

Quantum Mechanics

Basic formalism

$$\text{Canonical relations: } p = \frac{\hbar}{i} \frac{d}{dx}; \quad [x, p] = i\hbar$$

$$\text{Schrödinger equation: } i\hbar \frac{d|\psi_S\rangle}{dt} = H|\psi_S(t)\rangle; \quad U(t) = e^{-iHt/\hbar} \text{ for time-indep } H$$

$$\text{Heisenberg picture: } \mathcal{O}_H(t) = U^\dagger(t) \mathcal{O}_S U(t); \quad \frac{d\mathcal{O}_H}{dt} = \frac{1}{i\hbar} [\mathcal{O}_H(t), H_H(t)] + \frac{\partial \mathcal{O}}{\partial t} \Big|_H$$

$$\text{Plane-wave normalization: } \langle x|p\rangle = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$$

$$\text{Commutators: } [x, G(p)] = i\hbar \frac{\partial G}{\partial p}; \quad [p, F(x)] = -i\hbar \frac{\partial F}{\partial x}$$

$$\text{BCH lemma: } e^{iX} Y e^{-iX} = Y + i[X, Y] - \frac{1}{2!} [X, [X, Y]] - \frac{i}{3!} [X, [X, [X, Y]]] + \dots$$

$$\text{BCH formula: } e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\dots}$$

Quantum harmonic oscillator

$$\text{Hamiltonian: } H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 = \frac{1}{2} \hbar\omega (P^2 + X^2) = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

$$\text{Unitless coordinates: } X = x \left(\frac{m\omega}{\hbar} \right)^{1/2} = \frac{1}{\sqrt{2}} (a + a^\dagger)$$

$$P = \frac{p}{(m\omega\hbar)^{1/2}} = -\frac{i}{\sqrt{2}} (a - a^\dagger)$$

$$\text{Raising/lowering operators: } a = \frac{X + iP}{\sqrt{2}}; \quad a^\dagger = \frac{X - iP}{\sqrt{2}}; \quad [a, a^\dagger] = 1$$

$$\text{Eigenstates: } H|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle; \quad a|n\rangle = \sqrt{n} |n-1\rangle; \quad a^\dagger|n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\text{Ground state wave function: } \psi_0(X) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-X^2/2}$$

Time-independent nondegenerate perturbation theory

First-order energy correction: $E_n^{(1)} = \langle n^{(0)} | V | n^{(0)} \rangle$

Second-order energy correction: $E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k^{(0)} | V | n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$

First-order ket correction: $|n^{(1)}\rangle = \sum_{k \neq n} |k^{(0)}\rangle \frac{\langle k^{(0)} | V | n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$

General ket: $(H_0 - E_n^{(0)}) |n^{(j)}\rangle = (E_n^{(1)} - V) |n^{(j-1)}\rangle + E_n^{(2)} |n^{(j-2)}\rangle + \dots + E_n^{(j)} |n^{(0)}\rangle$

General energy: $E_n^{(j)} = \langle n^{(0)} | V | n^{(j-1)} \rangle$

Time-dependent perturbation theory

Interaction picture: $|\psi_I(t)\rangle = U_0^\dagger(t) |\psi_S(t)\rangle$; $\mathcal{O}_I(t) = U_0^\dagger(t) \mathcal{O}_S(t) U_0(t)$

Time evolution: $i\hbar \frac{d|\psi_I\rangle}{dt} = V_I(t) |\psi_I(t)\rangle$; $\frac{d\mathcal{O}_I}{dt} = \frac{1}{i\hbar} [\mathcal{O}_I(t), H_0(t)] + \frac{\partial \mathcal{O}}{\partial t} \Big|_I$

First-order transition amplitude: $\langle m | U(t) | n \rangle = \delta_{mn} - \frac{i}{\hbar} \int_{t_0}^t dt' e^{i(E_m - E_n)t'/\hbar} \langle m | V_S(t') | n \rangle$

Fermi's golden rule (steplike): $\frac{dP_m}{dt} = \frac{2\pi}{\hbar} |V_{mn}|^2 \delta(E_m - E_n) \approx \frac{2\pi}{\hbar^2} \rho(E_n) \langle |V_{mn}|^2 \rangle$

Fermi's golden rule (harmonic): $\frac{dP_m^\pm}{dt} = \frac{2\pi}{\hbar} |V_{mn}|^2 \delta(E_m - (E_n \pm \hbar\omega)) \approx \frac{2\pi}{\hbar^2} \rho(E_n \pm \hbar\omega) \langle |V_{mn}|^2 \rangle$

WKB approximation

Quantization (free on both sides): $\int_{x_L}^{x_R} dx \sqrt{2m(E - V(x))} = \pi\hbar \left(n + \frac{1}{2} \right), \quad n \geq 0$

Quantization (hard wall): $\int_0^{x_R} dx \sqrt{2m(E - V(x))} = \pi\hbar \left(n + \frac{3}{4} \right), \quad n \geq 0$

Classically allowed region: $\psi(x) \propto \frac{1}{\sqrt{k(x)}} \exp\left(\pm i \int^x dx' k(x')\right), \quad E = \frac{\hbar^2 k^2}{2m} + V(x)$

Classically forbidden region: $\psi(x) \propto \frac{1}{\sqrt{\kappa(x)}} \exp\left(\pm i \int^x dx' \kappa(x')\right), \quad E = \frac{\hbar^2 \kappa^2}{2m} - V(x)$

Scattering

General formalism

$$T\text{-matrix: } T = \frac{V}{1 - (E - H_0 + i\eta)^{-1}V} = V + V(E - H_0 + i\eta)^{-1}T$$

$$\text{Lippmann-Schwinger: } |f\rangle = |i\rangle + (E - H_0 + i\eta)^{-1}T|i\rangle = |i\rangle + (E - H_0 + i\eta)^{-1}V|f\rangle$$

$$\text{Green's function: } \frac{\hbar^2}{2m} \langle \mathbf{r} | (E - H_0 + i\eta)^{-1} | \mathbf{r}' \rangle = -\frac{1}{4\pi} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

$$\text{Scattering state (far-field approximation): } f(\mathbf{r}) = \frac{1}{L^{3/2}} \left(e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}', \mathbf{k}) \frac{e^{ikr}}{r} \right)$$

$$\text{Scattering amplitude: } f(\mathbf{k}', \mathbf{k}) = -\frac{mL^3}{2\pi\hbar^2} \langle \mathbf{k}' | V | f \rangle$$

$$\text{Differential cross-section: } \frac{d\sigma}{d\Omega} = |f(\mathbf{k}', \mathbf{k})|^2$$

$$\text{Total cross-section: } \sigma = \int d\Omega \frac{d\sigma}{d\Omega}$$

$$\text{Optical theorem: } \sigma = \frac{4\pi}{k} \text{Im} f(\mathbf{k}, \mathbf{k})$$

Born approximation

$$\text{Momentum transfer: } \mathbf{q} = \mathbf{k}' - \mathbf{k}; \quad |\mathbf{q}| = 2k \sin \frac{\theta}{2}$$

$$\text{First Born approximation: } f(\mathbf{k}', \mathbf{k}) \approx -\frac{m}{2\pi\hbar^2} \int d^3r' V(\mathbf{r}') e^{-i\mathbf{q}\cdot\mathbf{r}'}$$

$$\text{First Born approximation (isotropic } V): f(\mathbf{k}', \mathbf{k}) = -\frac{2m}{q\hbar^2} \int_0^\infty dr r \sin(qr) V(r)$$

Partial waves for isotropic potentials

$$\text{Radial Schrödinger equation: } -\frac{\hbar^2}{2m} \frac{d^2 u_\ell}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2} u_\ell(r) + V(r) u_\ell(r) = E u_\ell(r)$$

$$\text{Boundary conditions: } u_\ell(r) = r R_\ell(r) \quad \text{and} \quad u_\ell \rightarrow 0 \text{ as } r \rightarrow 0$$

$$\text{Wave functions in free-particle zone: } R_\ell(r) \propto \cos \delta_\ell j_\ell(kr) - \sin \delta_\ell n_\ell(kr)$$

$$R_\ell(r) \propto e^{2i\delta_\ell} h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)$$

$$u_0(r) \propto \cos \delta_0 \sin kr + \sin \delta_0 \cos kr = \sin(kr + \delta_0)$$

$$\text{Spherical Bessel functions: } j_\ell(kr) \sim \frac{\sin(kr - \ell\pi/2)}{kr}; \quad n_\ell(kr) \sim -\frac{\cos(kr - \ell\pi/2)}{kr}$$

Spherical Hankel functions: $h_\ell^{(1)}(kr) = j_\ell(kr) + in_\ell(kr); \quad h_\ell^{(2)}(kr) = j_\ell(kr) - in_\ell(kr)$

Scattering amplitude: $f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell P_\ell(\cos \theta) \quad \text{for} \quad f_\ell = \frac{e^{2i\delta_\ell} - 1}{2ik} = \frac{e^{i\delta_\ell} \sin \delta_\ell}{k}$

Cross-section: $\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell$

Angular momentum

Commutation relations: $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k; \quad J_\pm = J_x \pm iJ_y; \quad [J_+, J_-] = 2\hbar J_z$

Eigenvalues: $\mathbf{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle; \quad J_z |j, m\rangle = \hbar m |j, m\rangle$

Raising and lowering operators: $J_\pm |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$

Wigner-Eckart theorem: $\langle \alpha'; j', m' | T_q^{(k)} | \alpha; j, m \rangle = \langle j, k; m, q | j, k; j', m' \rangle \cdot \frac{C}{\sqrt{2j+1}}$

Orbital angular momentum

Definition: $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Spherical harmonics: $Y_0^0 = \sqrt{\frac{1}{4\pi}}$

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}; \quad Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} e^{\pm i\varphi} \sin \theta = \mp \sqrt{\frac{3}{8\pi}} \frac{x \pm iy}{r}$

$Y_\ell^0 = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta); \quad Y_\ell^m(-\mathbf{r}) = (-1)^\ell Y_\ell^m(\mathbf{r}); \quad Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$

Hydrogen atom eigenstates: $E_n = -\frac{m|e|^4}{2(4\pi\epsilon_0)^2 \hbar^2} \frac{1}{n^2} \approx \frac{-13.6 \text{ eV}}{n^2} \quad \text{for} \quad m = \frac{m_e m_p}{m_e + m_p} \approx m_e$

Bohr radius and fine structure constant: $a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m|e|^2}; \quad \alpha = \frac{|e|^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}$

Hydrogen atom radial wavefunctions: $R_{10}(r) = \frac{2}{a_0^{3/2}} e^{-r/a_0}$

$R_{20}(r) = 2 \left(\frac{1}{2a_0} \right)^{3/2} \left(1 - \frac{r}{2a_0} \right) e^{-r/(2a_0)}; \quad R_{21}(r) = \frac{1}{\sqrt{3}} \left(\frac{1}{2a_0} \right)^{3/2} \frac{r}{a_0} e^{-r/(2a_0)}$

If nuclear charge is equal to $Z|e|$, replace a_0 by a_0/Z

Spin angular momentum

Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Bloch sphere: $|\theta, \varphi\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + e^{i\varphi} \sin \frac{\theta}{2} |\downarrow\rangle$

Spin 1/2 in a magnetic field (SI): $H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\gamma \frac{\hbar}{2} \boldsymbol{\sigma} \cdot \mathbf{B} = -\frac{g}{2} \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$

Gyromagnetic ratio; Bohr magneton (SI): $\gamma = -\frac{g|e|}{2m}; \quad \mu_B = \frac{|e|\hbar}{2m}; \quad g_{\text{electron}} \approx 2$

Classical Mechanics

Newtonian mechanics

Nonrelativistic Doppler effect: $f_r = \frac{c \pm v_r}{c \pm v_s} f_s$

$+v_r$ if receiver moves toward source; $+v_s$ if source moves away from receiver

Centrifugal; Coriolis; Euler forces: $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}); -2m\boldsymbol{\omega} \times \dot{\mathbf{r}}; -m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$

Formation rules

	Lagrangian	Hamiltonian
Nonrelativistic Newtonian mass:	$+\frac{1}{2}m \dot{\mathbf{r}} ^2$	$+\frac{ \mathbf{p} ^2}{2m}$
Scalar potential:	$-V(\mathbf{r}, t)$	$+V(\mathbf{r}, t)$
Vector potential:	$+q\dot{\mathbf{r}} \cdot \mathbf{A}$	replace $\mathbf{p}$ by $\mathbf{p} - q\mathbf{A}$
Holonomic constraint:	$+\lambda(\text{constraint})$	$-\lambda(\text{constraint})$
Special relativistic mass:	$-mc^2\sqrt{1 - \frac{ \dot{\mathbf{r}} ^2}{c^2}}$	$+\sqrt{(mc^2)^2 + (\mathbf{p} c)^2}$

Lagrangian mechanics

Lagrangians

Stationary action principle: $\delta \left(\int dt L(q, \dot{q}, t) \right) = 0$

Euler-Lagrange equation: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$

Canonical momenta: $p = \frac{\partial L}{\partial \dot{q}}$

Energy: $E = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$

Lagrangian densities

Stationary action principle: $\delta \left(\int dt d^3x \mathcal{L}(\phi, \dot{\phi}, \nabla\phi, t) \right) = 0$

Euler-Lagrange equation: $\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

Energy continuity equation: $\frac{\partial}{\partial t} \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \frac{\partial \phi}{\partial t} - \mathcal{L} \right] + \nabla \cdot \left[\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} \frac{\partial \phi}{\partial t} \right] = 0 \quad \text{if} \quad \frac{\partial \mathcal{L}}{\partial t} = 0$

Energy-momentum tensor: $\partial_\mu T^\mu_\nu = 0 \quad \text{for} \quad T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$

Hamiltonian mechanics

$$\text{Hamiltonian: } H(p, q, t) = p\dot{q} - L = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$$

$$\text{Hamilton's equations of motion: } \dot{p} = -\frac{\partial H}{\partial q}; \quad \dot{q} = \frac{\partial H}{\partial p}$$

$$\text{Hamilton-Jacobi equation: } H\left(q, \frac{\partial W(q, P)}{\partial q}\right) = E$$

$$\text{Poisson bracket: } \{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

$$\text{Adiabatic invariant: } J \propto \oint p \, dq$$

Canonical transformations

$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i};$	$P_i = -\frac{\partial F_1}{\partial Q_i};$	$H' = H + \frac{\partial F_1}{\partial t}$
$F_2(q, P, t)$	$p_i = \frac{\partial F_2}{\partial q_i};$	$Q_i = \frac{\partial F_2}{\partial P_i};$	$H' = H + \frac{\partial F_2}{\partial t}$
$F_3(p, Q, t)$	$q_i = -\frac{\partial F_3}{\partial p_i};$	$P_i = -\frac{\partial F_3}{\partial Q_i};$	$H' = H + \frac{\partial F_3}{\partial t}$
$F_4(p, P, t)$	$q_i = -\frac{\partial F_4}{\partial p_i};$	$Q_i = \frac{\partial F_4}{\partial P_i};$	$H' = H + \frac{\partial F_4}{\partial t}$

Fluids

$$\text{Euler hydrodynamics: } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial S}{\partial t} + (\mathbf{v} \cdot \nabla)S = 0; \quad \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla P(\rho, s)$$

$$\text{Speed of sound: } c = \sqrt{\frac{\partial P}{\partial \rho}}$$

$$\text{Bernoulli's law: } \frac{v^2}{2} + gz + \frac{P}{\rho} = \text{constant}$$

$$\text{Kelvin circulation theorem: } \frac{D}{Dt} \oint \mathbf{v} \cdot d\boldsymbol{\ell} = 0 \quad \text{for} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

$$\text{Vortex: } \mathbf{v} = \frac{\kappa}{r} \hat{\boldsymbol{\theta}}$$

Special relativity

Sign convention: $(-1, +1, +1, +1)$

Relativistic factors: $\beta = \frac{v}{c}; \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$

Lorentz boost: $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$

Metric invariance: $\Lambda \eta \Lambda^T = \eta$

Time dilation; Lorentz contraction: $\Delta t = \gamma \Delta \tau_{\text{rest}}; \quad \ell = \frac{\ell_{\text{rest}}}{\gamma}$

Classical mechanics

Four-vectors: $x^\mu = (ct, \mathbf{x}); \quad p^\mu = (E, c\mathbf{p}); \quad F^\mu = \left(\frac{dE}{d\tau}, c \frac{d\mathbf{p}}{d\tau} \right); \quad \partial^\mu = \left(-\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$

Velocity transformation: $u' = \frac{u - v}{1 - uv/c^2}$

Energy and momentum: $E = \gamma mc^2; \quad \mathbf{p} = \gamma m\mathbf{v}; \quad \mathbf{v} = c^2 \frac{\mathbf{p}}{E}$

Energy-momentum relationship: $E = \sqrt{(mc^2)^2 + (|\mathbf{p}|c)^2}$

Relativistic Doppler effect: $\frac{\lambda_r}{\lambda_s} = \frac{f_s}{f_r} = \sqrt{\frac{1 + \beta}{1 - \beta}} \quad (\beta > 0 \text{ if source and receiver move apart})$

Electromagnetism

Four-vectors: $J^\mu = (c\rho, \mathbf{J}); \quad A^\mu = (\phi/c, \mathbf{A})$

Field transformations: $\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}; \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$

$\mathbf{E}'_{\perp} = \gamma (\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}); \quad \mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E} \right)$

Electromagnetic tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$

Four-vector Maxwell's equations: $(\partial_\mu \partial^\mu) A^\nu = \partial_\mu F^{\mu\nu} = \mu_0 J^\nu$

Statistical Mechanics

Elementary thermodynamics

$$\text{Euler relation: } E = TS - PV + \mu N$$

$$\text{Energy: } dE = \delta Q - \delta W = TdS - PdV + \mu dN$$

$$\text{Helmholtz free energy: } F = E - TS; \quad dF = -SdT - PdV + \mu dN$$

$$\text{Gibbs free energy: } G = E - TS + PV; \quad dG = -SdT + VdP + \mu dN$$

$$\text{Enthalpy: } H = E + PV; \quad dH = TdS + VdP + \mu dN$$

$$\text{Grand potential: } \Phi = -PV = E - TS - \mu N$$

$$\text{Maxwell: } \left. \frac{\partial T}{\partial V} \right|_S = - \left. \frac{\partial P}{\partial S} \right|_V; \quad \left. \frac{\partial T}{\partial P} \right|_S = \left. \frac{\partial V}{\partial S} \right|_P; \quad \left. \frac{\partial S}{\partial V} \right|_T = \left. \frac{\partial P}{\partial T} \right|_V; \quad \left. \frac{\partial S}{\partial P} \right|_T = - \left. \frac{\partial V}{\partial T} \right|_P$$

Stat mech definitions

$$\text{Entropy and temperature: } S = k \ln \Omega; \quad \frac{1}{T} = \frac{\partial S}{\partial E}; \quad \beta = (kT)^{-1}$$

$$\text{Stirling's formula: } N! \approx N^N e^{-N} \sqrt{2\pi N}; \quad \ln(N!) \approx N \ln N - N$$

$$\text{Heat capacity: } C_X = T \left. \frac{\partial S}{\partial T} \right|_X$$

$$\text{Poisson distribution: } P(N) = \frac{e^{-\lambda} \lambda^N}{N!}$$

Ensembles

$$\text{Partition function: } Z = \sum_r e^{-\beta E_r}; \quad E = - \frac{\partial \ln Z}{\partial \beta} = kT^2 \frac{\partial \ln Z}{\partial T}; \quad F = -kT \ln Z$$

$$\text{Grand partition function: } Q = \sum_r e^{-\beta(E_r - \mu N_r)}; \quad N = kT \frac{\partial \ln Q}{\partial \mu}$$

Monatomic ideal gas

$$\text{Partition function: } Z = \frac{V^N}{N! \lambda^{3N}} \quad \text{for } \lambda = \left(\frac{h^2}{2\pi m kT} \right)^{1/2}$$

$$\text{Equation of state: } PV = NkT = nRT$$

$$\text{Energy: } E = \frac{3}{2} NkT$$

Free energy: $F = -NkT \left(\ln \left[\frac{V}{N} \frac{1}{\lambda^3} \right] + 1 \right)$

Chemical potential: $\mu = -kT \ln \left[\frac{V}{N} \frac{1}{\lambda^3} \right] < 0$

Adiabatic expansion: $PV^\gamma = \text{constant}$ for $\gamma = \frac{C_P}{C_V} = \frac{Nk}{C_V} + 1$

Maxwell velocity distribution: $P(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT} \right)$

Heat engines

Conservation of energy: $Q_H - W = Q_C$

Efficiency of a heat engine: $\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H} < 1$; $\eta = 1 - \frac{T_C}{T_H}$ for a Carnot cycle

Quantum gases

Occupation numbers

Fermi-Dirac distribution: $n_k^{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$

Bose-Einstein distribution: $n_k^{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}$

Maxwell-Boltzmann distribution: $n_k^{MB}(\epsilon) = e^{-\beta(\epsilon_k - \mu)}$

Fermi gases

Equation of state: $P = \frac{2}{3} \frac{E}{V}$

Fermi energy (nonrelativistic 3-D electron gas): $E_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3}$

Energy at $T = 0$ (nonrelativistic 3-D electron gas): $E = \frac{3}{5} N E_F$

Bose gases

Equation of state: $P = \frac{2}{3} \frac{E}{V}$

Critical temperature: $N = gV \int \frac{d^3p}{h^3} \frac{1}{e^{p^2/(2mkT_c)} - 1}$

Number of particles in ground state: $\frac{N(\epsilon=0)}{N} = 1 - \left(\frac{T}{T_c} \right)^{3/2}$

Photon gases and blackbody radiation

$$E = \frac{4}{c}\sigma VT^4; \quad P = \frac{1}{3}\frac{E}{V}; \quad \frac{\text{Power radiated}}{\text{Area}} = \sigma T^4 \quad \text{for} \quad \sigma = \frac{\pi^2 k^4}{60\hbar^3 c^2}$$

Debye model

$$\omega_D = c_s \left(6\pi^2 \frac{N}{V} \right)^{1/3}$$

Other examples

Joule-Thomson process (throttling)

$$\Delta H = 0; \quad \mu = \left. \frac{\partial T}{\partial P} \right|_H = -\frac{1}{C_P} \left(T \left. \frac{\partial S}{\partial P} \right|_T + V \right) = -\frac{1}{C_P} V(1 - \alpha T) \quad \text{for} \quad \alpha = \left. \frac{1}{V} \frac{\partial V}{\partial T} \right|_P$$

Clausius-Claperyon equation

$$\frac{dP}{dT} = \frac{L}{T\Delta v} \quad L \text{ latent heat of vaporization per particle, } v \text{ volume per particle}$$

Magnetization

$$M = kT \frac{\partial \ln Z}{\partial B} = - \left. \frac{\partial F}{\partial B} \right|_{T,V,N}$$

Van der Waals gas

$$\left(P + \frac{an^2}{V^2} \right) (V - nb) = nRT$$

Electromagnetism

$$\begin{array}{ll} \text{Maxwell's equations:} & \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} & \nabla \cdot \mathbf{B} = 0 \\ & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \end{array}$$

$$\begin{array}{ll} \text{Maxwell's equations in matter:} & \nabla \cdot \mathbf{D} = \rho_f & \nabla \cdot \mathbf{B} = 0 \\ & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{array}$$

$$\text{Lorentz force law: } \mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}; \quad d\mathbf{F} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

$$\text{Continuity equation: } \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\text{Potentials: } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi; \quad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\text{Speed of light: } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

Statics

Electrostatics

$$\text{Integral formulas: } \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{r^2} \hat{\mathbf{z}}; \quad V = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{r}; \quad \mathbf{r} = \mathbf{r} - \mathbf{r}'$$

$$\text{Electric potential: } \Delta V = -\int \mathbf{E} \cdot d\boldsymbol{\ell}; \quad \mathbf{E} = -\nabla V$$

$$\text{Work required to assemble a charge collection: } W = \frac{1}{2} \sum q_i V_i = \frac{1}{2} \int d^3r' \rho V$$

Magnetostatics

$$\text{Vector potential (Coulomb gauge): } \nabla \cdot \mathbf{A} = 0; \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}; \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}}{r}$$

$$\text{Biot-Savart: } \mathbf{B} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J} \times \hat{\mathbf{z}}}{r^2} = \frac{\mu_0}{4\pi} \int \frac{I d\boldsymbol{\ell}' \times \hat{\mathbf{z}}}{r^2}$$

Multipole expansion

Electric multipole expansion: $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + Q_{ij} \frac{3r_i r_j - r^2 \delta_{ij}}{r^5} + \dots \right]$

$$Q = \int d^3r \rho(\mathbf{r}); \quad \mathbf{p} = \int d^3r \rho(\mathbf{r}) \mathbf{r}; \quad Q_{ij} = \frac{1}{2} \int d^3r \rho(\mathbf{r}) r_i r_j$$

Legendre polynomial expansions: $V_{\text{out}} = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} P_{\ell}(\cos \theta) \left[\int d^3r' \rho(\mathbf{r}') (r')^{\ell} P_{\ell}(\cos \theta') \right]$

$$V_{\text{in}} = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} r^{\ell} P_{\ell}(\cos \theta) \left[\int d^3r' \frac{1}{(r')^{\ell+1}} \rho(\mathbf{r}') P_{\ell}(\cos \theta') \right]$$

Electric dipoles

Electric dipole moment; potential $\mathbf{p} = \int d^3r' \rho(\mathbf{r}') \mathbf{r}'; \quad V = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$

Electric field, dipole at the origin: $\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) = \frac{1}{4\pi\epsilon_0} \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{p}}{r^3} \quad (r \neq 0)$

Ideal electric dipoles: $\mathbf{p} = q\mathbf{d}; \quad \boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}; \quad \mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}; \quad U = -\mathbf{p} \cdot \mathbf{E}$

Magnetic dipoles

Magnetic dipole moment; potential: $\mathbf{m} = \frac{1}{2} \int d^3r' \mathbf{r}' \times \mathbf{J}(\mathbf{r}'); \quad \mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$

Magnetic field, dipole at the origin: $\mathbf{B} = \frac{\mu_0 m}{4\pi r^3} (2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}) = \frac{\mu_0}{4\pi} \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad (r \neq 0)$

Ideal magnetic dipoles: $\mathbf{m} = Ia\hat{\mathbf{n}}; \quad \boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}; \quad \mathbf{F} = (\mathbf{m} \cdot \nabla) \mathbf{B}; \quad U = -\mathbf{m} \cdot \mathbf{B}$

Magnetic dipole moment from angular momentum: $\mathbf{m} = \frac{Q}{2M} \mathbf{L}$

Matter

Bound charges: $\rho_b = -\nabla \cdot \mathbf{P}; \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$

Electric displacement: $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$

Linear isotropic homogeneous dielectrics: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}; \quad \mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$

Bound currents: $\mathbf{J}_b = \nabla \times \mathbf{M}; \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$

Fictitious magnetic charges: $\rho_m = -\nabla \cdot \mathbf{M}; \quad \sigma_m = \mathbf{M} \cdot \hat{\mathbf{n}}$

$\mathbf{H}$ field: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$

Linear isotropic homogeneous magnetic materials: $\mathbf{M} = \chi_m \mathbf{H}; \quad \mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$

Potential theory

Solutions to Laplace's equation

Spherical—azimuthal symmetry: $V = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta)$

Cylindrical—azimuthal symmetry ($D_0 = 0$ in a simply connected region):

$$V = (A_0 + B_0 \ln s)(C_0 + D_0 \varphi) + \sum_{k=1}^{\infty} \left(A_k s^k + \frac{B_k}{s^k} \right) (C_k \sin(k\varphi) + D_k \cos(k\varphi))$$

Cylindrical—Bessel functions:

$$s \frac{\partial}{\partial s} \left(s \frac{\partial R}{\partial s} \right) + (s^2 \kappa^2 - \alpha^2) R = 0; \quad \frac{\partial^2 \Phi}{\partial \varphi^2} = -\alpha^2 \Phi; \quad \frac{\partial^2 Z}{\partial z^2} = \kappa^2 Z$$

$$R_{\alpha}^{\kappa} = \begin{cases} A_0 + B_0 \ln s & \text{for } \kappa = 0 \text{ and } \alpha = 0 \\ A_{\alpha} s^{\alpha} + B_{\alpha} s^{-\alpha} & \text{for } \kappa = 0 \text{ and } \alpha \neq 0 \\ A_{\alpha}^{\kappa} J_{\alpha}(\kappa s) + B_{\alpha}^{\kappa} Y_{\alpha}(\kappa s) & \text{for } \kappa^2 > 0 \text{ (exponential in } z, \text{ oscillatory in } s) \\ A_{\alpha}^{\kappa} I_{\alpha}(\kappa s) + B_{\alpha}^{\kappa} K_{\alpha}(\kappa s) & \text{for } \kappa^2 < 0 \text{ (oscillatory in } z, \text{ exponential in } s) \end{cases}$$

$$Y_{\alpha}, K_{\alpha} \rightarrow \infty \quad \text{as } s \rightarrow 0$$

Mathematical relationships

Sines and cosines: $\int_0^a dx \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) = \frac{a}{2} \delta_{nm}$

$$\int_0^{2\pi} d\varphi e^{im\varphi} e^{-in\varphi} = 2\pi \delta_{mn}$$

Legendre polynomials: $\int_{-1}^1 dx P_{\ell}(x) P_m(x) = \frac{2}{2\ell+1} \delta_{\ell m}$

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{3x^2 - 1}{2}; \quad P_3(x) = \frac{5x^3 - 3x}{2}; \quad P_{\ell}(1) = 1$$

Boundary conditions

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\epsilon_0}; \quad \hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f; \quad \hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0; \quad \hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K}_f; \quad \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$$

$\hat{\mathbf{n}}$ points from medium 1 to medium 2

Green's functions

$$\text{Definition: } \nabla^2 G(\mathbf{r}, \mathbf{r}') = -\frac{1}{\epsilon_0} \delta(\mathbf{r} - \mathbf{r}')$$

$$\text{Dirichlet Green's fn.: } \phi(\mathbf{r}) = \int d^3r' \rho(\mathbf{r}') G_D(\mathbf{r}, \mathbf{r}') - \epsilon_0 \int_S dS' \phi_S(\mathbf{r}') \frac{\partial G_D}{\partial n'}; \quad G_D(\mathbf{r}_S, \mathbf{r}') = 0$$

$$\text{Delta functions: } \delta(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(z-z')} = \frac{1}{\pi} \int_0^{\infty} dk \cos(k(z - z'))$$

$$\delta(\varphi - \varphi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\varphi - \varphi')}$$

Energy and momentum

Energy

$$\text{Mechanical power: } \frac{\partial u_{mech}}{\partial t} = \mathbf{J} \cdot \mathbf{E}$$

$$\text{Electromagnetic energy density: } u_{EM} = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \quad \text{in a vacuum}$$

$$\text{Electromagnetic energy density (time-harmonic field): } \langle u_{EM} \rangle = \frac{1}{2} \frac{\partial}{\partial \omega} (\epsilon \omega) E^2$$

$$\text{Poynting vector: } \mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{in a vacuum}$$

$$\text{Poynting's theorem: } \frac{\partial u_{mech}}{\partial t} + \frac{\partial u_{EM}}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

Momentum

$$\text{Mechanical force density: } \mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$$

$$\text{Electromagnetic momentum density: } \mathbf{g} = \frac{\mathbf{S}}{c^2} = \epsilon_0 \mathbf{E} \times \mathbf{B} \quad \text{in a vacuum}$$

$$\text{Stress-energy tensor (vacuum): } T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

$$\text{Momentum conservation: } \mathbf{f} + \frac{\partial \mathbf{g}}{\partial t} - \nabla \cdot \mathbf{T} = 0$$

Circuits

$$\text{Capacitors: } C = \frac{Q}{V}; \quad E = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C}; \quad Z = \frac{1}{i\omega C}$$

$$\text{Inductors: } \varepsilon = - \int \mathbf{E} \cdot d\boldsymbol{\ell} = -L \frac{dI}{dt}; \quad W = \frac{1}{2} L I^2; \quad Z = i\omega L$$

$$\text{Resistors: } V = IR; \quad P = I^2 R = IV; \quad Z = R$$

Waves

$$\text{Wave equation (vacuum): } \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\text{Plane waves: } \omega = \frac{c}{n} k; \quad n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}; \quad \langle \mathbf{S} \rangle = \frac{1}{2} \epsilon_0 c E^2; \quad \mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}$$

$$\text{Phase velocity; group velocity: } \mathbf{v}_p = \frac{\omega}{k} \hat{\mathbf{k}}; \quad \mathbf{v}_g = \frac{d\omega}{d\mathbf{k}}$$

$$\text{Impedance: } Z = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}$$

$$\text{Conductors: } \tilde{\epsilon} = \epsilon + i \frac{\sigma}{\omega}; \quad k^2 = \tilde{\epsilon} \mu \omega^2$$

$$\text{Plasma frequency: } \omega_p = \sqrt{\frac{q^2 n}{m \epsilon_0}}; \quad \epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right)$$

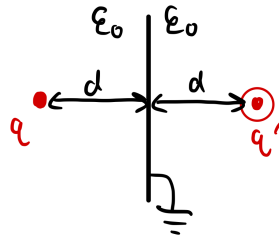
$$\text{Time-averaging theorem: } \langle \text{Re}(A) \text{Re}(B) \rangle = \frac{1}{2} \text{Re}(AB^*) = \frac{1}{2} \text{Re}(A^* B)$$

Waveguides

Transverse Electric (TE)	Transverse Magnetic (TM)
$E_z = 0$	$H_z = 0$
$(\nabla_{\perp}^2 + \gamma^2) H_z = 0$	$(\nabla_{\perp}^2 + \gamma^2) E_z = 0$
$\mathbf{H}_{\perp} = \frac{ih}{\gamma^2} \nabla_{\perp} H_z$	$\mathbf{E}_{\perp} = \frac{ih}{\gamma^2} \nabla_{\perp} E_z$
$\mathbf{E}_{\perp} = -\frac{\omega\mu}{h} \hat{\mathbf{z}} \times \mathbf{H}_{\perp}$	$\mathbf{H}_{\perp} = \frac{\omega\epsilon}{h} \hat{\mathbf{z}} \times \mathbf{E}_{\perp}$
Structural dispersion:	$\gamma^2 = \mu\epsilon\omega^2 - h^2; \quad \gamma^2 > 0$

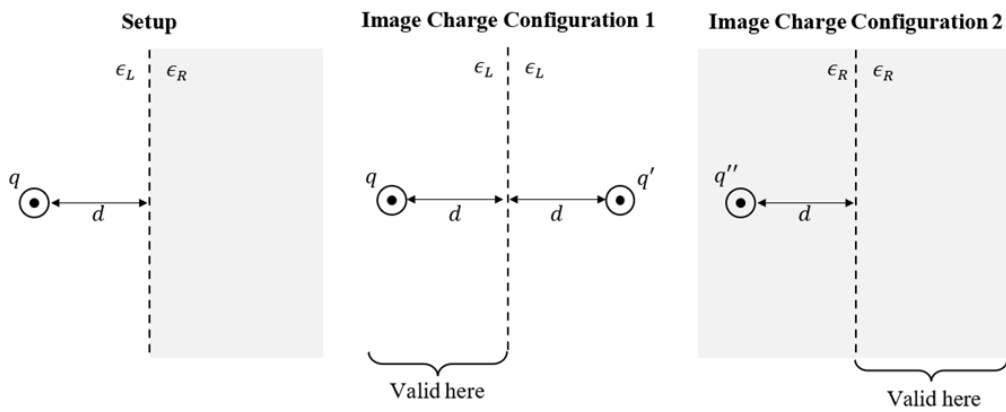
Method of images

Conducting plane



$$q' = -q$$

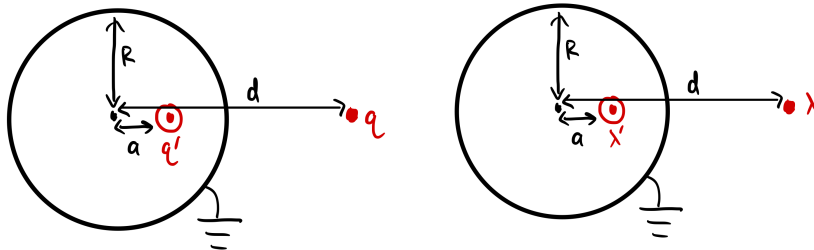
Dielectric plane



$$q' = \frac{\epsilon_L - \epsilon_R}{\epsilon_L + \epsilon_R} q; \quad q'' = \frac{2\epsilon_R}{\epsilon_L + \epsilon_R} q$$

Conducting sphere

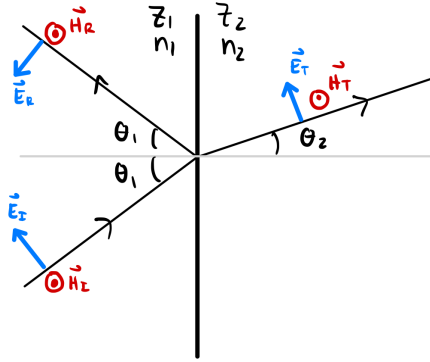
Conducting cylinder



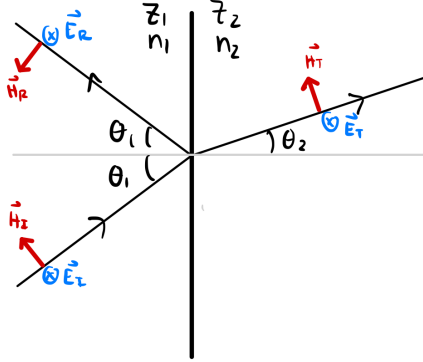
$$q' = -\frac{R}{d} q; \quad a = \frac{R^2}{d} \quad \Bigg| \quad \lambda' = -\lambda; \quad a = \frac{R^2}{d}$$

Fresnel relations

Snell's law: $n_1 \sin \theta_1 = n_2 \sin \theta_2$



p-polarization



s-polarization

$$r_p = \frac{Z_1 \cos \theta_1 - Z_2 \cos \theta_2}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2}$$

$$r_p = \frac{n_2 \cos \theta_1 - n_1 \cos \theta_2}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

$$t_p = \frac{2Z_2 \cos \theta_1}{Z_1 \cos \theta_1 + Z_2 \cos \theta_2}$$

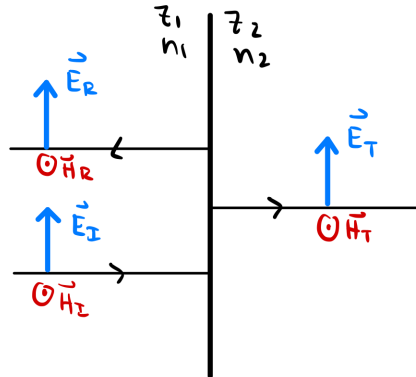
$$t_p = \frac{2n_1 \cos \theta_1}{n_2 \cos \theta_1 + n_1 \cos \theta_2}$$

$$r_s = \frac{Z_2 \cos \theta_1 - Z_1 \cos \theta_2}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2}$$

$$r_s = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$

$$t_s = \frac{2Z_2 \cos \theta_1}{Z_2 \cos \theta_1 + Z_1 \cos \theta_2}$$

$$t_s = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}$$



Normal incidence

$$r = \frac{Z_2 - Z_1}{Z_1 + Z_2} = \frac{n_1 - n_2}{n_1 + n_2}; \quad t = \frac{2Z_2}{Z_1 + Z_2} = \frac{2n_1}{n_1 + n_2}$$

Power reflection coefficients: $R = |r|^2$; $T = \frac{|\langle \mathbf{S}_T \rangle \cdot \hat{\mathbf{n}}|}{|\langle \mathbf{S}_I \rangle \cdot \hat{\mathbf{n}}|} = \frac{Z_1 \cos \theta_2}{Z_2 \cos \theta_1} |t|^2$

Electrodynamics

$$\text{Lorenz gauge condition: } \nabla \cdot \mathbf{A}_L + \frac{1}{c^2} \frac{\partial \phi_L}{\partial t} = 0$$

$$\text{Inhomogenous wave equations: } \nabla^2 \phi_L - \frac{1}{c^2} \frac{\partial^2 \phi_L}{\partial t^2} = -\frac{\rho}{\epsilon_0}; \quad \nabla^2 \mathbf{A}_L - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}_L}{\partial t^2} = -\mu_0 \mathbf{J}$$

$$\text{Retarded potentials: } \phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t - r/c)}{r}; \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}', t - r/c)}{r}$$

Radiation and scattering

Generally applicable—radiation

$$\text{Radiation vector: } \boldsymbol{\alpha}(\mathbf{r}, t) = \frac{\partial}{\partial t} \int d^3r' \mathbf{j}\left(\mathbf{r}', t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}\right) \quad (20.143)$$

$$\text{Fields: } c\mathbf{B}_{\text{rad}} = -\hat{\mathbf{r}} \times \frac{\partial \mathbf{A}_{\text{rad}}}{\partial t} = -\frac{\mu_0}{4\pi r} \hat{\mathbf{r}} \times \boldsymbol{\alpha} \quad (20.107)$$

$$\mathbf{E}_{\text{rad}} = -\hat{\mathbf{r}} \times c\mathbf{B}_{\text{rad}} = \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{\partial \mathbf{A}_{\text{rad}}}{\partial t} \right) \quad (20.108)$$

$$\hat{\mathbf{r}} \times \mathbf{E}_{\text{rad}} = c\mathbf{B}_{\text{rad}}; \quad \mathbf{E}_{\text{rad}} \times c\mathbf{B}_{\text{rad}} = \hat{\mathbf{r}} |\mathbf{E}_{\text{rad}}|^2; \quad |\mathbf{E}_{\text{rad}}| = c |\mathbf{B}_{\text{rad}}| \quad (20.109)$$

$$\text{Power radiated: } \frac{dP}{d\Omega} = r^2 \hat{\mathbf{r}} \cdot \mathbf{S}_{\text{rad}} = \frac{r^2}{c\mu_0} |\mathbf{E}_{\text{rad}}|^2 = \frac{r^2 c}{\mu_0} |\mathbf{B}_{\text{rad}}|^2 = \frac{\mu_0}{(4\pi)^2 c} |\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 \quad (20.81) \quad (20.110)$$

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} (\mathbf{E}_{\text{rad}} \times c\mathbf{B}_{\text{rad}}) = \frac{|\mathbf{E}_{\text{rad}}|^2}{\mu_0 c} = \frac{c}{\mu_0} |\mathbf{B}_{\text{rad}}|^2$$

$$\text{For } \hat{\mathbf{j}}(\mathbf{k}, \omega) = \int d^3r' \mathbf{j}(\mathbf{r}', \omega) e^{-i\mathbf{k} \cdot \mathbf{r}'}, \quad \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 \omega^2}{2(4\pi)^2 c} \left| \hat{\mathbf{r}} \times \hat{\mathbf{j}}(\mathbf{k}, \omega) \right|^2 \quad (20.120)$$

Generally applicable—scattering

$$|\langle \mathbf{S}_{\text{inc}} \rangle| = \frac{1}{2} \epsilon_0 c |E_0|^2 = \frac{1}{2} \frac{1}{\mu_0 c} |E_0|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{\langle dP/d\Omega \rangle}{|\langle \mathbf{S}_{\text{inc}} \rangle|} = \frac{r^2 \langle \mathbf{S}_{\text{rad}} \cdot \hat{\mathbf{r}} \rangle}{|\langle \mathbf{S}_{\text{inc}} \rangle|} = r^2 \frac{|\mathbf{E}_{\text{rad}}|^2}{|\mathbf{E}_{\text{inc}}|^2} \quad (21.4)$$

Radiation—specific cases

Nonrelativistic particle (Larmor formula)

$$\mathbf{j}(\mathbf{r}, t - r/c + (\hat{\mathbf{r}} \cdot \mathbf{r}_0)/c) \approx \mathbf{j}(\mathbf{r}, t - r/c) = q \mathbf{v}(t - r/c) \delta(\mathbf{r} - \mathbf{r}_0(t - r/c)) \quad (20.111)$$

$$\boldsymbol{\alpha} = q \mathbf{a}(t - r/c) = q \mathbf{a}_{\text{ret}}$$

$$\mathbf{E}_{\text{rad}} = \frac{\mu_0 q}{4\pi r} \hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times \mathbf{a}(t - r/c)] \quad (20.113)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} q^2 |\mathbf{a}_{\text{ret}}|^2 \sin^2 \theta \quad (20.114)$$

$$P = \frac{\mu_0}{4\pi c} \frac{2}{3} q^2 |\mathbf{a}_{\text{ret}}|^2 = \frac{1}{4\pi \epsilon_0} \frac{2}{3} \frac{q^2 |\mathbf{a}_{\text{ret}}|^2}{c^3} \quad (20.115)$$

Electric dipole radiation $\mathbf{p} = \int d^3r' \rho(\mathbf{r}')\mathbf{r}'$

$$\boldsymbol{\alpha} = \ddot{\mathbf{p}}_{\text{ret}} \quad (20.151)$$

$$c\mathbf{B}_{\text{rad}} = -\frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}}{r} \quad (20.152)$$

$$\mathbf{E}_{\text{rad}} = \frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}_{\text{ret}}) - \ddot{\mathbf{p}}_{\text{ret}}}{r} \quad (20.153)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c} |\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\text{ret}}|^2 \quad (20.154)$$

$$P = \frac{\mu_0}{4\pi c} \frac{2}{3} |\ddot{\mathbf{p}}_{\text{ret}}|^2 \quad (20.155)$$

Magnetic dipole radiation $\mathbf{m} = \frac{1}{2} \int d^3r' \mathbf{r}' \times \mathbf{j}(\mathbf{r}')$

$$\boldsymbol{\alpha} = \frac{1}{c} \ddot{\mathbf{m}}_{\text{ret}} \times \hat{\mathbf{r}} \quad (20.172)$$

$$\mathbf{E}_{\text{rad}} = \frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\text{ret}}}{r} \quad (20.174)$$

$$c\mathbf{B}_{\text{rad}} = \frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}}_{\text{ret}}) - \ddot{\mathbf{m}}_{\text{ret}}}{r} \quad (20.173)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3} |\hat{\mathbf{r}} \times \ddot{\mathbf{m}}_{\text{ret}}|^2 \quad (20.179)$$

$$P = \frac{\mu_0}{4\pi c^3} \frac{2}{3} |\ddot{\mathbf{m}}_{\text{ret}}|^2 \quad (20.180)$$

Scattering—specific cases

Thomson scattering

(EM plane wave incident on a free electron $-e$; long-wavelength approximation)

$$\mathbf{j}(\mathbf{r}, t) = \frac{ie^2 E_0}{m\omega} \exp[i(\mathbf{k}_0 \cdot \mathbf{r}_0 - \omega t)] \delta(\mathbf{r}(t) - \mathbf{r}_0(t)) \hat{\mathbf{e}}_0 \quad (21.10), \text{ computed via eqn. of motion}$$

$$\mathbf{p}(t) = -\frac{e^2 E_0}{m\omega^2} \hat{\mathbf{e}}_0 e^{-i\omega t} \quad \text{for } r_0 \approx 0 \quad (21.13)$$

$$\frac{d\sigma}{d\Omega} = r_e^2 |\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0|^2 = r_e^2 \left(1 - |\hat{\mathbf{k}} \cdot \hat{\mathbf{e}}_0|^2\right) \quad \text{for } r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} \quad (21.11)$$

$$\sigma = \frac{8\pi}{3} r_e^2 \quad \text{for linearly polarized or unpolarized} \quad (21.18)$$

Rayleigh scattering

(EM plane wave incident on a *small* object; $\mathbf{p}$ and $\mathbf{m}$ must be calculated; $k_0 = \omega/c$)

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0 c} \right)^2 |\hat{\mathbf{k}} \times \mathbf{m} + \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times c\hat{\mathbf{p}})|^2 \quad (21.20)$$

$$\text{If } \mathbf{p} = \alpha\epsilon_0 E_0 \hat{\mathbf{e}}_0 \text{ and } \mathbf{m} = 0, \quad \frac{d\sigma}{d\Omega} = \left(\frac{k_0^2 \alpha}{4\pi} \right)^2 (1 - |\mathbf{k} \cdot \hat{\mathbf{e}}_0|^2) = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0} \right)^2 |\hat{\mathbf{k}} \times \mathbf{p}|^2 \quad (21.21)$$

$$\hat{\mathbf{j}} = -i\omega\mathbf{p} \quad \text{for a small dipole at the origin}$$

Born approximation $\chi_e \ll 1$

(EM plane wave—weak dielectric response)

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} \approx -i\omega\epsilon_0\chi [\hat{\mathbf{e}}_0 E_0 e^{i(\mathbf{k}_0 \cdot \mathbf{r} - \omega t)}] \quad (21.46)$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{k_0^2}{4\pi} \right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0|^2 \left| \int d^3r' \chi e^{-i\mathbf{q} \cdot \mathbf{r}'} \right|^2 \quad \text{for } \mathbf{q} = \mathbf{k} - \mathbf{k}_0 \quad (21.47)$$

Mathematics

Trigonometry

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}; \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin^2 x + \cos^2 x = 1; \quad 1 + \tan^2 x = \sec^2 x; \quad 1 + \cot^2 x = \csc^2 x$$

$$\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos x; \quad \cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin x$$

$$\sin(x + \pi) = -\sin x; \quad \cos(x + \pi) = -\cos x$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

$$\cos a \cos b = \frac{1}{2} [\cos(a - b) + \cos(a + b)]$$

$$\sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]$$

$$\sin a \cos b = \frac{1}{2} [\sin(a + b) + \sin(a - b)]$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}; \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$(\sin x)' = \cos x; \quad (\cos x)' = -\sin x; \quad (\tan x)' = \sec^2 x$$

$$(\csc x)' = -\csc x \cot x; \quad (\sec x)' = \sec x \tan x; \quad (\cot x)' = -\csc^2 x$$

$$(\ln(\sec x + \tan x))' = \sec x; \quad \left(\ln\left(\tan \frac{x}{2}\right)\right)' = (-\ln(\csc x + \cot x))' = \csc x$$

$$\left(\arcsin\left(\frac{x}{c}\right)\right)' = \frac{1}{\sqrt{c^2 - x^2}}; \quad \left(\arctan\left(\frac{x}{c}\right)\right)' = \frac{c}{c^2 + x^2}$$

Thrigonometry

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}; \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\cosh^2 x - \sinh^2 x = 1; \quad \operatorname{sech}^2 x = 1 - \tanh^2 x; \quad \operatorname{csch}^2 x = \coth^2 x - 1$$

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$$

$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$$

$$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$$

$$\sinh(2x) = 2 \sinh x \cosh x$$

$$\cosh(2x) = \sinh^2 x + \cosh^2 x = 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1$$

$$\tanh(2x) = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\sinh^2 x = \frac{\cosh(2x) - 1}{2}; \quad \cosh^2 x = \frac{\cosh(2x) + 1}{2}$$

$$\operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1}); \quad \operatorname{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$$

$$(\sinh x)' = \cosh x; \quad (\cosh x)' = \sinh x; \quad (\tanh x)' = \operatorname{sech}^2 x$$

$$\left(\operatorname{arcsinh}\left(\frac{x}{c}\right)\right)' = \frac{1}{\sqrt{x^2 + c^2}}; \quad \left(\operatorname{arctanh}\left(\frac{x}{c}\right)\right)' = \frac{c}{c^2 - x^2}$$

Calculus

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \frac{\pi^{1/2}}{\alpha^{1/2}}; \quad \int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = \frac{1}{2} \frac{\pi^{1/2}}{\alpha^{3/2}}; \quad \int_{-\infty}^{\infty} dx x^4 e^{-\alpha x^2} = \frac{3}{4} \frac{\pi^{1/2}}{\alpha^{5/2}}$$

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2 + \beta x} = \frac{\pi^{1/2}}{\alpha^{1/2}} e^{\beta^2/(4\alpha)}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx}; \quad \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx}; \quad \int_{-\infty}^{\infty} dx |f(x)|^2 = \int_{-\infty}^{\infty} dk |\tilde{f}(k)|^2$$

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x_0)}; \quad \delta(ax) = \frac{\delta(x)}{|a|}; \quad \delta(g(x)) = \sum_{g(x_i)=0} \frac{\delta(x - x_i)}{|g'(x_i)|}$$

$$\delta(\mathbf{r}) = \frac{\delta(r)\delta(\theta)\delta(\varphi)}{r^2 \sin \theta}$$

$$\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi \delta(\mathbf{r} - \mathbf{r}')$$

Unit vectors and vector algebra

$$\hat{\mathbf{s}} = \hat{\mathbf{x}} \cos \varphi + \hat{\mathbf{y}} \sin \varphi$$

$$\hat{\varphi} = -\hat{\mathbf{x}} \sin \varphi + \hat{\mathbf{y}} \cos \varphi$$

$$\hat{\mathbf{x}} = \hat{\mathbf{s}} \cos \varphi - \hat{\varphi} \sin \varphi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{s}} \sin \varphi + \hat{\varphi} \cos \varphi$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin \theta \cos \varphi + \hat{\mathbf{y}} \sin \theta \sin \varphi + \hat{\mathbf{z}} \cos \theta$$

$$\hat{\theta} = \hat{\mathbf{x}} \cos \theta \cos \varphi + \hat{\mathbf{y}} \cos \theta \sin \varphi - \hat{\mathbf{z}} \sin \theta$$

$$\hat{\varphi} = -\hat{\mathbf{x}} \sin \varphi + \hat{\mathbf{y}} \cos \varphi$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}} \sin \theta \cos \varphi + \hat{\theta} \cos \theta \cos \varphi - \hat{\varphi} \sin \varphi$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin \theta \sin \varphi + \hat{\theta} \cos \theta \sin \varphi + \hat{\varphi} \cos \varphi$$

$$\hat{\mathbf{z}} = \hat{\mathbf{r}} \cos \theta - \hat{\theta} \sin \theta$$

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\boldsymbol{\tau} = dx dy dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\boldsymbol{\tau} = r^2 \sin \theta dr d\theta d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

Curl : $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}}$

$$+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\boldsymbol{\tau} = s ds d\phi dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} - (\mathbf{A} \cdot \nabla) \mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A}) d\boldsymbol{\tau} = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$