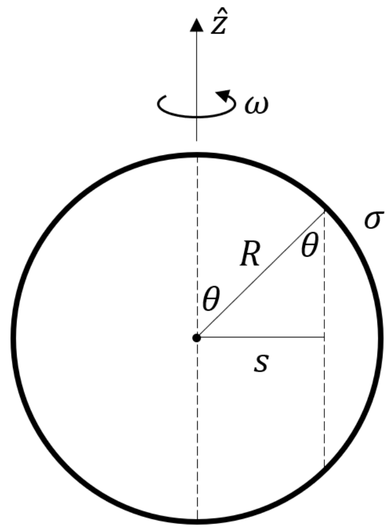
11. (Electromagnetism)

A thin spherical shell of radius R has a constant surface charge density σ and is rotating with angular frequency ω. Find the magnetic field inside and outside the shell.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*



This is a magnetostatic potential theory problem. Inside and outside the shell, there is no volume current, so

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} = 0 \tag{61}$$

Since the curl of $\mathbf{B}$ is zero, we can write $\mathbf{B}$ inside and outside the shell as the gradient of a magnetic scalar potential $\psi(\mathbf{r})$ (adding a negative sign to strengthen the analogy with the electric potential):

$$\mathbf{B} = -\nabla\psi \tag{62}$$

By the no-magnetic-monopoles Maxwell's equation,

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{B} \\ &= \nabla \cdot (-\nabla\psi) \\ 0 &= -\nabla^2\psi \quad \text{inside and outside the shell} \end{aligned} \tag{63}$$

Therefore, Laplace's equation holds inside and outside the shell. All that remains is to determine the boundary condition on the shell, which comes from the surface current on the shell. The surface current $\mathbf{K}$ is defined by

$$\mathbf{K} = \sigma\mathbf{v} \tag{64}$$

where σ is the surface charge density and $\mathbf{v}$ is the velocity of the charges. In this case, the charges are spinning about the $\mathbf{z}$-direction with angular frequency ω. Therefore,

$$\mathbf{v} = \omega s\,\hat{\varphi} \quad \text{where } s \text{ is the distance to the } \hat{\mathbf{z}}\text{-axis} \tag{65}$$

From the diagram above, we can relate s to the radius of the shell R and the polar angle θ:

$$s = R\sin\theta \tag{66}$$

Therefore, the velocity of the charges is

$$\mathbf{v} = \omega R \sin\theta\,\hat{\varphi} \tag{67}$$

Plugging this into (64), we get

$$\mathbf{K} = \sigma\omega R\sin\theta\,\hat{\varphi} \quad \text{on the shell} \tag{68}$$

The boundary conditions for the magnetic field on the shell are

$$\hat{\mathbf{n}}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0 \quad \text{and} \quad \hat{\mathbf{n}}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\mathbf{K} \tag{69}$$

where the unit vector $\hat{\mathbf{n}}$ points from region 1 to region 2. In this case, if region 1 is inside the shell and region 2 is outside the shell, these boundary conditions boil down to

$$B_r\big|_{r\to R^-} = B_r\big|_{r\to R^+} \quad \text{and} \quad \hat{\mathbf{r}}\times\left(\mathbf{B}\big|_{r\to R^+} - \mathbf{B}\big|_{r\to R^-}\right) = \mu_0\sigma\omega R\sin\theta\,\hat{\varphi} \tag{70}$$

Thus, we have a full mathematical setup for a boundary value problem for the magnetic scalar potential $\psi(\mathbf{r})$:

$$\begin{aligned}
\nabla^2\psi(\mathbf{r}) = 0 \quad \text{for} \quad r \neq R & \qquad \text{(Laplace's equation)} \\
-\nabla\psi(\mathbf{r}) = \mathbf{B} & \qquad \text{(relation between potential and magnetic field)} \\
B_r\big|_{r\to R^-} = B_r\big|_{r\to R^+} & \qquad \text{(boundary cond. for normal component of } \mathbf{B}\text{)} \\
\left(\mathbf{B}\big|_{r\to R^+} - \mathbf{B}\big|_{r\to R^-}\right) = \mu_0\sigma\omega R\sin\theta\,\hat{\varphi} & \qquad \text{(boundary cond. for parallel component of } \mathbf{B}\text{)}
\end{aligned} \tag{71}$$

To solve this problem, use the general solution to Laplace's equation in spherical coordinates with azimuthal symmetry:

$$\psi(\mathbf{r}) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{72}$$

Here, $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, and so on.

To simplify the algebra, note that

In potential theory problems, only the multipole moments in the setup will be present in the solution.

In this case, the only multipole moment in the setup comes from the boundary condition for the parallel component of $\mathbf{B}$: $\left(\mathbf{B}\big|_{r\to R^+} - \mathbf{B}\big|_{r\to R^-}\right) = \mu_0\sigma\omega R\sin\theta\,\hat{\varphi}$. This tells us that the magnetic field $\mathbf{B}$ has a θ dependence given by $\sin\theta$. Since $\mathbf{B}$ is the gradient of ψ, this implies that ϕ has a θ dependence given by $\cos\theta$. Since $P_1(\cos\theta) = \cos\theta$, ψ therefore has a contribution only from the first multipole moment, $\ell = 1$.

Our ansatz for $\psi(\mathbf{r})$ is therefore

$$\psi(\mathbf{r}) = \begin{cases} \left(A'r + \frac{C'}{r^2}\right)\cos\theta & \text{for} \quad r < R \\ \left(D'r + \frac{B'}{r^2}\right)\cos\theta & \text{for} \quad R < r \end{cases} \tag{73}$$

Any part of the magnetic scalar potential that goes like $\frac{1}{r^2}\cos\theta$ as $r \to 0$ produces an infinite magnetic field at $r = 0$. The magnetic field is finite everywhere, so we must have $C' = 0$.

Similarly, any part of the magnetic scalar potential that goes like $r\cos\theta = z$ as $r \to \infty$ produces a constant magnetic field in the $\hat{\mathbf{z}}$-direction. There is no such constant magnetic field infinitely far away from the shell, so we must have $D' = 0$.

This gives us

$$\psi(\mathbf{r}) = \begin{cases} A'r\cos\theta & \text{for} \quad r < R \\ \frac{B'}{r^2}\cos\theta & \text{for} \quad R < r \end{cases} \tag{74}$$

A' and B' do not have the same units, and this problem has only one length scale (R), so let's redefine the constants so that they do have the same units:

$$\psi(\mathbf{r}) = \begin{cases} A\frac{r}{R}\cos\theta & \text{for} \quad r < R \\ B\frac{R^2}{r^2}\cos\theta & \text{for} \quad R < r \end{cases} \tag{75}$$

Using the gradient operator in polar coordinates, $\nabla = \hat{\mathbf{r}}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial\theta}$, we get that

$$\mathbf{B}(r,\theta) = -\nabla\psi(r,\theta) = \begin{cases} -\frac{A}{R}\cos\theta\,\hat{\mathbf{r}} + \frac{A}{R}\sin\theta\,\hat{\theta} & \text{for} \quad r < R \\ 2B\frac{R^2}{r^3}\cos\theta\,\hat{\mathbf{r}} + B\frac{R^2}{r^3}\sin\theta\,\hat{\theta} & \text{for} \quad R < r \end{cases} \tag{76}$$

Now to apply the boundary conditions at $r = R$. The boundary condition for the normal component of $\mathbf{B}$ at the shell is

$$B_r\big|_{r\to R^-} = B_r\big|_{r\to R^+}$$

Plugging in the ansatz (76), we get

$$\begin{aligned} -\frac{A}{R}\cos\theta &= 2B\frac{R^2}{R^3}\cos\theta \\ A &= -2B \end{aligned} \tag{77}$$

The boundary condition for the parallel component of $\mathbf{B}$ at the shell is

$$\hat{\mathbf{r}} \times \left(\mathbf{B}\big|_{r\to R^+} - \mathbf{B}\big|_{r\to R^-}\right) = \mu_0\sigma\omega R\sin\theta\,\hat{\varphi}$$

Plugging in the parallel component of the ansatz (76), we get

$$\begin{aligned} \hat{\mathbf{r}} \times \left(B\frac{R^2}{R^3}\sin\theta\,\hat{\theta} - \frac{A}{R}\sin\theta\,\hat{\theta}\right) &= \mu_0\sigma\omega R\sin\theta\,\hat{\varphi} \\ \frac{B-A}{R}\sin\theta\left(\hat{\mathbf{r}}\times\hat{\theta}\right) &= \mu_0\sigma\omega R\sin\theta\,\hat{\varphi} \\ \frac{B-A}{R}\sin\theta\,\hat{\varphi} &= \mu_0\sigma\omega R\sin\theta\,\hat{\varphi} \quad \text{since } \hat{\mathbf{r}}\times\hat{\theta} = \hat{\varphi} \\ B - A &= \mu_0\sigma\omega R^2 \end{aligned} \tag{78}$$

(77) and (78) are two simultaneous linear equations for A and B. We can use them to solve for both constants:

$$\begin{aligned} B - A &= \mu_0\sigma\omega R^2 \quad \text{by (78)} \\ B - (-2B) &= \mu_0\sigma\omega R^2 \quad \text{by (77)} \\ B &= \frac{\mu_0\sigma\omega R^2}{3} \end{aligned} \tag{79}$$

$$\begin{aligned} A &= -2B \quad \text{by (77)} \\ A &= -\frac{2\mu_0\sigma\omega R^2}{3} \end{aligned} \tag{80}$$

Plugging this into our ansatz (76), we get an expression for the magnetic field

$$\begin{aligned} \mathbf{B}(r,\theta) &= \begin{cases} -\frac{A}{R}\cos\theta\,\hat{\mathbf{r}} + \frac{A}{R}\sin\theta\,\hat{\theta} & \text{for} \quad r < R \\ 2B\frac{R^2}{r^3}\cos\theta\,\hat{\mathbf{r}} + B\frac{R^2}{r^3}\sin\theta\,\hat{\theta} & \text{for} \quad R < r \end{cases} \\ &= \begin{cases} \frac{2\mu_0\sigma\omega R}{3}\cos\theta\,\hat{\mathbf{r}} - \frac{2\mu_0\sigma\omega R}{3}\sin\theta\,\hat{\theta} & \text{for} \quad r < R \\ \frac{2\mu_0\sigma\omega R^4}{3}\frac{1}{r^3}\cos\theta\,\hat{\mathbf{r}} + \frac{\mu_0\sigma\omega R^4}{3}\frac{1}{r^3}\sin\theta\,\hat{\theta} & \text{for} \quad R < r \end{cases} \end{aligned}$$

$$\mathbf{B}(r,\theta) = \begin{cases} \frac{2\mu_0\sigma\omega R}{3}\left(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta}\right) & \text{for} \quad r < R \\ \frac{\mu_0\sigma\omega R^4}{3}\frac{1}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right) & \text{for} \quad R < r \end{cases} \tag{81}$$

Since $\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\theta} = \hat{\mathbf{z}}$, we can simplify this to get

$$\boxed{\mathbf{B}(r,\theta) = \begin{cases} \frac{2\mu_0\sigma\omega R}{3}\hat{\mathbf{z}} & \text{for} \quad r < R \\ \frac{\mu_0\sigma\omega R^4}{3}\frac{1}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right) & \text{for} \quad R < r \end{cases}} \tag{82}$$

The magnetic field inside the shell is uniform. It is nice to observe that the magnetic field outside the shell is the magnetic field of a magnetic dipole in the $\hat{\mathbf{z}}$-direction:

$$\mathbf{B}(r,\theta)|_{r>R} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right) \quad \text{for} \quad m = \frac{4\pi}{3}\sigma\omega R^4 \tag{83}$$