Chapter 13

Thermodynamics & Statistical Mechanics—Solutions

Solution 4.1. a) We first write down the number of available states using the binomial distribution:

$$g = \frac{N!}{n_0!\, n_1!}, \tag{13.1}$$

and then use Stirling's approximation to express the entropy, given by $S = k \ln g$, as

$$S = k[N \ln N - n_1 \ln n_1 - n_0 \ln n_0], \tag{13.2}$$

where k is the Boltzmann constant. We know that n_0 and n_1 satisfy the conditions

$$N = n_0 + n_1 \quad \text{and} \quad U = n_1 E. \tag{13.3}$$

Solving these for n_0 and n_1 and substituting into our equation for S gives

$$S = k\left[N \ln N - \frac{U}{E} \ln \frac{U}{E} - \left(N - \frac{U}{E}\right) \ln \left(N - \frac{U}{E}\right)\right]. \tag{13.4}$$

b) For a constant number of particles, the temperature can be found from

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N . \tag{13.5}$$

Using the expression for entropy (13.4) gives us the temperature:

$$T = \frac{E}{k \ln(EN/U - 1)} . \tag{13.6}$$

To find the range of n_0 for which $T < 0$, we switch variables from U and N to n_0 and n_1, using equation (13.3). This gives

$$\frac{1}{T} = \frac{k}{E}(\ln n_0 - \ln n_1) . \tag{13.7}$$

We can see that $T < 0$ when $n_0 < n_1$, so that the temperature is negative for $0 < n_0 < N/2$.

c) As the systems approach thermal equilibrium, ΔS_{total} must be greater than zero. We know that in each system, $\Delta S = \Delta Q/T$. Suppose system 1 has $T < 0$ and system 2 has $T > 0$. If heat flows to system 1 from system 2, then $\Delta Q_1 > 0$ and $\Delta Q_2 < 0$, implying that $\Delta S < 0$ in both subsystems. This cannot be true. Conversely, if heat flows from system 1 to system 2, $\Delta S > 0$ in both systems, which is allowed. Thus heat must flow from the system with negative temperature to the system with positive temperature. This makes sense, because most of the energy is in the system with negative temperature.

22. The plasma is electrically neutral, as a whole. Nevertheless local deviations in density appear. Consider the electrical potential $\phi(r)$ in the vicinity of a particular ion. The energy of another ion, of charge e, in that potential, is $e\phi(r)$. Therefore the density near the ion has the dependence

$$n(r) = ne^{-e\phi(r)/kT}. \tag{1}$$

The constant n must be the average density of the plasma, because the influence of the potential energy is expected to disappear as the thermal energy kT increases indefinitely. Each species of ion obeys an equation of the form (1) with density $n_\alpha(r)$ and charge e_α.

Another relation between $\phi(r)$ and n_α is provided by Poisson's equation relating the potential to the charge density:

$$\nabla^2\phi(r) = -4\pi \sum_\alpha e_\alpha n_\alpha. \tag{2}$$

On the assumption that the plasma is very hot, we may write, for Eq. (1),

$$n_\alpha(r) = n_\alpha\left[1 - \frac{e_\alpha\phi}{kT}\right].$$

We substitute in Eq. (2) and obtain the Helmholtz equation

$$\nabla^2\phi(r) = 4\pi\left\{\sum_\alpha \frac{n_\alpha e_\alpha^2}{kT}\right\}\phi \equiv k^2\phi,$$

which has the solution:

$$\phi(r) = e_{\text{ion}}\frac{e^{-kr}}{r}.$$

The effects of the electromagnetic interaction are therefore limited to a sphere of radius k^{-1} (Debye-Hückel radius).

12. From $\langle (E - \langle E\rangle)^2\rangle = \langle E^2 - 2E\langle E\rangle + \langle E\rangle^2\rangle$, we have $\langle (E - \langle E\rangle)^2\rangle = \langle E^2\rangle - \langle E\rangle^2$. In addition

$$\langle E\rangle = \frac{\sum E_n e^{-E_n/kT}}{\sum e^{-E_n/kT}} = -\frac{1}{Z}\frac{\partial Z}{\partial \theta},$$

where E_n are the energy states of the system,

$$Z = \sum e^{-E_n/kT} \quad \text{and} \quad \left(\frac{1}{\theta}\right) \equiv kT.$$

Similarly

$$\langle E^2\rangle = \frac{\sum E_n^2 e^{-E_n/kT}}{\sum e^{-E_n/kT}} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \theta^2};$$

thus

$$\langle E^2\rangle - \langle E\rangle^2 = \frac{\partial}{\partial\theta}\left(\frac{1}{Z}\frac{\partial Z}{\partial\theta}\right) = -\frac{\partial}{\partial\theta}\langle E\rangle,$$

but

$$\frac{\partial}{\partial\theta} = -kT^2\frac{\partial}{\partial T} \quad \text{and} \quad \frac{\partial\langle E\rangle}{\partial T} = C_v.$$

Finally one obtains $\langle (E - \langle E\rangle)^2\rangle = kT^2C_v$.

Now consider a macroscopic system with mean energy $\langle E\rangle$; then the fractional deviation in energy of the system is

$$\left[\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2}\right]^{1/2} = \left[\frac{kT^2 C_v}{\langle E\rangle^2}\right]^{1/2}.$$

To estimate the size of this number, one expects the energy $\langle E\rangle$ to be of the magnitude NkT (especially at high temperatures); then $C_v = Nk$, and we have

$$\left[\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2}\right]^{1/2} \approx N^{-1/2},$$

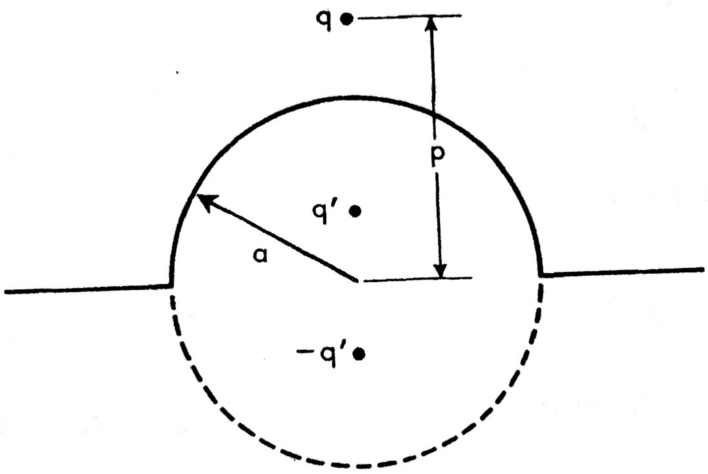
which is very small for systems of macroscopic size, i.e., $N \approx 10^{23}$.

7. The linearity of Maxwell's equations allows us to think of the magnetic field as arising from two current densities:

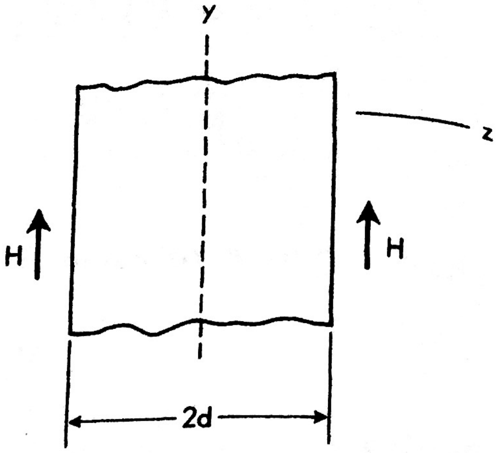
(1) A current density $j = I/\pi(b^2 - a^2)$, carried by the cylinder of radius b, and

(2) a current density $-j$ carried by a cylinder of radius a.

The sum of the current densities (1) and (2) is the current distribution of the bored-out cylinder. From Ampère's circuital law $\oint \mathbf{H}\cdot d\mathbf{l} = (4\pi/c)\int \mathbf{j}\cdot d\mathbf{A}$, one finds that (1) produces a magnetic field $H = 2Id/c(b^2 - a^2)$ at the center of the hole, while (2) produces no magnetic field at the center of the hole. The resultant magnetic field, H, is thus given by $H = 2Id/c(b^2 - a^2)$.



23. **J** is constant in time, since $\mathbf{E} = 0$ everywhere. Because the slab is infinite, **H** and **J** can be functions of z only. From Maxwell's equation



$$\mathbf{curl\ B} = \frac{4\pi \mathbf{J}}{c},$$

one obtains

$$\nabla^2 \mathbf{B} - \frac{4\pi \mathbf{B}}{\lambda} c^2 = 0,$$

where we have used the identity **curl curl** = **grad** div − ∇^2. A solution must be found obeying the boundary condition $\mathbf{B}(\pm d) = \mathbf{H}_0$. The solution desired is

$$\mathbf{B}(z) = \mathbf{H}_0 \frac{(e^{kz} + e^{-kz})}{(e^{kd} + e^{-kd})} = \mathbf{H}_0 \frac{\cosh(kz)}{\cosh(kd)},$$

where $k^2 = 4\pi/\lambda c^2$. The current density is determined from

$$\frac{4\pi \mathbf{J}}{c} = \mathbf{curl}\ B = -\hat{\mathbf{x}} \frac{\partial B}{\partial z} = -\hat{\mathbf{x}} k H_0 \frac{\sinh(kz)}{\cosh(kd)}.$$

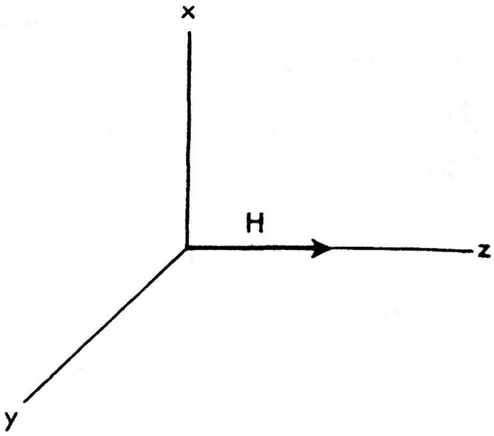
The field **H** has only the external currents as its sources and hence $\mathbf{H} = \mathbf{H}_0$ everywhere. This is no contradiction since $\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}$, where **M** is the magnetization per unit volume, and should satisfy $c\nabla \times \mathbf{M} = \mathbf{J}$. This is easily checked in this problem since $\nabla \times \mathbf{H} = 0$. Then $\nabla \times \mathbf{M} = (1/4\pi)\ \nabla \times \mathbf{B} = (1/4\pi)\ (4\pi\ \mathbf{J}/c) = \mathbf{J}/c$, and everything is consistent. One sees that **B** is confined to a region of the surface of skin depth $1/k$ and that a superconductor does not allow **B** to penetrate the interior.

pure imaginary).

21. We choose coordinates as in the figure. We ignore the magnetic field of the traveling wave, in comparison with that of the earth's field. The ionospheric electrons have the equation of motion

$$m\left(\frac{d\mathbf{v}}{dt}\right) = e\mathbf{E}e^{-i\omega t} + e\mathbf{v} \times \frac{\mathbf{H}}{c}\,.$$

We regard $\mathbf{E}$ as a superposition of right- and left-hand polarized beams, $E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i\omega t}$. Motion of the electrons is in the $z = 0$-plane, and must have the same time dependence as $\mathbf{E}$. This motivates putting $\mathbf{v}$ equal to $v_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i\omega t}$. Then



$$\mathbf{v} \times \mathbf{H} = \pm iHv_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{-i\omega t}$$

and

$$v_0(-im\omega \mp ieH/c) = eE_0,$$

hence

$$v_0 = \frac{ieE_0}{m(\omega \pm \omega_0)}, \qquad \text{where} \qquad \omega_0 = \frac{eH}{mc}\,.$$

The current density is $J_0 = Nev_0 = iNe^2E_0/m(\omega \pm \omega_0)$. But

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi \mathbf{J}}{c} = -\frac{i\omega}{c}\left[1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_0)}\right]\mathbf{E},$$

where $\omega_p^2 = 4\pi Ne^2/m$ is the square of the plasma frequency. On the other hand, in the absence of a current, but in a dielectric medium,

$$\text{curl } \mathbf{H} = \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t} = -\left(\frac{i\omega\epsilon}{c}\right)\mathbf{E}.$$

By comparison, $\epsilon_\pm = n_\pm^2 = 1 - \omega_p^2/\omega(\omega \pm \omega_0)$. Right- and left-hand polarized beams travel with different phase velocities c/n_+ and c/n_-, rotating $\mathbf{E}$. If at $z = 0$, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$ is in the x-direction, then after propagating a distance z,

$$\mathbf{E}_+ + \mathbf{E}_- = E_0\{\hat{\mathbf{x}}[e^{i\omega((n_+z/c)-t)} + e^{i\omega((n_-z/c)-t)}] + i\hat{\mathbf{y}}[e^{i\omega((n_+z/c)-t)} - e^{i\omega((n_-z/c)-t)}]\},$$

which implies a rotation through an angle θ such that

$$\tan\theta = \frac{i(e^{i\omega n_+z/c} - e^{i\omega n_-z/c})}{(e^{i\omega n_+z/c} + e^{i\omega n_-z/c})}.$$

Putting $n_+ - n_- = \delta n$, one finds $\tan\theta = -\tan(\omega\delta nz/2c)$ or $\theta = -\omega z\delta n/2c$.

$$I = I_0[(1 - v)^2/(1 + v)^2].$$

42. In the rest frame the force per unit length F is given by $\mathbf{F} = \lambda\mathbf{E}$, where $\mathbf{E}$ is the electric field at one wire produced by the other. This is easily found from Gauss' Law,

$$\int \mathbf{E}\cdot d\mathbf{A} = 4\pi \text{ (charge enclosed)}.$$

Thus $E = 2\lambda/a$, and the force $F = 2\lambda^2/a$ (repulsive).

In a frame in which the rods are seen to move with velocity v, there is a magnetic field $\mathbf{B} = v \times \mathbf{E}'/c$, in addition to the electric field $\mathbf{E}'$. The total force per unit length $\mathbf{F}'$ is then

$$\mathbf{F}' = \lambda'\left(\mathbf{E}' + \frac{\mathbf{v} \times \mathbf{B}'}{c}\right) = \lambda'\left(1 - \frac{v^2}{c^2}\right)\mathbf{E}'.$$

However, $\mathbf{E}' = 2\lambda'/a$ where λ' is the charge as seen in the new frame ($\lambda' = \gamma\lambda$ because of the Lorentz contraction of lengths). Thus

$$F' = \frac{2(\lambda')^2(1 - v^2/c^2)}{a} = \frac{2\lambda^2}{a} = F.$$

The fact that $F' = F$ may be seen easily by an alternate argument. If in its rest frame, one of the rods is allowed to move under the action of the force FL on it, it would gain momentum $dp = FL\,dt$, while in the frame in which the rods move, the gain is $dp' = F'L'dt'$. But $dp = dp'$ because momenta normal to the direction of a Lorentz transformation are invariant under such a transformation, and $dt' = \gamma dt$. Hence $LF = \gamma F'L'$. In addition $L' = L/\gamma$, due to Lorentz contraction; hence $F = F'$.