**3. (Quantum Mechanics)**

The purpose of this problem is to show that a spin zero particle with electric charge $e$ in the presence of a certain radial magnetic field $\mathbf{B}$ effectively behaves as a particle with spin $\frac{1}{2}$. The classical Lagrangian for the spin 0 particle is given by (here $\mathbf{v} = \dot{\mathbf{r}}$ and $r = |\mathbf{r}|$),

$$L = \frac{1}{2}m\mathbf{v}^2 + e\mathbf{A}\cdot\mathbf{v} \qquad \mathbf{B} = \nabla \times \mathbf{A} = g\frac{\mathbf{r}}{r^3}$$

where $m$ is the mass of the particle and $g$ is a real parameter.

(a) Compute the canonical momenta $\mathbf{p}$ conjugate to the position variables $\mathbf{r}$.

(b) Write down the Euler-Lagrange equation for the system in terms of $\mathbf{r}$ and $\mathbf{v}$.

(c) Using the results of (b) above, show that the combination $\mathbf{L} = (L_x, L_y, L_z)$ defined by

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} - eg\frac{\mathbf{r}}{r}$$

is time-independent.

(d) Compute the commutators $[L_i, r_j]$ (i.e. the commutators of the components of the vectors $\mathbf{L}$ and $\mathbf{r}$). An analogous result—which you are not asked to derive—for $[L_i, p_j]$ establishes that $\mathbf{L}$ represents angular momentum.

(e) Compute the quantum operator $L_z$ in spherical coordinates $r$, $\theta$, $\phi$ using the results of (a).

(f) Show that the eigenvalues of $L_z$ are half-odd-integer multiples of $\hbar$ when the electric charge $e$ and the parameter $g$ are related by $eg = \frac{\hbar}{2}$.

[Hint: In a convenient gauge, the vector potential $\mathbf{A}$ for the field $\mathbf{B}$ is given by $\mathbf{A} = g\mathbf{n}_\phi(1-\cos\theta)/(r\sin\theta)$ where $\mathbf{n}_\phi$ is the unit vector given by $\mathbf{n}_\phi = (-\sin\phi, \cos\phi, 0)$ in spherical coordinates where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.]

**Solution:** *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

This problem mixes classical and quantum mechanics. It attempts to walk you step-by-step through its solution. Since we are working with a fair amount of vector algebra and calculus, it is useful to rewrite the Lagrangian in Einstein summation notation (repeated indices summed over):

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\mathbf{A}\cdot\dot{\mathbf{r}} = \frac{1}{2}m\dot{r}_i\dot{r}_i + eA_i\dot{r}_i \tag{1}$$

(a) The definition of the canonical momentum is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} \quad \text{or} \quad p_i = \frac{\partial L}{\partial \dot{r}_i} \tag{2}$$

Taking this partial derivative in (1), we get

$$\mathbf{p} = m\dot{\mathbf{r}} + e\mathbf{A} \quad \text{or} \quad p_i = m\dot{r}_i + eA_i \tag{3}$$

Here, $\mathbf{A}$ is the vector potential for the field $\mathbf{B}$ given in the hint. In other words,

$$\boxed{\mathbf{p} = m\mathbf{v} + e\mathbf{A}} \tag{4}$$

(b) The Euler-Lagrange equation for this system is just the Newton's second law equation for the system. This is just a particle of charge $e$ in a magnetic field, so the Newton's second law equation is given by the Lorentz force law:

$$m\mathbf{a} = e\mathbf{v}\times\mathbf{B} \tag{5}$$

If you (like the author) didn't think of this, you can derive this from the definition of the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \tag{6}$$

The first term is equal to $\frac{d\mathbf{p}}{dt}$. Using (3) and taking a *total* time derivative (applying the chain rule to the vector potential $\mathbf{A}(\mathbf{r})$), we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) &= \frac{d}{dt}\left(m\dot{r}_i + eA_i\right)\\ &= m\ddot{r}_i + e\left(\partial_j A_i\right)\dot{r}_j\end{aligned} \tag{7}$$

The other term in the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial r_i} = e\left(\partial_i A_j\right)\dot{r}_j \tag{8}$$

Thus, the Euler-Lagrange equation is

$$\begin{aligned}0 &= m\ddot{r}_i + e\left(\partial_j A_i\right)\dot{r}_j - e\left(\partial_i A_j\right)\dot{r}_j\\ \Longrightarrow m\ddot{r}_i &= e\left[\left(\partial_i A_j\right)\dot{r}_j - \left(\partial_j A_i\right)\dot{r}_j\right]\end{aligned} \tag{9}$$

The quantity in brackets might remind you of the right-hand side of the BAC-CAB identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}) \quad \text{or} \quad \left(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\right)_i = b_i a_j c_j - c_i a_j b_j$$

Pattern-matching to (9), and noting that $\partial_i \dot{r}_j = 0$, we can identify $a_i \longleftrightarrow \dot{r}_i$, $b_i \longleftrightarrow \partial_i$, and $c_i \longleftrightarrow A_i$. This means that (9) becomes

$$m\ddot{r}_i = e\left[\dot{\mathbf{r}}_i \times (\nabla\times\mathbf{A})\right]_i \tag{10}$$

Since $\ddot{\mathbf{r}} = \mathbf{a}$ (acceleration), $\dot{\mathbf{r}} = \mathbf{v}$ (velocity), and $\nabla \times \mathbf{A} = \mathbf{B}$, this becomes

$$m\mathbf{a} = e\mathbf{v} \times \mathbf{B} \tag{11}$$

In this problem, $\mathbf{B} = g\frac{\mathbf{r}}{r^3}$. In terms of $\mathbf{r}$ and $\mathbf{b}$, the Euler-Lagrange equation boils down to

$$\boxed{m\dot{\mathbf{v}} = eg\,\mathbf{v} \times \frac{\mathbf{r}}{r^3}} \tag{12}$$

(c) To show that $\mathbf{L}$ is time-independent, we need only show that its *total* time derivative is zero. Using the product and chain rules, we get

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\mathbf{r} \times m\mathbf{v} - eg\frac{\mathbf{r}}{r}\right) \\
&= \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} - eg\frac{1}{r}\frac{d\mathbf{r}}{dt} - eg\mathbf{r}\frac{d}{dt}\left(\frac{1}{r}\right) \\
&= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} - eg\frac{\mathbf{v}}{r} + eg\frac{\mathbf{r}}{r^2}\frac{dr}{dt} \\
\frac{d\mathbf{L}}{dt} &= \mathbf{r} \times m\dot{\mathbf{v}} - eg\frac{\mathbf{v}}{r} + eg\frac{\mathbf{r}}{r^2}\frac{dr}{dt} \quad \text{since } \mathbf{v} \times \mathbf{v} = 0
\end{aligned} \tag{13}$$

The problem asks us to use our answer from part (b). Using (12) and substituting for $m\dot{\mathbf{v}}$, we get

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \mathbf{r} \times \left(eg\,\mathbf{v} \times \frac{\mathbf{r}}{r^3}\right) - eg\frac{\mathbf{v}}{r} + eg\frac{\mathbf{r}}{r^2}\frac{dr}{dt} \\
&= eg\left[\frac{1}{r^3}\mathbf{r} \times (\mathbf{v} \times \mathbf{r}) - \frac{\mathbf{v}}{r} + \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right]
\end{aligned} \tag{14}$$

The triple product $\mathbf{r} \times (\mathbf{v} \times \mathbf{r})$ can be simplified using the BAC-CAB rule:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$$

This gets us

$$\begin{aligned}
\mathbf{r} \times (\mathbf{v} \times \mathbf{r}) &= \mathbf{v}\,(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}\,(\mathbf{r} \cdot \mathbf{v}) \\
&= r^2\,\mathbf{v} - \mathbf{r}\,(\mathbf{r} \cdot \mathbf{v})
\end{aligned} \tag{15}$$

Plugging this back into (14), we get

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= eg\left[\frac{1}{r^3}\left(r^2\,\mathbf{v} - \mathbf{r}\,(\mathbf{r} \cdot \mathbf{v})\right) - \frac{\mathbf{v}}{r} + \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right] \\
&= eg\left[\frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^3}\,(\mathbf{r} \cdot \mathbf{v}) - \frac{\mathbf{v}}{r} + \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right] \\
&= eg\left[-\frac{\mathbf{r}}{r^3}\,(\mathbf{r} \cdot \mathbf{v}) + \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right]
\end{aligned} \tag{16}$$

One way to show that the term in brackets vanishes is to remember that the radial component of the velocity vector is the change in the radius:

$$\hat{\mathbf{r}} \cdot \mathbf{v} = \frac{dr}{dt} \tag{17}$$

Then,

$$\frac{\mathbf{r}}{r^2}\frac{dr}{dt} = \frac{\mathbf{r}}{r^2}\,(\hat{\mathbf{r}} \cdot \mathbf{v}) = \frac{\mathbf{r}}{r^3}\,(\mathbf{r} \cdot \mathbf{v}) \tag{18}$$

Another way to show that the term in brackets vanishes is to take the antiderivative of $\mathbf{r}\cdot\mathbf{v}$:

$$\begin{aligned}\frac{\mathbf{r}}{r^3}(\mathbf{r}\cdot\mathbf{v}) &= \frac{\mathbf{r}}{r^3}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)\\ &= \frac{\mathbf{r}}{r^3}\left(\frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})\right)\\ &= \frac{\mathbf{r}}{r^3}\left(\frac{1}{2}\frac{d}{dt}(r^2)\right)\\ &= \frac{\mathbf{r}}{r^3}\left(r\frac{dr}{dt}\right)\\ &= \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\end{aligned} \tag{19}$$

Either way, we get that $\frac{\mathbf{r}}{r^3}(\mathbf{r}\cdot\mathbf{v}) = \frac{\mathbf{r}}{r^2}\frac{dr}{dt}$, so $-\frac{\mathbf{r}}{r^3}(\mathbf{r}\cdot\mathbf{v}) + \frac{\mathbf{r}}{r^2}\frac{dr}{dt} = 0$. Plugging this result into (16), we establish the result

$$\frac{d\mathbf{L}}{dt} = 0 \tag{20}$$

so $\mathbf{L}$ is time-independent.

(d) To calculate the commutators $[L_i, r_j]$, we need to use the canonical commutation relations

$$[r_i, r_j] = 0 \quad \text{and} \quad [r_i, p_j] = i\hbar\,\delta_{ij} \tag{21}$$

This means that we need to write $\mathbf{L}$ in terms of $\mathbf{r}$ and $\mathbf{p}$. Since all components of $\mathbf{r}$ commute with each other, any terms in $\mathbf{L}$ that are independent of $\mathbf{p}$ commute with $\mathbf{r}$ automatically. From part (a), $\mathbf{p} = m\mathbf{v} + e\mathbf{A}$, and $\mathbf{A}$ depends only on $\mathbf{r}$, so

$$\begin{aligned}\mathbf{L} &= \mathbf{r}\times m\mathbf{v} - eg\frac{\mathbf{r}}{r}\\ &= \mathbf{r}\times(\mathbf{p} - e\mathbf{A}) - eg\frac{\mathbf{r}}{r}\\ &= \mathbf{r}\times\mathbf{p} - e\,\mathbf{r}\times\mathbf{A} - eg\frac{\mathbf{r}}{r}\end{aligned} \tag{22}$$

In components, this becomes

$$L_i = \epsilon_{ijk}\,r_j p_k + f_i(\mathbf{r}) \tag{23}$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol and $f_i(\mathbf{r})$ is a function of $\mathbf{r}$. Since $f_i$ is only a function of $\mathbf{r}$, this term commutes with all components of $\mathbf{r}$. Therefore, applying the commutation rules and relations, we get

$$\begin{aligned}[L_i, r_j] &= [\epsilon_{ik\ell}\,r_k p_\ell, r_j] + [f_i(\mathbf{r}), r_j]\\ &= [\epsilon_{ik\ell}\,r_k p_\ell, r_j] \quad \text{since } [r_i, r_j] = 0\\ &= \epsilon_{ik\ell} r_k\,[p_\ell, r_j] + \epsilon_{ik\ell}\,[r_k, r_j]\,p_k \quad \text{as } [AB, C] = A[B,C] + [A,C]B\\ &= \epsilon_{ik\ell} r_k\,[p_\ell, r_j] \quad \text{since } [r_i, r_j] = 0\\ &= -\epsilon_{ik\ell} r_k\,[r_j, p_\ell] \quad \text{since } [A,B] = -[B,A]\\ &= -\epsilon_{ik\ell} r_k\,(i\hbar\delta_{j\ell}) \quad \text{since } [r_i, p_j] = i\hbar\delta_{ij}\\ &= -i\hbar\,\epsilon_{ikj} r_k\\ &= i\hbar\,\epsilon_{ijk} r_k \quad \text{since } \epsilon_{ijk} = -\epsilon_{ijk}\end{aligned} \tag{24}$$

We have found that

$$\boxed{[L_i, r_j] = i\hbar\,\epsilon_{ijk} r_k} \tag{25}$$

(e) To write the quantum operators in spherical coordinates, we need to use the canonical quantization prescription

$$p_x = \frac{\hbar}{i}\frac{\partial}{\partial x} \quad \text{and} \quad p_y = \frac{\hbar}{i}\frac{\partial}{\partial y} \tag{26}$$

From (22), we can write $L_z$ in terms of $x$, $y$, $p_x$, and $p_y$:

$$\begin{aligned} L_z &= (\mathbf{r} \times \mathbf{p})_z - e\,(\mathbf{r} \times \mathbf{A})_z - eg\left(\frac{\mathbf{r}}{r}\right)_z \\ &= (xp_y - yp_x) - e\,(xA_y - yA_x) - eg\,\frac{z}{r} \end{aligned} \tag{27}$$

Canonically quantizing the first term, we get

$$xp_y - yp_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{28}$$

The best way to convert this to spherical coordinates is to guess (from previous experience) that the partial derivative $\frac{\partial}{\partial \varphi}$ is involved. In spherical coordinates, we have

$$x = r\sin\theta\cos\varphi \quad \text{and} \quad y = r\sin\theta\sin\varphi \tag{29}$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial \varphi} &= \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} \\ &= -r\sin\theta\sin\varphi\frac{\partial}{\partial x} + r\sin\theta\cos\varphi\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \end{aligned} \tag{30}$$

Therefore, plugging into (28), we get

$$xp_y - yp_x = \frac{\hbar}{i}\left(\frac{\partial}{\partial \varphi}\right) = -i\hbar\frac{\partial}{\partial \varphi} \tag{31}$$

For the second term in (27), we need to use the vector potential given in the hint:

$$\begin{aligned} \mathbf{A} &= g\hat{\varphi}\,\frac{1-\cos\theta}{r\sin\theta} \\ &= g\left(-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}\right)\,\frac{1-\cos\theta}{r\sin\theta} \end{aligned} \tag{32}$$

Therefore, in spherical coordinates, the second term in (27) simplifies to

$$\begin{aligned} -e\,(xA_y - yA_x) &= -e\left[(r\sin\theta\cos\varphi)\left(g\cos\varphi\frac{1-\cos\theta}{r\sin\theta}\right) - (r\sin\theta\sin\varphi)\left(-g\sin\varphi\frac{1-\cos\theta}{r\sin\theta}\right)\right] \\ &= -eg\,r\sin\theta\,\frac{1-\cos\theta}{r\sin\theta}\,\left[\cos^2\varphi + \sin^2\varphi\right] \\ &= -eg(1-\cos\theta) \end{aligned} \tag{33}$$

The last term in (27) simplifies to

$$\begin{aligned} -eg\frac{z}{r} &= -eg\frac{r\cos\theta}{r} \\ &= -eg\cos\theta \end{aligned} \tag{34}$$

Putting all three terms together using (31), (33), and (34), we get

$$L_z = -i\hbar\frac{\partial}{\partial\varphi} - eg(1-\cos\theta) - eg\cos\theta$$

$$\boxed{L_z = -i\hbar\frac{\partial}{\partial\varphi} - eg} \tag{35}$$

(f) If $eg = \frac{\hbar}{2}$, then

$$L_z = \hbar\left(-i\frac{\partial}{\partial\varphi} - \frac{1}{2}\right) \tag{36}$$

To find the eigenvalues of this operator, note that any wave function must be single-valued upon taking $\varphi \longrightarrow \varphi + 2\pi$. Therefore, by Fourier analysis, any wave function must be a linear combination of functions of the form $e^{in\varphi}$, where $n$ is an integer. Applying the operator $L_z$ to such a function, we get

$$\begin{aligned} L_z\, e^{in\varphi} &= \hbar\left(-i\frac{\partial}{\partial\varphi} - \frac{1}{2}\right)e^{in\varphi} \\ &= \hbar\left(-i\,(in) - \frac{1}{2}\right)e^{in\varphi} \\ &= \hbar\left(n - \frac{1}{2}\right)e^{in\varphi} \\ &= \hbar\,\underbrace{\left(\frac{2n-1}{2}\right)}_{\text{half odd-integer}}\,e^{in\varphi} \end{aligned} \tag{37}$$

This means that $e^{in\varphi}$ is an eigenfunction of $L_z$ with eigenvalue $\hbar\left(\frac{2n-1}{2}\right)$. Thus, when $eg = \frac{\hbar}{2}$, the eigenvalues of $L_z$ are half-odd-integer multiples of $\hbar$.

This result implies that in the presence of this magnetic field, this spin-zero particle acts like a spin-$\frac{1}{2}$ particle, in that its $L_z$ angular momentum eigenvalues are half-odd-integer multiples of $\hbar$.