0 \quad \frac{1}{\sqrt{2}}\binom{1}{-1},\ \frac{1}{\sqrt{2}}\binom{1}{1}$

$\mu_0 B_0 = \alpha \qquad |\chi_-\rangle \quad |\chi_+\rangle$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$
$$= C_1 e^{-iHt}|\chi_+\rangle + C_2 e^{-iHt}|\chi_-\rangle = C_1 e^{-i\alpha t}|\chi_+\rangle + C_2 e^{i\alpha t}|\chi_-\rangle$$

$$|\psi(0)\rangle = \binom{0}{1} = C_1\frac{1}{\sqrt{2}}\binom{1}{1} + \frac{C_2}{\sqrt{2}}\binom{1}{-1}$$

$$0 = \frac{C_1}{\sqrt{2}} + \frac{C_2}{\sqrt{2}} \qquad 1 = \frac{C_1}{\sqrt{2}} - \frac{C_2}{\sqrt{2}}$$

$$C_1 = -C_2$$

$$1 = \frac{C_1}{\sqrt{2}} + \frac{C_1}{\sqrt{2}} = 1 = \frac{2C_1}{\sqrt{2}} \qquad C_1 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$C_2 = -\frac{1}{\sqrt{2}}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\alpha t}|\chi_+\rangle - \frac{1}{\sqrt{2}}e^{i\alpha t}|\chi_-\rangle$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\alpha t} - e^{i\alpha t} \\ e^{-i\alpha t} + e^{i\alpha t}\end{pmatrix} = \begin{pmatrix} -i\sin(\omega t) \\ \cos(\omega t)\end{pmatrix} \qquad \omega=\alpha$$

a) $P_{z\uparrow} = |\langle + | \psi(t)\rangle|^2 = \left|(1\ \ 0)\begin{pmatrix}-i\sin(\omega t)\\ \cos(\omega t)\end{pmatrix}\right|^2 = \sin^2(\omega t)$

$P_{z\downarrow} = \cos^2(\omega t)$

b) $P_{x\uparrow} = |\langle \chi_+ | \psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1\ \ 1)\begin{pmatrix}-i\sin(\omega t)\\ \cos(\omega t)\end{pmatrix}\right|^2 = \frac{1}{2}$

$P_{x\downarrow} = \frac{1}{2}\,|-i\sin(\omega t) - \cos(\omega t)|^2 = \frac{1}{2}$

A Stern-Gerlach apparatus is adjusted so that the z-component of the spin of an electron (spin 1/2) transmitted through it is $-\hbar/2$. A uniform magnetic field in the x-direction is then switched on at time $t=0$.

(a) what are the probabilities associated with finding the different allowed values of the z-component of the spin after time T?

an electron in an external magnetic field has the Hamiltonian (due to the B-field) of

$$H = -\vec{\mu}\cdot\vec{B} = \frac{g\mu_0}{2}\vec{B}\cdot\vec{S}$$

for our case $\vec{B} = B_0\hat{x}$, so, we have

$$H = \frac{g\mu_0 B_0}{2}\sigma_x = \frac{g\mu_0 B_0}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

this is the Hamiltonian at $t \geq 0$. Before that, we are told the eigenvalue of the electron is $-\hbar/2$ (for the z-component). This eigenvalue corresponds to the eigenstate

$$|\psi(t=0)\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we want to write the $|\psi(t=0)\rangle$ in terms of the eigenstates of H. That is, first let's find the eigenstates of H. (set $\hbar = 1$)

eigenvalues: $\det\begin{pmatrix} -\lambda & \frac{g\mu_0 B_0}{2} \\ \frac{g\mu_0 B_0}{2} & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - \mu_0^2 B_0^2 = 0 \therefore \lambda = \pm\mu_0 B_0$

where $g \equiv 2$ (since electron)

eigenstates: • $|\lambda = +\mu_0 B_0\rangle$: $\begin{pmatrix} -\mu_0 B_0 & \mu_0 B_0 \\ \mu_0 B_0 & -\mu_0 B_0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow -\mu_0 B_0\phi_1 + \mu_0 B_0 \phi_2 = 0$
$\Rightarrow \phi_1 = \phi_2$

$$\Rightarrow |\lambda = +\mu_0 B_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• $|\lambda = -\mu_0 B_0\rangle$; $\begin{pmatrix} \mu_0 B_0 & \mu_0 B_0 \\ \mu_0 B_0 & \mu_0 B_0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \Rightarrow -\phi_1 = \phi_2$

$$\Rightarrow |\lambda = -\mu_0 B_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So,

$$|\psi(t=0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|\lambda = \mu_0 B_0\rangle - \frac{1}{\sqrt{2}}|\lambda = -\mu_0 B_0\rangle$$

$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +1 \end{pmatrix}\right] - \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now, apply the time evolution operator.

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \frac{1}{\sqrt{2}}e^{-i\mu_0 B_0 t}|\lambda = \mu_0 B_0\rangle - \frac{1}{\sqrt{2}}e^{i\mu_0 B_0 t}|\lambda = -\mu_0 B_0\rangle$$

Now, we want to write $|\psi(t)\rangle$ in terms of the eigenstates of S_z. So, we have

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\mu_0 B_0 t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}e^{i\mu_0 B_0 t}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{-1}{2}\begin{pmatrix} -e^{-i\mu_0 B_0 t} + e^{i\mu_0 B_0 t} \\ -e^{-i\mu_0 B_0 t} - e^{i\mu_0 B_0 t} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2i\sin(\mu_0 B_0 t) \\ -2\cos(\mu_0 B_0 t) \end{pmatrix}$$

$$= -\frac{1}{2}2i\sin(\mu_0 B_0 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2}{2}\cos(\mu_0 B_0 t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore \boxed{|\psi(t)\rangle = -i\sin(\mu_0 B_0 t)|\uparrow\rangle + \cos(\mu_0 B_0 t)|\downarrow\rangle}$$

Thus,

$$\boxed{\begin{aligned} P_{z\uparrow} &= |\langle \uparrow | \psi(t) \rangle|^2 = \sin^2(\mu_0 B_0 t) \\ P_{z\downarrow} &= |\langle \downarrow | \psi(t) \rangle|^2 = \cos^2(\mu_0 B_0 t) \end{aligned}}$$

$$P_z = \sin^2(\ ) + \cos^2(\ ) = 1 \quad \checkmark$$

(b) what are the probabilities associated with finding the different allowed values of the x-component of the spin after time T?

From eq (1), we have

$$\boxed{\begin{aligned} P_{x\uparrow} &= |\langle \uparrow | \psi(t) \rangle|^2 = \frac{1}{2} \\ P_{x\downarrow} &= |\langle \downarrow | \psi(t) \rangle|^2 = \frac{1}{2} \end{aligned}}$$

$$P_x = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

The Hamiltonian for a spinless charged particle in a magnetic field is:

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2$$

where the magnetic field B is related to the vector potential $\vec{A}$ by $\vec{B} = \vec{\nabla}\times\vec{A}$. Here, e is the charge of the particle, m the mass, c the velocity of light and, $\vec{p} = (p_x, p_y, p_z)$ is the momentum of the particle. Let $\vec{A} = -B_0 y\hat{x}$, corresponding to the magnetic field $\vec{B} = B_0\hat{z}$.

(a) Find the energy levels of the particle.

(b) Would the energy levels change if we chose $\vec{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$? Give reasons for your answer.

(a) $H\psi = \frac{1}{2m}\left[\vec{p}^2\psi - \frac{e}{c}\vec{p}\cdot(\vec{A}\psi) - \frac{e}{c}\vec{A}\cdot(\vec{p}\psi) + \frac{e^2}{c^2}\vec{A}^2\psi\right]$

$\vec{p} = -i\hbar\vec{\nabla}$; $\vec{A} = -B_0 y\hat{x} \Rightarrow \vec{A}^2 = B_0^2 y^2$

also $\vec{p}\cdot(\vec{A}\psi) = -i\hbar\vec{\nabla}\cdot(\vec{A}\psi) = -i\hbar\left[\psi\overset{\nearrow 0}{(\vec{\nabla}\cdot\vec{A})} + \vec{A}\cdot(\vec{\nabla}\psi)\right] = \vec{A}\cdot(\vec{p}\psi)$

and using the dot product: $\vec{A}\cdot(\vec{p}\psi) = -B_0 y p_x\psi$

so

$$H\psi = \frac{1}{2m}\left[\vec{p}^2\psi + \frac{2eB}{c}y p_x\psi + \frac{e^2}{c^2}\vec{A}^2\psi\right]$$

$$= \frac{p^2}{2m}\psi + \frac{eB}{mc}y p_x\psi + \frac{1}{2}\frac{e^2B_0^2}{mc^2}y^2\psi$$

define $\omega = \frac{eB}{mc}$

$$= \frac{p^2}{2m}\psi + \frac{1}{2}m\omega^2 y^2\psi + \omega y p_x\psi$$

as $[p_x, H] = 0 = [p_z, H]$ one possible form of $\psi = e^{ik_x x}Y(y)e^{ik_z z}$

as x: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E_x\psi \Rightarrow E_x = \frac{\hbar^2}{2m}k_x^2$

z: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2}{2m}k_z^2$

y: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m\omega^2 y^2\psi + \underbrace{\omega y p_x\psi}_{\omega y(-i\hbar)(ik_x)\psi = \omega\hbar k_x y} = E_y\psi$

completing the square:

$$\frac{1}{2}m\omega^2\Big[\underbrace{y + \frac{\hbar k_x}{m\omega}}_{=y'}\Big]^2 = \frac{1}{2}m\omega^2 y^2 + \omega\hbar k_x y + \frac{\hbar^2 k_x^2}{2m}$$

so $\frac{1}{2}m\omega^2 y^2 + \omega\hbar k_x y = \frac{1}{2}m\omega^2(y')^2 - \frac{\hbar^2 k_x^2}{2m}$

$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y'^2}\psi + \frac{1}{2}m\omega^2(y')^2\psi = E'_y\psi = \Big[E_y + \frac{\hbar^2 k_x^2}{2m}\Big]\psi$

$E'_y = (n_y + 1/2)\hbar\omega \Rightarrow E_y = (n_y+1/2)\hbar\omega - \frac{\hbar^2 k_x^2}{2m}$

so $E = E_x + E_y + E_z = \cancel{\frac{\hbar^2}{2m}k_x^2} + (n_y+1/2)\hbar\omega \cancel{- \frac{\hbar^2}{2m}k_x^2} + \frac{\hbar^2}{2m}k_z^2$

$$= (n_y + 1/2)\hbar\omega + \frac{\hbar^2}{2m}k_z^2$$

(b) As energy is a measurable observable it can't depend upon the gauge you choose - hence the energy levels will not change.

A more detailed look:

$\vec{A} = -\frac{B_0}{2}y\hat{x} + \frac{B_0}{2}x\hat{y} \Rightarrow A^2 = \frac{B_0^2}{4}(x^2+y^2)$

$H\psi = \frac{1}{2m}\Big[\vec{p}^2\psi + \underbrace{\frac{2e}{c}\vec{A}\cdot(\vec{p}\psi)}_{\frac{2e}{c}\left[-\frac{B_0}{2}y p_x\psi + \frac{B_0}{2}x p_y\psi\right]} + \frac{e^2 B_0^2}{c^2 4}(x^2+y^2)\psi\Big]$

$$H\psi = \frac{p^2}{2m}\psi + \frac{eB_0}{2mc}\underbrace{(xp_y - yp_x)}_{L_z}\psi + \frac{e^2B_0^2}{2mc^2 4}(x^2+y^2)\psi$$

$$\omega = \frac{eB_0}{mc} \Rightarrow \omega' = \frac{\omega}{2}$$

$$H\psi = \frac{p^2}{2m}\psi + \omega' L_z\psi + \frac{1}{2}m(\omega')^2 x^2\psi + \frac{1}{2}m(\omega')^2 y^2\psi$$

again $\psi = X(x)\,Y(y)\,e^{ik_z z}$

z: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi = E_z\psi \Rightarrow E_z = \frac{\hbar^2}{2m}k_z^2$ and $\omega' L_z\psi = \omega'\hbar m_\ell = \frac{\omega\hbar}{2}m_\ell$

x: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \frac{1}{2}m(\omega')^2x^2\psi = E_x\psi \Rightarrow E_x = (n_x+\tfrac{1}{2})\hbar\omega' = (n_x+\tfrac{1}{2})\frac{\hbar\omega}{2}$

y: $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\psi + \frac{1}{2}m(\omega')^2y^2\psi = E_y\psi \Rightarrow E_y = (n_y+\tfrac{1}{2})\frac{\hbar\omega}{2}$

So

$$E = E_x + E_y + E_z = (n_x+\tfrac{1}{2})\frac{\hbar\omega}{2} + (n_y+\tfrac{1}{2})\frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 + \frac{\omega\hbar}{2}m_\ell$$

from part (a) $E = (n_y+\tfrac{1}{2})\hbar\omega + \frac{\hbar^2}{2m}k_z^2$

if $n_y = 0$ $\quad E = \frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2$

in our case if $n_x = n_y = m_\ell = 0$

$$E = \frac{1}{2}\frac{\hbar\omega}{2} + \frac{1}{2}\frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 = \frac{\hbar\omega}{2} + \frac{\hbar^2}{2m}k_z^2 \text{ (as above)}$$

so indeed the energy doesn't change (the degeneracy does though)

Fall 2002 #2 (p 1 of 1)

The Hamiltonian for a spinless charged particle in a magnetic field is

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2$$

where the magnetic field $\vec{B}$ is related to $\vec{A}$ via $\vec{B} = \nabla \times \vec{A}$. let $\vec{A} = -B_0 y\,\hat{x}$ corresponding to $\vec{B} = B_0\hat{z}$

(a) Find the energy levels of the particle.

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e}{c}\vec{A}(\vec{r})\right]^2 = \frac{1}{2m}\left[p^2 - \frac{e}{c}\vec{A}\cdot\vec{p} - \frac{e}{c}\vec{p}\cdot\vec{A} + \frac{e^2}{c^2}A^2\right]$$

if this was a weak $\vec{B}$ field, we could ignore the A^2 term. But, it is not, So, using $\vec{A} = -B_0 y\,\hat{x}$, we get

$$H = \frac{1}{2m}\left[p^2 + \underbrace{\frac{2B_0 e}{c}\,y\,p_x}_{\text{since } [y, p_x] = 0} + \frac{e^2}{c^2}B_0^2 y^2\right]$$

recall from $\frac{mv^2}{R} = qvB \underset{\omega = vR}{\Rightarrow} \omega = \frac{qB}{m} = \frac{-eB}{m}$. So, (for $c=1$)

$$H = \frac{p^2}{2m} - \omega\, y\, p_x + \frac{m\omega^2}{2}y^2$$

$$\text{let } u = \frac{-\omega p_x}{m\omega^2} + y$$

So,

$$\frac{1}{2}m\omega^2 u^2 - \frac{\omega^2 p_x^2}{2m\omega^2} = \frac{1}{2}m\omega^2 y^2 - \omega y\, p_x$$

Thus, our Hamiltonian is of the form

$$H = \underbrace{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 u^2}_{H_0} - \underbrace{\frac{p_x^2}{2m}}_{H'}$$

we know that $E_0 = \langle H_0 \rangle = (n + \frac{1}{2})\,\omega = (n + \frac{1}{2})\frac{qB}{m}$

and we know that E' can be found from

$$-\frac{P_x^2}{2m}\psi = E'\psi \quad \Rightarrow \quad \frac{d^2\psi}{dx^2} + K_x^2\psi = 0 \quad , K_x^2 = 2mE$$

$$\Rightarrow \quad E = \frac{K_x^2}{2m}$$

Thus,

$$E = \langle H \rangle = \langle H_0 \rangle + \langle H' \rangle = \boxed{(n + \tfrac{1}{2})\frac{qB}{m} + \frac{K_x^2}{2m}}$$

(b) would the energy levels change if we choose $\vec{A}$ to be $\frac{B_0}{2}(-y\hat{x} + x\hat{y})$?

note: $\nabla \cdot \vec{A} = \frac{B_0}{2}\left[\frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x)\right] = 0$

This is the Coulomb gauge. From gauge invariance, the energy levels will not change!

Fall 2002 # 3 (p 1 of 2)

A charge particle of charge, q, and mass, m, is bound in a one-dimensional H.O. potential $V = \frac{1}{2} m\omega^2 x^2$. The system is then placed in an electric field E that is constant in space and time. (see Spring 2005 #1)

(a)

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - q E_0 x$$

, where the problem tells us that we are in 1-D $\Rightarrow \vec{E} = E_0 \hat{x}$

$$\Delta E_0^{(1)} = -q E_0 \langle n | x | n \rangle = 0$$

due to x being odd and $|n\rangle$ have same parity with $|n\rangle$

$$\Delta E_0^{(2)} = \sum_{m \neq n} \frac{|\langle m | H' | n \rangle|^2}{E_n^{\circ} - E_m^{\circ}}$$

where $E_n^{\circ} - E_m^{\circ} = (n + \frac{1}{2})\omega - (m + \frac{1}{2})\omega = \omega(n-m)$

and

$$\langle m | H' | n \rangle = -q E_0 \langle m | x | n \rangle = \frac{-q E_0}{\sqrt{2m\omega}} \langle m | (a + a^+) | n \rangle$$

$$= \frac{-q E_0}{\sqrt{2m\omega}} \left[\langle m | a | n \rangle + \langle m | a^+ | n \rangle \right]$$

$$= \frac{-q E_0}{\sqrt{2m\omega}} \left[\sqrt{n}\, \delta_{m,n-1} + \sqrt{n+1}\, \delta_{m,n+1} \right]$$

So,

$$\Delta E_0^{2} = \frac{q^2 E_0^2}{2m\omega} \left[\frac{n}{\omega(n-n+1)} + \frac{n+1}{\omega(n-n-1)} \right] = \boxed{\frac{-q^2 E_0^2}{2m\omega^2}}$$

Fall 2002 #3 (pcorc

(b) What are the third and higher order shifts (in E) in the ground state energy?

All higher order terms are zero. This has to be since if you solved this problem exactly (see spring 2005 #1 (b)) you get exactly the ground state energy for an unperturbed H.O. plus the term calculated in part (a). That is,

$$\boxed{E_0 = \frac{\omega}{2} - \frac{q^2 \mathcal{E}_0^2}{2m\omega^2}} \leftarrow \text{exact!}$$

Fall 2002 #4

4. Quantum Mechanics.

Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r} \qquad (1)$$

(a) What is the ground state energy of this Hamiltonian?

(b) What is the expectation value of the potential energy $\left\langle -\frac{Ze^2}{r}\right\rangle$ in the ground state?

(c) What is the expectation value of the kinetic energy $\left\langle -\frac{\hbar^2}{2m}\nabla^2\right\rangle$ in the ground state? ($\hbar = 1$)

Fall 2005 #4

$$H = \frac{-\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r} = \frac{p^2}{2m} - \frac{Ze^2}{r}$$

$$E = -\frac{Z^2\alpha^2 m}{2n^2} \qquad a = \text{bohr radius}$$

b) $2\langle T\rangle = \langle r\frac{\partial V}{\partial r}\rangle \qquad r\frac{\partial}{\partial r}\left(\frac{-Ze^2}{r}\right) = r\left(\frac{Ze^2}{r^2}\right) = \langle\frac{Ze^2}{r}\rangle$

$$\langle\psi|H|\psi\rangle = \langle\psi|T|\psi\rangle + \langle\psi|V|\psi\rangle$$

$$E_0 = \langle\psi|T|\psi\rangle + \langle\psi|V|\psi\rangle \qquad r\frac{\partial V}{\partial r} = -V$$

$$= \frac{\langle\psi|r\frac{\partial V}{\partial r}|\psi\rangle}{2} + \langle\psi|V|\psi\rangle$$

4

$$E_0 = \langle\frac{-V}{2}\rangle + \langle V\rangle$$

$2\langle T\rangle = -\langle V\rangle$

$\langle T\rangle = \frac{-\langle V\rangle}{2}$

$$E_0 = \frac{1}{2}\langle V\rangle \qquad \langle V\rangle = 2E_0 = \frac{-Z^2\alpha^2 m}{n^2}$$

c) $\langle T\rangle = -E_0 = \frac{Z^2\alpha^2 m}{2n^2}$

Fall 2002 # 4 (p 1 of 1)

Consider the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{r}$$

(a) what is the ground state energy of this Hamiltonian?

This is just the Hamiltonian for a hydrogen-like atom. Thus, we immediately know the energy levels are (in natural units)

$$\boxed{E_n = -\frac{\alpha^2 m}{2n^2} Z^2} \quad , n = 1, 2, 3 \ldots$$

$n=1$ for ground state

(b) what is the expectation value of the potential energy $\langle -\frac{Ze^2}{r} \rangle$ in the ground state?

from Virial theorem (see Griffiths' QM problem 4.41), we know that

$$\langle V \rangle = 2E_n$$

So,

$$\langle V \rangle \Big|_{n=1} = 2\left(\frac{-\alpha^2 m Z^2}{2n^2}\right)_{n=1} = \boxed{-\alpha^2 m Z^2}$$

(c) what is the expectation value of the kinetic energy $\langle -\frac{\hbar^2}{2m}\nabla^2 \rangle$ in the ground state?

again from Virial theorem, we know

$$\langle T \rangle = -E_n$$

So,

$$\boxed{\langle T \rangle = \frac{\alpha^2 m Z^2}{2}}$$

8. *Electricity and Magnetism*

Radiating charges

(a) A point charge q under acceleration $a(t)$ emits electromagnetic radiation. Giveq ualitative physical arguments why the radiated power, P, should beof the form $P = Bq^2a^2$, where B is a proportionality constant. Determine by dimensional analysis the dependence of B on fundamental physical constants. Explain how and why the exact expression for B differs from this estimate.

(b) A point charge $q$has mass m and is attached to a spring (of spring constant κ) hanging from a fixed support above an infinite horizontal conducting plane. The charge is set in motion with amplitude $A <$ h, the equilibrium height of the charge abovethe conducting plane. Calculate its instantaneous radiating power.

a) It should have that form, since it matches the radiation of an electric dipole because the monopole does not radiate, but rather its dipole term. It is like it makes its own dipole as it moves.

$$P = \text{Watts} = \frac{J}{s} = \frac{Nm}{s} = B\,\overset{\text{columbs}}{C^2}\,\frac{m^2}{s^4} = B\,\frac{A^2 m^2}{s^2}$$

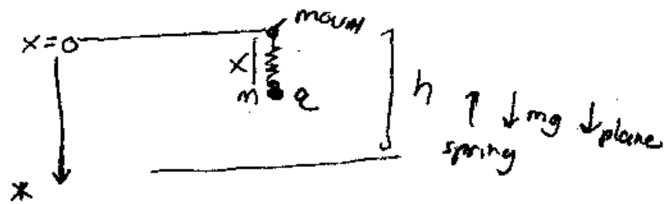
$$B = \frac{N m s^2}{s A^2 m^2} = \frac{N s}{A^2 m} = \frac{\mu_0}{\underset{\text{speed of light}}{c}}$$

$$P \propto \frac{\mu_0}{c} q^2 a^2$$

The exact expression varies by factors of π's that can't be found by demisional anaysis

b) $x(t) = x_0 \cos(\omega t)$ $\qquad x_0 = A$

position of q



$F_s = -kx = -k[A\cos(\omega t)]$

$F_{plane} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2(h-x))^2}$ Griffiths 3.12

$F_{gravity} = mg$

$F_{net} = -k[A\cos(\omega t)] + \frac{1}{4\pi\epsilon_0}\frac{q^2}{(4(h-x)^2)} + mg$

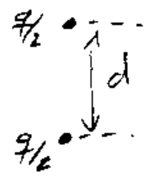
$\frac{F_{net}}{m} = a$

$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2}{6\pi c}\left[\frac{-k}{m}[A\cos(\omega t)] + \frac{1}{4\pi\epsilon_0 m}\frac{q^2}{(4(h-x)^2)} + g\right]^2$

Fall 2002 # 8 (p 1of2)

(a) A point charge q under acceleration $a(t)$ emits electromagnetic radiation. Give qualitative physical arguments why the radiated power, P, should be of the form $P = B q^2 a^2$, where B is a proportionality constant. Determine by dimensional analysis the dependence of B on fundamental physical constants. Explain how and why the exact expression for B differs from this estimate.

when a point charge accelerates, there is an imbalance of internal electromagnetic forces (see Griffiths section 11.2.3). You can think of the charge as a dumbell with each the total charge q divided on the two halves separated by a distance d A-like so



the net force on the dumbbell is

$$\vec{F}_{self} = \frac{q}{2}(\vec{E}_1 + \vec{E}_2) \propto q^2 a^2$$

Now, consider the Larmor Formula (valid when $v \ll c$)

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \Rightarrow \boxed{B = \frac{\mu_0}{6\pi c}}$$

$$\left.\begin{array}{l} [P] = J/s = \dfrac{Kg\, m^2/s^2}{s} \\ [q^2 a^2] = C^2 (m/s^2)^2 \end{array}\right\} \quad \frac{Kg\, m^2}{s^3} = [B]\,\frac{C^2 m^2}{s^4} \Rightarrow \boxed{[B] = \frac{Kg \cdot s}{C^2}} \quad (1)$$

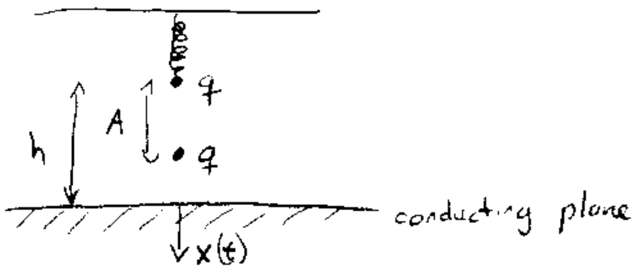
where $C = A \cdot s$

$$[\mu_0] = N/A^2 = \frac{N}{C^2/s^2} = \frac{Kg\; m/s^2}{C^2/s^2} = \frac{Kg \cdot m}{C^2}$$

$$\left[\frac{\mu_0}{c}\right] = \frac{\frac{Kg \cdot m}{C^2}}{m/s} = \frac{Kg \cdot s}{C^2} \checkmark \quad \text{matches eq (1)}$$

Fall 2002 # 8 (p 2 of 2)

(b) A point charge q has mass m and is attached to a spring (w/ spring constant k) hanging from a fixed support above an infinite horizontal conducting plane. The charge is set in motion with amplitude $A < h$, the equilibrium height of the charge above the conducting plane. Calculate its instantaneous radiating power.



We are told that the charge is set in motion with amplitude A. So set $t=0$ to when charge has amplitude A. That is,

$$x(t) = A\cos(\omega t)$$

where A and ω are a bit complicated due to the force from gravity and image "dipole"

If this was an exact dipole of two charges separated by a distance d the amplitude would be qd.

So, the acceleration is given by

$$a = \ddot{x}(t) = -A\omega^2\cos(\omega t) \qquad (2)$$

Now, from part (a), we are told that the radiated power is

$$P = Bq^2a^2$$

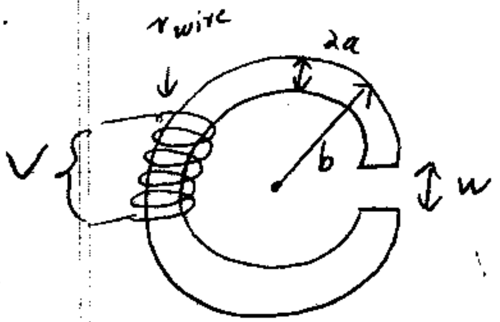
Since we must also consider the image, we have

$$P = 2Bq^2a^2$$

and substituting in for a from eq (2), we have

$$\boxed{P = 2Bq^2A^2\omega^4\cos^2(\omega t)}$$

E.M. F'02 #9



$r_{wire} \ll a \ll b$

a) To determine the field in the gap:

$$\oint \vec{H} \cdot d\vec{\ell} = I_f = NI$$

$$H_{in}(2\pi b - w) + H_{gap} w = NI$$

$$H_{in} = \frac{B}{\mu} \;;\; H_{gap} = \frac{B}{\mu_0}$$

so $\frac{B}{\mu}(2\pi b - w) + \frac{B}{\mu_0} w = NI \Rightarrow B = \dfrac{NI}{\frac{1}{\mu}(2\pi b - w) + \frac{w}{\mu_0}}$

$I = \frac{V}{R}$; $R = \rho \dfrac{N 2\cancel{\pi} a}{\cancel{\pi} r_w^2} = \dfrac{2Na\rho}{r_w^2}$ so $B = \dfrac{V r_w^2}{2a\rho\left[\frac{1}{\mu}(2\pi b - w) + \frac{w}{\mu_0}\right]}$

b) $\tau = \frac{L}{R}$; $L = \frac{N\Phi}{I}$; $\Phi = \int B \cdot da = B \cdot \pi a^2$

so $L = \dfrac{NB\pi a^2}{I}$

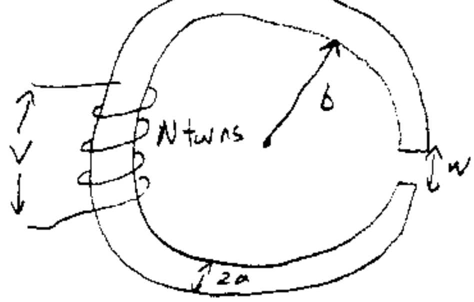
$$\tau = \frac{L}{R} = \frac{NB\pi a^2}{\underbrace{IR}_{=V}} = \frac{N\pi a^2}{V} B = \frac{N^2 \pi a^2}{\frac{1}{\mu}(2\pi b - w) + \frac{w}{\mu_0}} \underbrace{\frac{\cancel{V}}{R}}_{R = \frac{2Na\rho}{r_w^2}} \cdot \frac{1}{\cancel{V}}$$

$$= \frac{\cancel{N^2}\pi \cancel{a^2}}{\frac{1}{\mu}(2\pi b - w) + \frac{w}{\mu_0}} \cdot \frac{r_w^2}{2\cancel{Na}\rho} = \frac{N\pi a r_w^2}{2\rho\left[\frac{1}{\mu}(2\pi b - w) + \frac{w}{\mu_0}\right]}$$

$$= \frac{N\pi a \mu\mu_0 r_w^2}{2\rho\left[\mu_0(2\pi b - w) + \mu w\right]}$$

Spring 2002 # 9 (p 1 of 2)

A D.C. electromagnet is constructed from a cylindrical soft-iron bar with radius a. The relative magnetic permeability of the iron is μ. The bar is bent into a C-shape as shown below with radius b. The width of the small gap is w. The magnet is energized by winding a coil of copper wire N turns tightly around the bar and connecting the coil to a D.C. power supply with voltage V. The copper wire has a resistivity ρ and radius r_{wire}. Assume $r_{wire} << a << b$ and ignore fringe-field effects.



(a) what is the steady-state value of the magnetic field B in the gap?

Ampère's circuital law is given by

$$\oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} N I$$

$$\Rightarrow \quad H_c \, (2\pi b - w) + H_G \, W = N I \left(\frac{4\pi}{c} \right)$$

where H_c is the field from the C-shape magnet and H_G is the field from the gap. So,

$$H_c = \frac{B}{\mu} \quad \text{and} \quad H_G = B$$

Making these substitutions, we get

$$B \left[\frac{1}{\mu} (2\pi b - w) + w \right] = N I \left(\frac{4\pi}{c} \right)$$

$$\Rightarrow \quad B = \frac{\left(\frac{4\pi}{c}\right) N I \mu}{2\pi b - w + w\mu} \qquad (1)$$

Now we want to replace I with what we are given. So, we know that

$$V = IR \Rightarrow I = \frac{V}{R} = \frac{V}{\rho \frac{L}{A}}$$

where L and A are the length and area of the wire. We are told that the radius of the wire is r_w width of the magnet is $2a$ ($\Rightarrow L_w = N 2\pi a$)

$$I = \frac{V \pi r_w^2}{\rho N 2\pi a} = \frac{V r_w^2}{\rho N 2a}$$

substituting this result into our expression for B yields

$$\boxed{B = \frac{V \mu r_w^2 \left(\frac{4\pi}{c}\right)}{\rho\, 2a \left[2\pi b - w + w\mu\right]}}$$

b) what is the time constant governing the response of the current in the coil when the voltage is turned on?

$$\tau = \frac{L}{R}$$, where the self inductance, L, is

$$L = \frac{\Phi_B}{I} = \frac{\int \vec{B} \cdot d\vec{a}}{I}$$

This is actually the flux through a single turn. The flux through N turns is

$$N \int \vec{B} \cdot d\vec{a}$$

So,

$$\tau = \frac{1}{\underbrace{RI}_{=V}} N |\vec{B}| \underbrace{\pi a^2}_{\text{just where wires are}} = \boxed{\frac{N \mu \pi a r_w^2 \left(\frac{4\pi}{c}\right)}{2\rho \left[2\pi b - w + w\mu\right]}}$$

10. ***Electricity and Magnetism***

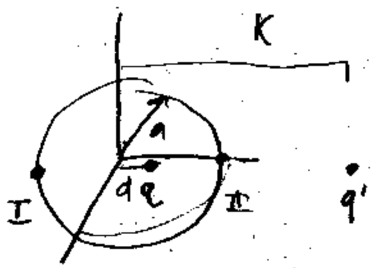
A point charge q is inside a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

(a) the potential inside the sphere;

(b) the induced surface-charge density at the point on the sphere nearest to q;

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surface?



$\phi(\mathrm{I}, \mathrm{II}) = 0$

At I

$$\frac{q}{a+d} + \frac{q'}{K+a} = 0 \qquad q' = -\frac{q(K+a)}{(a+d)}$$

II

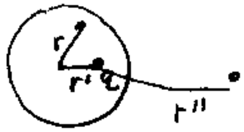
$$\frac{q}{a-d} + \frac{q'}{K-a} = 0 \qquad q' = -\frac{q(K-a)}{(a-d)} \Rightarrow \frac{(K+a)}{(a+d)} = \frac{(K-a)}{(a-d)}$$

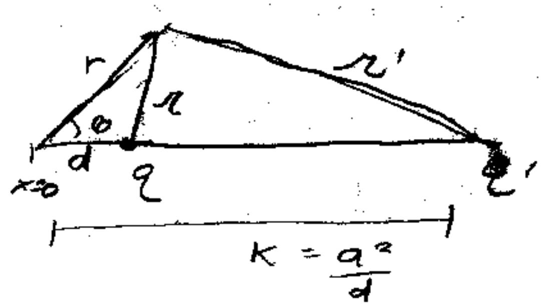
$$\Rightarrow Kd = a^2$$

$$K = \frac{a^2}{d}$$ (Fall 99 # 3)

$$q' = -\frac{q\left(\frac{a^2}{d}+a\right)}{(a+d)} = -\frac{qa}{d}\left(\frac{\cancel{a+d}}{\cancel{a+d}}\right) = -\frac{qa}{d}$$

a) $\Phi = \dfrac{q/4\pi\varepsilon_0}{|\mathfrak{r}-r'|} + \dfrac{\frac{qa}{d}\frac{1}{4\pi\varepsilon_0}}{|\bar{r}-\mathfrak{r}''|}$





b) $\sigma = -\varepsilon_0 \dfrac{\partial V}{\partial n} = -\varepsilon_0 \dfrac{\partial V}{\partial r}\Big|_{r=a}$

$\mathfrak{r} = r^2 + d^2 - 2rd\cos\theta$

$\mathfrak{r}' = r^2 + \dfrac{a^4}{d^2} - 2r\dfrac{a^2}{d}\cos\theta$

$$\sigma = -\frac{q}{4\pi}\frac{\partial}{\partial r}\left(\frac{1}{r^2+d^2-2rd\cos\theta} - \frac{\frac{a}{d}}{r^2+\left(\frac{a^2}{d}\right)^2 - \frac{2ra^2}{d}\cos\theta}\right)$$

$$= \frac{-q}{4\pi}\left[\frac{-1}{(r^2+d^2-2rd\cos\theta)^2}\cdot(2r-2d\cos\theta) + \frac{\frac{a}{d}\left(2r-2\frac{a^2}{d}\cos\theta\right)}{\left(r^2+\left(\frac{a^2}{d}\right)^2-\frac{2ra^2}{d}\cos\theta\right)^2}\right]\Bigg|_{r=a}$$

$$= \frac{-q}{4\pi}\left[\frac{-1(2a-2d\cos\theta)}{(a^2+d^2-2ad\cos\theta)^2} + \frac{\frac{a}{d}\left(2a-\frac{2a^2}{d}\cos\theta\right)}{\left(a^2+\left(\frac{a^2}{d}\right)^2-\frac{2a^3}{d}\cos\theta\right)}\right]$$

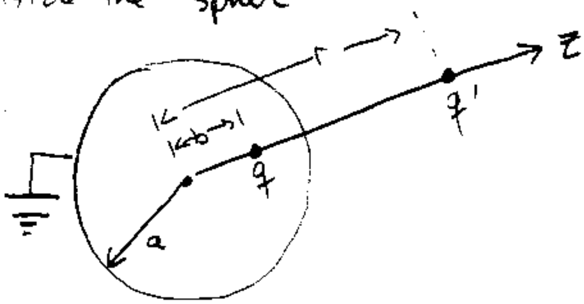
c) $$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(K-d)^2} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-\frac{q^2a}{d}}{\left(\frac{a^2}{d}-d\right)^2}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{-q^2ad}{(a^2-ad)^2}\right]$$

Pulled twards sphere surface. $\hat{r}$

d) There is no change since the charge is inside the conductor, and information of outside fields is not related in.

A point charge q is inside a hollow, grounded, conducting sphere of inner radius a. Use the method of images to find

(a) the potential inside the sphere



save yourself much time and memorize the following for image charge sphere problems (works whether charge is inside sphere or outside).

$$q' = -q\frac{a}{b} \quad ; \quad r = \frac{a^2}{b}$$

So, the potential inside sphere is

$$\Phi(r,\theta) = \frac{q}{\sqrt{x^2+y^2+(z-b)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-r)^2}}$$

$$= \frac{q}{\sqrt{x^2+y^2+(z-b)^2}} - \frac{q(a/b)}{\sqrt{x^2+y^2+(z-\frac{a^2}{b})^2}}$$

note: the convention is to write

$$x^2+y^2+(z-\alpha)^2 = \underbrace{x^2+y^2+z^2}_{r^2} - 2z\cdot\alpha + \alpha^2 = r^2+\alpha^2-2z\alpha\cos\theta$$

So, we have

$$\boxed{\Phi(r,\theta) = \frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{q(a/b)}{\sqrt{r^2+\left(\frac{a^2}{b}\right)^2-2r\left(\frac{a^2}{b}\right)\cos\theta}}}$$

(b) the induced surface-charge density at the point on the sphere nearest to q.

$$\sigma = -\frac{1}{4\pi}\frac{\partial \Phi}{\partial r}\bigg|_{r=a}$$

So,

$$-4\pi\sigma = \frac{-q(2r-2b\cos\theta)}{2(r^2+b^2-2rb\cos\theta)^{3/2}}\bigg|_{r=a} - \frac{q(a/b)\left(2r-2\left(\frac{a^2}{b}\right)\cos\theta\right)}{2\left[r^2+\left(\frac{a^2}{b}\right)^2-2r\left(\frac{a^2}{b}\right)\cos\theta\right]^{3/2}}\bigg|_{r=a}$$

$$\therefore \boxed{\sigma = \frac{q}{4\pi}\left\{\frac{a-2b\cos\theta}{[a^2+b^2-2ab\cos\theta]^{3/2}} + \frac{a^2-a^3\cos\theta}{b\left[a^2+\left(\frac{a^2}{b}\right)^2-2\left(\frac{a^3}{b}\right)\cos\theta\right]^{3/2}}\right\}}$$

(c) the magnitude and direction of the force acting on q

note: if the question asked for force acting on surface of sphere it would be

$$F_z = 2\pi\int \sigma^2\,\hat{z}\cdot d\vec{a}$$

but it does not. So, we simply have (Griffiths' eq 3.18)

$$\vec{F} = \frac{qq'}{(r-b)^2}$$

$$\therefore \boxed{F = \frac{-q^2(a/b)}{\left[\frac{a^2}{b}-b\right]^2} = \frac{-q^2ab}{(a^2-b^2)^2}}$$

Force must be in the $+\hat{z}$ direction

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

The answer to both questions is that there will be no change in the solution. Let's show this separately.

(i) if sphere is kept at a fixed potential V.

So, we have

$$\Phi' = \Phi + V \quad \text{, where } V \text{ is constant.}$$

$$\Rightarrow \sigma' = \frac{-1}{4\pi} \frac{\partial \Phi'}{\partial r}\bigg|_{r=a} = \frac{-1}{4\pi}\left[\frac{\partial \Phi}{\partial r}\bigg|_{r=a} + \cancel{\frac{\partial V}{\partial r}}^{\,0}\bigg|_{r=a} \right] = \sigma \checkmark$$

&

$$\vec{E}' = -\nabla\Phi' = -\nabla\Phi - \cancel{\nabla V}^{\,0} = \vec{E} \Rightarrow \vec{F}' = q\vec{E}' = q\vec{E} = \vec{F} \checkmark$$

(ii) if the sphere has a total charge Q on its inner and outer surfaces.

since Q is also constant wrt r, θ, ϕ, we get the same result as above, that is

$$\Phi' = \Phi + \frac{Q+q}{R} \quad \text{, where } R \text{ is outer radius}$$

since the second term is constant wrt r, θ, ϕ, no change will occur in σ and $\vec{F}$.

Fall 2002 #10 (p 1 of 2)

(note: this is Jackson problem 2.2!)

In this case, we can fix an image charge somewhere outside the conducting shell. This image charge will function to hold the potential everywhere on the spherical surface at zero potential; that is, the image charge takes the place of the charge density that's present on the spherical shell.

Note that this problem is entirely equivalent to the one exhibited in Jackson section 2.2 except that the image charge needs to be at a distance *greater* than a. Let d denote charge q's radial distance from the center of the sphere; then you should find that the image charge's magnitude and distance are:

$$q_{im} = -\frac{qa}{d} \qquad |\vec{x}_{im}| = \frac{a^2}{d}$$

Note that $(a^2/d) > a$ since $d < a$. So the image charge is indeed outside the conducting sphere. The potential inside the sphere then is given by the following

$$\Phi_{in}(\vec{x}) = \frac{q}{|\vec{x} - d\hat{x}_q|} - \frac{qa}{d\left|\vec{x} - \frac{a^2}{d}\hat{x}_q\right|}$$

where $\hat{x}_q$ is the direction vector from the origin (center of sphere) to charge q's location.

(part b) Find the induced surface-charge density.

Just use the relation $\sigma = -(1/4\pi)(\partial\Phi/\partial x)_{x=a}$. After differentiation, algebra, and evaluation of the result at $x = a$, you should find

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{d}{a}\right)^2\left[\frac{1-(a/d)^2}{(1+(d/a)^2 - 2(d/a)\cos\gamma)^{3/2}}\right]$$

where γ is the angle between $\hat{x}_q$ and the position vector to some arbitrary field point $\vec{x}$.

(part c) Find the magnitude and direction of the force acting on q.

Since we're dealing with a point charge, we can use Coulomb's law:

$$\vec{F} = q\left(-\vec{\nabla}\Phi_{in}\right) = \frac{q^2}{d^2}\left(\frac{d}{a}\right)^3\left[1-\frac{d^2}{a^2}\right]^{-2}$$

Fall 2002 #10 (p 2 of 2)

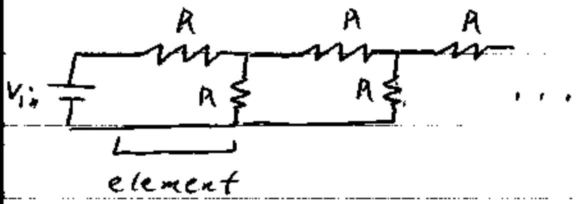
(part d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q on its inner and outer surfaces?

If there's a fixed potential V on the surface, then we'de need to account for this potential inside the sphere by using a point charge of magnitude Va located at the center of the sphere. This will produce an extra piece in the potential given by

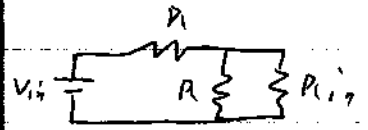
$$\Phi_{extra} = \frac{Va}{|\vec{x}|}$$

The total potential inside the sphere now will have a singularity at the origin. On the other hand, if there's a total charge Q on the surface of the condcuting sphere, there'll be no change in the potential expression for the interior of the sphere. This extra charge Q will be distributed uniformly on the sphere's surface; the isotropic nature of this surface charge density ensures that the interior E-field inside the sphere due to Q is zero (as you can readily verify using Gauss' law) and so the potential due to Q everywhere inside the sphere is an arbitrary constant which can be set to zero.

A voltage is applied to the infinitely long resistor network shown below. Each resistor has the same resistance R. Calculate the power dissipated in each resistor.



Define $R_{in} = \frac{V_{in}}{I_{in}}$, in order to find R_{in} add an additional element to the chain (which we replace by R_{in}) – Feynman V. II p22-13.



$$R_{eq} = R + \frac{R R_{in}}{R+R_{in}} = R_{in}$$ (as the chain is infinite)

$$\frac{R R_{in}}{R+R_{in}} = R_{in} - R \Rightarrow R R_{in} = (R_{in} - R)(R + R_{in})$$

$$= \cancel{R R_{in}} + R_{in}^2 - R^2 - \cancel{R R_{in}}$$

then $R_{in}^2 - R R_{in} - R^2 = 0$ which is in the form

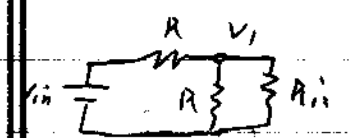
$$ax^2 + bx + c = 0$$

$$R_{in} = -\frac{(-R)}{2} \pm \frac{1}{2}\sqrt{R^2 - 4(-R^2)} \qquad x = \frac{-b}{2a} \pm \frac{1}{2}\sqrt{b^2 - 4ac}$$

$$= \frac{R}{2} \pm \frac{R}{2}\sqrt{5} = \frac{R}{2}(1 + \sqrt{5})$$

can't be "–" as $R_{in} > 0$ by definition

Now label the nodes:



to get V_1 just find R_{eq}, divide V_{in} to get the current and then multiply by $\frac{R R_1}{R+R_1}$:

$$R_{eq} = R + \frac{R R_{in}}{R+R_{in}} \Rightarrow \frac{V_{in}}{R_{eq}} = I \quad \Rightarrow V_1 = \frac{R R_1}{R+R_1} I = \frac{\frac{R R_1}{R+R_1}}{R + \frac{R R_1}{R+R_1}} V_{in}$$

so $\dfrac{V_1}{V_{in}} = \dfrac{R R_{in}}{R^2 + R R_{in} + R R_{in}} = \dfrac{R R_{in}}{R^2 + 2 R R_{in}} = \dfrac{\frac{\cancel{R^2}}{2}(1+\sqrt{5})}{\cancel{R^2} + \cancel{R^2}(1+\sqrt{5})} = \dfrac{1+\sqrt{5}}{2+2(1+\sqrt{5})} = \dfrac{1+\sqrt{5}}{4+\sqrt{5}} \equiv \beta$

so $\dfrac{V_1}{V_{in}} = \beta$

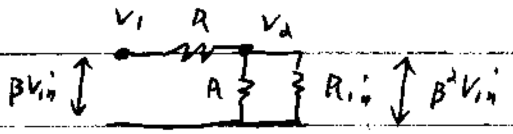
The power dissipated by the parallel/shunt resistor is just

$$P = IV = \frac{\cancel{V}^2}{R} = \frac{V_1^2}{R} = \frac{V_{in}^2}{R}\beta^2$$

As for the series resistor

$$P = \frac{[V_{in} - V_1]^2}{R} = \frac{V_{in}^2\left(1 - \frac{V_1}{V_{in}}\right)^2}{R} = \frac{V_{in}^2}{R}(1-\beta)^2 = \frac{V_{in}^2}{R}(\beta-1)^2$$

For the second circuit element (series, shunt combo.)



so $\dfrac{V_2}{V_{in}} = \beta^2 \Rightarrow \dfrac{V_N}{V_{in}} = \beta^N$

$$P_{2,shunt} = \frac{V_2^2}{R} = \frac{V_{in}^2}{R}(\beta^2)^2 \Rightarrow P_{N,shunt} = \frac{V_N^2}{R} = \frac{V_{in}^2}{R}(\beta^N)^2 = \boxed{\beta^{2N}\frac{V_{in}^2}{R}}$$

$$P_{2,series} = \frac{[V_1 - V_2]^2}{R} = \frac{V_{in}^2}{R}\left[\frac{V_1}{V_{in}} - \frac{V_2}{V_{in}}\right]^2 = \frac{V_{in}^2}{R}\left[\beta - \beta^2\right]^2 = \frac{V_{in}^2\beta^2}{R}[1-\beta]^2$$

$$= \frac{V_{in}^2}{R}\beta^2(\beta-1)^2$$

so $P_{N,series} = \dfrac{V_{in}^2}{R}\left[\dfrac{V_{N-1}}{V_{in}} - \dfrac{V_N}{V_{in}}\right]^2 = \dfrac{V_{in}^2}{R}\left[\beta^{N-1} - \beta^N\right]^2 = \dfrac{V_{in}^2}{R}\beta^{2(N-1)}\left[1-\beta\right]^2$

$$= \boxed{\frac{V_{in}^2}{R}\beta^{2(N-1)}[\beta-1]^2}$$

In summary

$$P_{N,shunt} = \frac{V_{in}^2}{R}\beta^{2N}$$

$$P_{N,series} = \frac{V_{in}^2}{R}\beta^{2(N-1)}(\beta-1)^2$$

13. *Statistical Mechanics and Thermodynamics*

Consider an ideal monoatomic gas in which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperature T.

Find the a) chemical potential, b) free energy, c) entropy, d) pressure and e) heat capacity at constant pressure.

$$Z_1 = \sum_E e^{-\beta E} = e^{-\beta E - \beta(E+\Delta)} = e^{-\beta E} e^{-\beta E} e^{\Delta}$$

$$= e^{-2\beta E} e^{-\Delta}$$

$$Z = \frac{(Z_1)^N}{N!} \qquad \ln Z = N(\ln Z_1 - \ln N + 1)$$
$$= N(-2\beta E - \Delta - \ln N + 1)$$

$$F = -kT \ln Z = -kTN(1 - 2\beta E - \Delta - \ln N)$$

a) $$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T} = -kT(1 - 2\beta E - \Delta - \ln N) - kTN\left(-\frac{1}{N}\right)$$
$$= kT(2\beta E + \Delta + \ln N - 1) + kT$$
$$= kT(2\beta E + \Delta + \ln N)$$

b) $F = -kT \ln Z$

c) $$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = kN(1 - 2\beta E - \Delta - \ln N) + kTN\left(\frac{+2E}{kT^2}\right)$$

$kN(2E\beta)$

$$= kN(1 - \Delta - \ln N)$$

d) $p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = 0$

e) $= C_p = Nk\left(1 + \frac{f}{R}\right)$ degrees of freedom $\Rightarrow C_p = Nk\left(1 + \frac{3}{R}\right)$

Spring 2002 #13 (p 1 of 2)

Consider an ideal monoatomic gas n which each atom has two internal energy states, one an energy Δ above the other. There are N atoms in a volume V at temperature T.

Find

(a) chemical potential

As usual start with the partition function. Let ϵ be the energy of the lower state. Then $\epsilon + \Delta$ is the energy of the other state.
So, the partition function of one atom is

$$Z_1 = \sum_r e^{-\beta \epsilon_r} = e^{-\beta \epsilon} + e^{-\beta(\epsilon+\Delta)}$$

since this is a monatomic ideal gas, we know the energy is

$$\epsilon = \frac{3}{2} KT$$

The partition function for N indistinguishable atoms is

$$Z = \frac{1}{N!}(Z_1)^N = \frac{1}{N!}\left[e^{-\beta\epsilon} + e^{-\beta(\epsilon+\Delta)}\right]^N$$

$$\Rightarrow \quad \ln Z = -\ln N! + N \ln\left[e^{-\beta\epsilon} + e^{-\beta(\epsilon+\Delta)}\right]$$

since $N \gg 1$
use stirling's formula

$$\cong -N\ln N + N + N\ln\left[e^{-\beta\epsilon}(1+e^{-\beta\Delta})\right]$$

$$\therefore \quad \ln Z \cong -N\ln N + N - N\beta\epsilon + N\ln\left[1+e^{-\beta\Delta}\right]$$

Then, the free energy is

$$F = -KT\ln Z$$

$$= KTN\ln N - KTN + N\epsilon - KTN\ln\left[1+e^{-\beta\Delta}\right]$$

Now, we know that the chemical potential is

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T} = KT\ln N + KT - KT + \epsilon - KT\ln\left[1+e^{-\beta\Delta}\right]$$

$$\Rightarrow \boxed{\mu = kT\left[\ln N + \frac{3}{2} - \ln\left(1+e^{-\beta\Delta}\right)\right]} \quad , \ \epsilon = \frac{3}{2}kT$$

(b) Free energy

see part (e)

$$\boxed{F = -kTN\left[-\ln N + 1 - \frac{3}{2} + \ln\left(1+e^{-\beta\Delta}\right)\right]}$$

(c) entropy

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = -kN\ln N + kN + kN\ln\left(1+e^{-\beta\Delta}\right) + kTN\,\frac{\frac{\Delta}{kT^2}e^{-\Delta/kT}}{1+e^{-\beta\Delta}}$$

$$\Rightarrow \boxed{S = kN\left[-\ln N + 1 + \ln\left(1+e^{-\beta\Delta}\right) + \frac{\Delta}{kT}\frac{1}{1+e^{\beta\Delta}}\right]}$$

(d) pressure

$$\boxed{p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = 0}$$

(e) heat capacity at constant pressure

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p = T\left[\underbrace{\frac{\frac{\Delta}{kT^2}e^{-\beta\Delta}}{1+e^{-\beta\Delta}} - \frac{\Delta}{kT^2}\frac{1}{1+e^{\beta\Delta}}}_{=0} + \frac{\Delta}{kT}\frac{\frac{\Delta}{kT^2}e^{\beta\Delta}}{\left(1+e^{\beta\Delta}\right)^2}\right]$$

$$\therefore \boxed{C_p = \left(\frac{\Delta}{kT}\right)^2\frac{e^{\beta\Delta}}{\left(1+e^{\beta\Delta}\right)^2}}$$