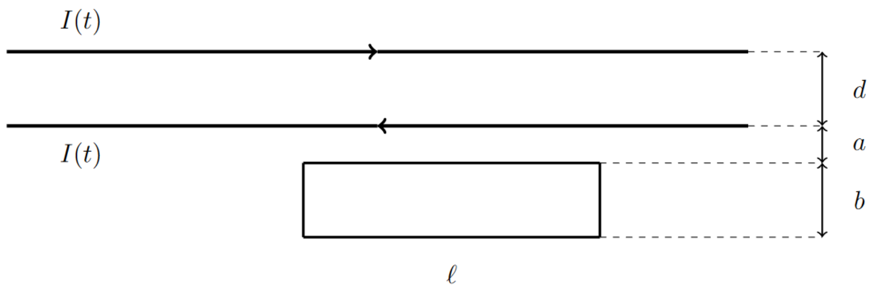
7. (Electromagnetism)

Electromagnetic induction makes it possible to carry out a wiretap on a landline phone without cutting any wires. Here is how it works. The telephone wires are represented by two infinitely long straight parallel wires separated by a distance d, carrying time-dependent currents $\pm I(t)$ in the two wires of equal magnitude and opposite direction, as indicated by the arrows in the figure. Parallel to and in the same plane as the wires, we install a small rectangular closed circuit of length ℓ and width b representing the wiretapping device, separated from the closest telephone wire by a distance a, as shown in the figure below.



(a) Write down the magnetic field produced by a single wire traversed by a current $I(t)$.

(b) Compute the magnetic flux $\Phi(t)$ through the rectangular loop as a function of the geometrical data specified in the figure and the current $I(t)$.

(c) Compute the electromotive force $\varepsilon(t)$ generated by this flux.

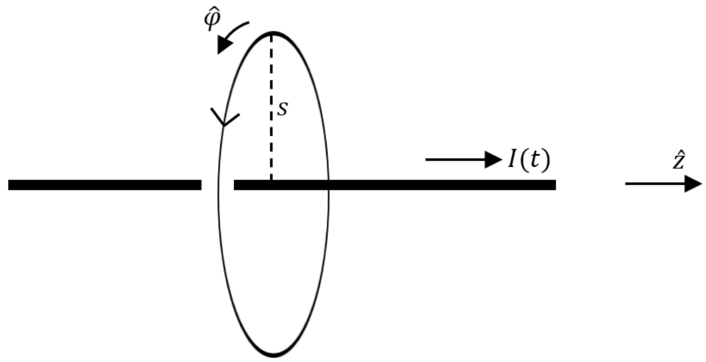
(d) Is the current induced in the rectangular loop clockwise or counterclockwise? Justify your answer in terms of the sign of $I(t)$ and its derivative at any give time t.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) While the problem doesn't say so explicitly, you are permitted to use the quasi-static approximation for this entire problem. Finding the magnetic field for a single wire in this approximation requires Ampere's law without the displacement current term:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}; \qquad \int_{\text{loop}} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enclosed}} \tag{105}$$

Here, the appropriate Amperian loop is a circular loop of radius s, centered on and perpendicular to the wire:



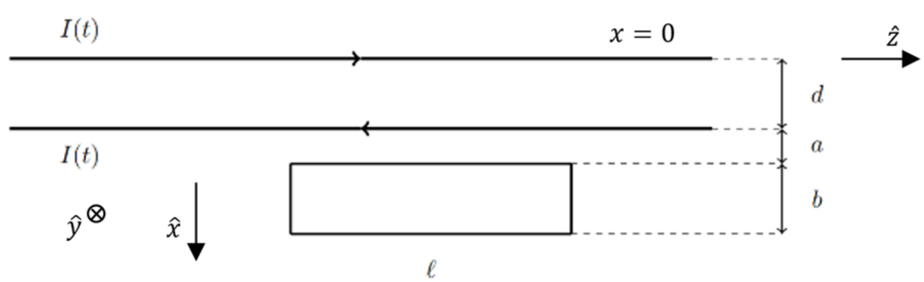
By the azimuthal symmetry of the setup, the magnetic field of a single wire points in the $\hat{\varphi}$ direction and depends only on the distance to the wire s: $\mathbf{B}_1 = B_1(s,t)\hat{\varphi}$. Applying the integral version of Ampere's law, we get

$$\begin{aligned} 2\pi s\, B_1(s,t) &= \int_{\text{loop}} \mathbf{B}_1 \cdot d\boldsymbol{\ell} \\ &= \mu_0 I_{\text{enclosed}} \\ &= \mu_0 I(t) \quad \text{(positive by the right-hand rule)} \end{aligned} \tag{106}$$

This gives us $B_1(s,t) = \mu_0 I(t)/(2\pi s)$, or

$$\boxed{\mathbf{B}_1(s,t) = \frac{\mu_0 I(t)}{2\pi s}\hat{\varphi} \quad \text{where } s \text{ is the distance to the wire}} \tag{107}$$

(b) The first step for this problem is to superpose the magnetic fields of both wires to get the total magnetic field in the area of the loop. Let the top wire correspond to $x = 0$, and let $\hat{\mathbf{z}}$ be in the direction of the top wire:



By the right-hand rule, the direction of the magnetic fields in the area of the loop can be determined (assuming $I(t) > 0$): The magnetic field due to the top wire points in the $+\hat{\mathbf{y}}$

direction (into the page), and the magnetic field due to the bottom wire points in the $-\hat{\mathbf{y}}$ direction (out of the page). The distance to each of the wires in the area of the loop ($d+a \leq x \leq d+a+b$) is defined by

$$\begin{aligned} \text{Distance to top wire } &= s_{\text{top}} = x \\ \text{Distance to bottom wire } &= s_{\text{bottom}} = x - d \end{aligned} \tag{108}$$

Therefore, in the area of the wire, the magnetic field due to both wires is

$$\begin{aligned} \mathbf{B}(x,t) &= \mathbf{B}_{\text{top}}(s_1,t) + \mathbf{B}_{\text{bottom}}(s_2,t) && (109) \\ &= +\frac{\mu_0 I(t)}{2\pi s_{\text{top}}}\hat{\mathbf{y}} - \frac{\mu_0 I(t)}{2\pi s_{\text{bottom}}}\hat{\mathbf{y}} \\ &= \hat{\mathbf{y}}\,\frac{\mu_0 I(t)}{2\pi}\left(\frac{1}{x} - \frac{1}{x-d}\right) \quad \text{by (108)} \\ &= -\hat{\mathbf{y}}\,\frac{\mu_0 I(t)}{2\pi}\left(\frac{1}{x-d} - \frac{1}{x}\right) && (110) \end{aligned}$$

In the area of the loop, the magnetic field due to the bottom wire is stronger than the magnetic field due to the top wire, since the loop is closer to the bottom wire than the top wire. Therefore, assuming $I(t) > 0$, the flux through the loop is in the $-\hat{\mathbf{y}}$ direction (out of the page). Therefore, taking $d\mathbf{a} = -\hat{\mathbf{y}}\,dA$ for the loop and using the formula for magnetic flux, we will get a positive flux when $I(t) > 0$:

$$\begin{aligned} \Phi(t) &= \int \mathbf{B}\cdot d\mathbf{a} \\ &= \int dA\,\mathbf{B}\cdot(-\hat{\mathbf{y}}) \\ &= \ell\int_{x=d+a}^{x=d+a+b} dx\,\frac{\mu_0 I(t)}{2\pi}\left(\frac{1}{x-d} - \frac{1}{x}\right) \quad \text{since the loop is at } d+a \leq x \leq a+b \\ &= \frac{\mu_0 \ell I(t)}{2\pi}\left[\ln(x-d) - \ln x\right]_{x=d+a}^{x=d+a+b} \\ &= \frac{\mu_0 \ell I(t)}{2\pi}\left[\ln(a+b) - \ln a - (\ln(d+a+b) - \ln(d+a))\right] \end{aligned}$$

$$\boxed{\Phi(t) = \frac{\mu_0 \ell I(t)}{2\pi}\ln\left(\frac{(d+a)(a+b)}{(d+a+b)a}\right)} \tag{111}$$

(c) The definition for electromotive force (emf) in terms of magnetic flux is

$$\varepsilon = -\frac{d\Phi}{dt} \tag{112}$$

Therefore, we have

$$\boxed{\varepsilon(t) = -\frac{\mu_0 \ell}{2\pi}\frac{dI}{dt}\ln\left(\frac{(d+a)(a+b)}{(d+a+b)a}\right)} \tag{113}$$

The sign of the emf is based on our sign convention. Instead of worrying about the interpretation of the sign now, we will do so in the next part, when we discuss the direction of the induced current.

(d) Lenz's law states that the induced current produces an induced magnetic field that tends to oppose changes in the magnetic field. In this case, the discussion around (111) tells us that, when

$\frac{dI}{dt} > 0$, the magnetic flux through the loop $\Phi(t)$ increases in the $-\hat{\mathbf{y}}$ direction (out of the page). Therefore, the induced current in the loop produces a magnetic field in the $+\hat{\mathbf{y}}$ direction (into the page) to counteract this change. This corresponds to a clockwise induced current in the loop.

Similarly, when $\frac{dI}{dt} < 0$, there is a counterclockwise induced current in the loop.