1. *Quantum Mechanics* (Spring 2003)

In one-dimension, a particle is subject to a harmonic oscillator potential with a time dependent origin,

$$V(x) = \frac{1}{2}m\omega^2(x - \epsilon(t))^2$$

where

$$\epsilon(t) = \epsilon e^{-t^2/\tau^2}, \quad \epsilon \ll 1$$

Suppose the particle is in the ground state at $t = -\infty$. What states can the particle be in at $t = +\infty$, and what are the probabilities for each? Work to lowest order in ϵ.

2. *Quantum Mechanics* (Spring 2003)

Consider two $s = 1/2$ spins interacting through the Hamiltonian

$$H = J\sigma_1^z\sigma_2^z + h(\sigma_1^x + \sigma_2^x)$$

What is the ground state energy?

3. *Quantum Mechanics* (Spring 2003)

A *coherent state* of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalue basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

4. *Quantum Mechanics* (Spring 2003)

Consider two electrons which are constrained to live on two sites. There is an interaction energy U when both electrons are on the same site. When they are on different sites, there is no interaction energy. There is an amplitude t for an electron to hop from one site to the other. In other words, the Hamiltonian is of the form:

$$H = -t(|1\uparrow, 1\downarrow\rangle\langle 1\uparrow, 2\downarrow| + \text{h.c.} + |2\uparrow, 2\downarrow\rangle\langle 1\uparrow, 2\downarrow| + \text{h.c.}) + U(|1\uparrow, 1\downarrow\rangle\langle 1\uparrow, 1\downarrow| + |2\uparrow, 2\downarrow\rangle\langle 2\uparrow, 2\downarrow|)$$

where $\left|1\sigma, 2\sigma'\right\rangle$ is the state with an electron of spin $\sigma = \uparrow, \downarrow$ at site 1 and an electron of spin $\sigma' = \uparrow, \downarrow$ at site 2 while $|1\sigma, 1\ -\sigma\rangle$ is the state with two electrons (of spins σ and $-\sigma$) at site 1. What are the energies and degeneracies of the ground and first excited states of the system to lowest order in t for $t \ll U$?

5. *Quantum Mechanics* (Spring 2003)

Consider a particle of mass m which moves in the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ ax & \text{for } x > 0 \end{cases}$$

Estimate the ground state energy.

6. *Statistical Mechanics and Thermodynamics* (Spring 2003)

Consider a system with N lattice sites and N atoms. These atoms may move around, but each site may be occupied by only 0,1, or 2 atoms at a time. Note that if there are n unoccupied sites, then there must also be n doubly-occupied sites; therefore, macrostates may be denoted by the value of n. The energy of a site is 0 if unoccupied, 0 if singly occupied, and e if doubly occupied; the total energy is thus $U(n) = ne$.

(a) Find the number of microstates vs. n.

(b) Find the temperature using part (a).

(c) What is the grand canonical partition function for a single site? Using this and the constraints of the problem, deduce the chemical potential.

(d) Find the average energy per site using part (c); confirm your answer using part (b).

7. *Statistical Mechanics and Thermodynamics* (Spring 2003)

Consider two atoms, A and B, of mass m_A and m_B. There atoms can form a molecule $C = AB$ with binding energy Δ. Initially, a certain number, N_A and N_B, of A and B atoms is placed in a box of volume V. If the system is brought to thermal equilibrium at temperature T, how many atoms A and B and molecules C will be found in the box? You can assume that A, B, and C are noninteracting (more precisely, the only effect of the interaction is to give rise to the bound state C.) You may also assume a dilute concentration of atoms and molecules.

8. *Electricity and Magnetism* (Spring 2003)

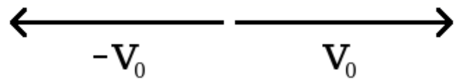
A cylindrical capacitor of length L is composed of an inner cylindrical conductor of radius r and a concentric outer conducting cylindrical shell of radius R.

(a) What is the capacitance of this arrangement (you may ignore fringing fields at the ends)?

(b) The two conductors are held at a constant potential difference, V, using a battery. A cylindrical shell of dielectric material of length L and which just fits in between the conductors (inner radius $\sim r$ and outer radius $\sim R$) is inserted so that half is inside of the capacitor (i.e. $L/2$ of the length of the capacitor is now filled with dielectric). What is the force on the dielectric in this position (magnitude and direction)?

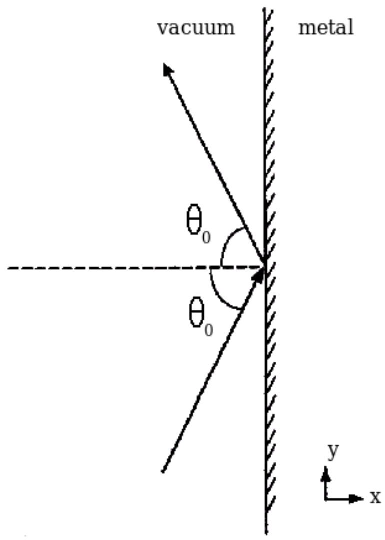
9. *Electricity and Magnetism* (Spring 2003)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



10. *Electricity and Magnetism* (Spring 2003)

X-Ray Mirror: X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. As shown below, the metal occupies the region $x > 0$. The X-rays propagate in the x–y plane (the plane of the picture) and their polarization is in the z direction, coming out of the page. Assume that the metal contains n free electrons per unit volume and is non-magnetic. Derive an expression for the critical angle θ_0.



11. *Electricity and Magnetism* (Spring 2003)

Secret Circuit: A two-terminal "black box" is given to you. Inside the box a circuit is attached to the terminals which is known to contain a lossless inductor L, a lossless capacitor C, and a resistor R. When a 1.5 Volt battery is connected across the terminals, a current of 1.5 milliamperes flows. When an AC voltage of 1.0 Volt (RMS) at a frequency of 60 Hz is connected, a current of 0.01 amperes (RMS) flows. As the AC frequency is increased while the applied voltage is maintained constant, the current is found to go through a maximum exceeding 100 amperes at $\nu = 1000$ Hz. What is the circuit inside the box? What are the values of R, L, and C?

12. *Electricity and Magnetism* (Spring 2003)

(a) Show that the field inside a sphere of uniformly magnetized material ($\mathbf{M} = M\hat{\mathbf{z}}$) is:

$$\mathbf{B} = \frac{2}{3}\mu_0 M\hat{\mathbf{z}}$$

(b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_0\hat{\mathbf{z}}$. Using the above result, find the magnetic field inside the sphere.

13. *Statistical Mechanics and Thermodynamics* (Spring 2003)

Consider a d-dimensional material in which the important excitation are non-conserved bosons, and assume that the dispersion relation for these bosons is $\omega = ak^3$, where k is the wave vector's amplitude and a is a constant. The low temperature specific heat goes as T^q. What is the value of the power q?

Note: The dimensionality, d, of the material is not necessarily equal to three.

14. *Statistical Mechanics and Thermodynamics* (Spring 2003)

A system can exchange energy and volume with a large reservoir.

(a) Show that the entropy of this combined system (system & reservoir) is maximized when the temperature of the system is equal to the temperature of the reservoir and the pressure of the system is equal to the pressure of the reservoir.

(b) Assume that the reservoir is much larger than the system. Expand to second order in the energy and volume of the system. Find the inequalities which must be satisfied in order that the entropy of the combined system is a maximum at the extremum point.

Selected Answers — Spring 2003

10) use snell's law

$$\theta_0 = \sin^{-1}\left[\sqrt{1 - \frac{4\pi n e^2 \hbar^2}{m E^2}}\right]$$

11) $C = 0.10\,\mu F$

$L \simeq 265\,mH$

$R = 1000\,\Omega$

1. *Quantum Mechanics* (Spring 2003)

In one-dimension, a particle is subject to a harmonic oscillator potential with a time dependent origin,

$$V(x) = \frac{1}{2}m\omega^2(x - \epsilon(t))^2$$

where

$$\epsilon(t) = \epsilon e^{-t^2/\tau^2}, \quad \epsilon \ll 1$$

Suppose the particle is in the ground state at $t = -\infty$. What states can the particle be in at $t = +\infty$, and what are the probabilities for each? Work to lowest order in ϵ.

$$V(x) = \frac{1}{2} m\omega^2 \left(x^2 - 2x\,\epsilon(t) + \cancel{\epsilon^2(t)}\right)$$

$$\Rightarrow H'(t) = -m\omega^2 x\,\epsilon(t) = -m\omega^2 x\,\epsilon\, e^{-t^2/\tau^2}$$

$$\langle \psi_f | \psi \rangle = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{\infty} \langle \phi_f | H'(t') | \phi_i \rangle e^{-i\omega_{fi}t'} dt'$$

$$= \delta_{n'0} + \frac{i}{\hbar} m\omega^2 \epsilon \int_{-\infty}^{\infty} \langle n' | x | 0 \rangle e^{-in'\omega t'} e^{-t'^2/\tau^2} dt'$$

$$= \delta_{n'0} + \frac{i}{\hbar} m\omega^2 \epsilon \int_{-\infty}^{\infty} \langle n' | \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) | 0 \rangle e^{-in'\omega t'} e^{-t'^2/\tau^2} dt'$$

$$\Rightarrow \langle \psi_1 | \psi \rangle = \frac{i}{\hbar} m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} e^{-t'^2/\tau^2 - i\omega t'} dt'$$

$$\int_{-\infty}^{\infty} e^{-t'^2/\tau^2 - i\omega t'} dt' = \int_{-\infty}^{\infty} e^{-\frac{1}{\tau^2}\left(t'^2 + i\omega\tau^2 t' - \frac{1}{4}\omega^2\tau^4\right) - \frac{1}{4}\omega^2\tau^2} dt'$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{\tau^2}\left(t' - \frac{i}{2}\omega\tau^2\right)^2 - \frac{1}{4}\omega^2\tau^2} dt' = e^{-\frac{1}{4}\omega^2\tau^2} \int_{-\infty}^{\infty} e^{-u^2/\tau^2} du$$

$$= e^{-\frac{1}{4}\omega^2\tau^2}\, \tau \int_{-\infty}^{\infty} e^{-y^2} dy = \tau e^{-\frac{1}{4}\omega^2\tau^2}\sqrt{\pi}$$

$$\Rightarrow \langle \psi_1 | \psi \rangle = \frac{i}{\hbar} m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}}\, \tau e^{-\frac{1}{4}\omega^2\tau^2}\sqrt{\pi}$$

$$\Rightarrow P(n=1) = |\langle \psi_1 | \psi \rangle|^2 = \frac{m^2\omega^4}{\hbar^2}\epsilon^2 \frac{\hbar}{2m\omega}\tau^2 e^{-\frac{1}{2}\omega^2\tau^2}\pi$$

$$= \frac{\pi m \omega^3 \tau^2 \epsilon^2}{2\hbar} e^{-\frac{1}{2}\omega^2\tau^2} \quad \text{to first order}$$

and $P(n=0) = 1$ to first order

and $P(n) = 0$ to first order for all $n = 0, 1$.

2. *Quantum Mechanics* (Spring 2003)

Consider two $s = 1/2$ spins interacting through the Hamiltonian

$$H = J\sigma_1^z\sigma_2^z + h(\sigma_1^x + \sigma_2^x)$$

What is the ground state energy?

We choose to work in the basis $\{ |++\rangle, |+-\rangle, |-+\rangle, |--\rangle \}$

so we express $|\psi\rangle$ as the 4-component spinor $|\psi\rangle = \begin{pmatrix} \langle ++|\psi\rangle \\ \langle +-|\psi\rangle \\ \langle -+|\psi\rangle \\ \langle --|\psi\rangle \end{pmatrix}$

No we consider the action of the terms of the Hamiltonian on the base kets. Note $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\sigma_1^z\sigma_2^z|++\rangle = |++\rangle \qquad (\sigma_1^x+\sigma_2^x)|++\rangle = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = |-+\rangle + |+-\rangle$$

$$\sigma_1^z\sigma_2^z|+-\rangle = -|+-\rangle \qquad (\sigma_1^x+\sigma_2^x)|+-\rangle = \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = |--\rangle + |++\rangle$$

$$\sigma_1^z\sigma_2^z|-+\rangle = -|-+\rangle \qquad (\sigma_1^x+\sigma_2^x)|-+\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 = |++\rangle + |--\rangle$$

$$\sigma_1^z\sigma_2^z|--\rangle = |--\rangle \qquad (\sigma_1^x+\sigma_2^x)|--\rangle = \begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2 + \begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2 = |+-\rangle + |-+\rangle$$

$$H \doteq J\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + h\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} J & h & h & 0 \\ h & -J & 0 & h \\ h & 0 & -J & h \\ 0 & h & h & J \end{pmatrix}$$

Now we find the energy eigenvalues by solving $\det(H-\lambda I)=0$

$$0 = \det(H-\lambda I) = \begin{vmatrix} J-\lambda & h & h & 0 \\ h & -J-\lambda & 0 & h \\ h & 0 & -J-\lambda & h \\ 0 & h & h & J-\lambda \end{vmatrix} = \begin{vmatrix} J-\lambda & h & h & 0 \\ h & -J-\lambda & 0 & h \\ 0 & J+\lambda & -J-\lambda & 0 \\ 0 & h & h & J-\lambda \end{vmatrix}$$

$$= (J-\lambda)\begin{vmatrix} -J-\lambda & 0 & h \\ J+\lambda & -J-\lambda & 0 \\ h & h & J-\lambda \end{vmatrix} - h\begin{vmatrix} h & h & 0 \\ J+\lambda & -J-\lambda & 0 \\ h & h & J-\lambda \end{vmatrix}$$

$$= (J-\lambda)\{(-J-\lambda)[(-J-\lambda)(J-\lambda)] + h[h(J+\lambda) - h(-J-\lambda)]\}$$
$$-h\{h[(-J-\lambda)(J-\lambda)] - h[(J+\lambda)(J-\lambda)]\}$$

$$= (J+\lambda)^2(J-\lambda^2) + 2h^2(J+\lambda)(J-\lambda) + 2h^2(J+\lambda)(J-\lambda)$$

$$= (J+\lambda)^2(J-\lambda)^2 + 4h^2(J+\lambda)(J-\lambda)$$

We see the two roots $(J+\lambda)=0$ and $(J-\lambda)=0$, so divide by $(J+\lambda)(J-\lambda)$:

$$(J+\lambda)(J-\lambda) + 4h^2 = 0 \Rightarrow J^2 - \lambda^2 + 4h^2 = 0 \Rightarrow \lambda = \pm\sqrt{J^2+4h^2}$$

Therefore the four energy eigenvalues are $\pm J, \pm\sqrt{J^2+4h^2}$

The lowest of these is $E_0 = -\sqrt{J^2+4h^2}$ which is the ground state energy.

3. *Quantum Mechanics* (Spring 2003)

A *coherent state* of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalue basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$.

See Griffiths Problem 3.35

First we write $|\alpha\rangle$ as an eigenvalue expansion

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n a|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle \quad \text{since it's zero for } n=0$$

Now let $n \to n+1 \Rightarrow a|\alpha\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle$

So $a|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,|n\rangle = \alpha\sum_{n=0}^{\infty} c_n|n\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle$

Now take the projection with $\langle m|$, $\sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,\langle m|n\rangle = \sum_{n=0}^{\infty} \alpha c_n \langle m|n\rangle$

$$\Rightarrow \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\,\delta_{mn} = \sum_{n=0}^{\infty} \alpha c_n \delta_{mn} \Rightarrow c_{m+1}\sqrt{m+1} = \alpha c_m$$

$$\Rightarrow c_{m+1} = \frac{\alpha c_m}{\sqrt{m+1}} \quad \text{So} \quad c_1 = \frac{\alpha c_0}{\sqrt{1}},\ c_2 = \frac{\alpha^2 c_0}{\sqrt{2}},\ c_3 = \frac{\alpha^3 c_0}{\sqrt{6}}$$

$$\Rightarrow c_m = \frac{\alpha^m c_0}{\sqrt{m!}} \quad \Rightarrow |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n c_0}{\sqrt{n!}}|n\rangle$$

Now we can determine c_0 by normalizing.

$$1 = \langle\alpha|\alpha\rangle = \left(\sum_{n=0}^{\infty} \frac{(\alpha^*)^n c_0^*}{\sqrt{n!}}\langle n|\right)\left(\sum_{n=0}^{\infty} \frac{\alpha^n c_0}{\sqrt{n!}}|n\rangle\right)$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2}$$

Therefore $$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

4. *Quantum Mechanics* (Spring 2003)

Consider two electrons which are constrained to live on two sites. There is an interaction energy U when both electrons are on the same site. When they are on different sites, there is no interaction energy. There is an amplitude t for an electron to hop from one site to the other. In other words, the Hamiltonian is of the form:

$$H = -t(|1\uparrow,1\downarrow\rangle\langle 1\uparrow,2\downarrow| + \text{h.c.} + |2\uparrow,2\downarrow\rangle\langle 1\uparrow,2\downarrow| + \text{h.c.}) + U(|1\uparrow,1\downarrow\rangle\langle 1\uparrow,1\downarrow| + |2\uparrow,2\downarrow\rangle\langle 2\uparrow,2\downarrow|)$$

where $|1\sigma, 2\sigma'\rangle$ is the state with an electron of spin $\sigma=\uparrow,\downarrow$ at site 1 and an electron of spin $\sigma'=\uparrow,\downarrow$ at site 2 while $|1\sigma, 1-\sigma\rangle$ is the state with two electrons (of spins σ and $-\sigma$) at site 1. What are the energies and degeneracies of the ground and first excited states of the system to lowest order in t for $t \ll U$?

Note that there can't be two of the same spin in one site by the Pauli exclusion principle, and there is nothing to cause the spins to flip, so we assume that one electron is always spin up and the other is always spin down. Thus there are 4 distinct states $|1\uparrow,1\downarrow\rangle, |1\uparrow,2\downarrow\rangle, |1\downarrow,2\uparrow\rangle, |2\uparrow,2\downarrow\rangle$, so $|\psi\rangle = \begin{pmatrix}\langle 1\uparrow,1\downarrow|\psi\rangle \\ \langle 1\uparrow,2\downarrow|\psi\rangle \\ \langle 1\downarrow,2\uparrow|\psi\rangle \\ \langle 2\uparrow,2\downarrow|\psi\rangle\end{pmatrix}$

The Hamiltonian above seems to be missing transitions to and from the $|1\downarrow,2\uparrow\rangle$ state. We will assume this is an error. In the basis described above, H is

$$H \doteq -t\begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix} + U\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix} = \begin{pmatrix}U&-t&-t&0\\-t&0&0&-t\\-t&0&0&-t\\0&-t&-t&U\end{pmatrix}$$

We find the energy eigenvalues by solving $\det(H-\lambda I)=0$

$$0=\det(H-\lambda I) = \begin{vmatrix}U-\lambda&-t&-t&0\\-t&-\lambda&0&-t\\-t&0&-\lambda&-t\\0&-t&-t&U-\lambda\end{vmatrix} = \begin{vmatrix}U-\lambda&-t&-t&0\\-t&-\lambda&0&-t\\0&\lambda&-\lambda&0\\0&-t&-t&U-\lambda\end{vmatrix}$$

$$= (U-\lambda)\begin{vmatrix}-\lambda&0&-t\\\lambda&-\lambda&0\\-t&-t&U-\lambda\end{vmatrix} + t\begin{vmatrix}-t&-t&0\\\lambda&-\lambda&0\\-t&-t&U-\lambda\end{vmatrix}$$

$$= (U-\lambda)\left[-\lambda(-\lambda(U-\lambda)) - t(-\lambda t - \lambda t)\right] + t(U-\lambda)(\lambda t + \lambda t)$$

$$= (U-\lambda)\left[\lambda^2(U-\lambda) + 4\lambda t^2\right]$$

$$\Rightarrow \lambda = U \text{ or } \lambda = 0 \text{ or } \lambda(U-\lambda) + 4t^2 = 0$$

$$\lambda^2 - \lambda U - 4t^2 = 0 \Rightarrow \lambda = \tfrac{1}{2}\left[+U \pm \sqrt{U^2+16t^2}\right] = \frac{U}{2} \pm \frac{U}{2}\sqrt{1+\frac{16t^2}{U^2}}$$

$$\text{or } \lambda \approx \frac{U}{2} \pm \frac{U}{2}\left(1 - \frac{8t^2}{U^2}\right) = \frac{U}{2} \pm \frac{U}{2} \pm \frac{4t^2}{U}$$

The ground state energy is the lowest of $\{U, 0, U+\frac{4t^2}{U}, -\frac{4t^2}{U}\}$ which is $E_0 = -\frac{4t^2}{U}$ and the first excited state is $E_1 = 0$.
Both have degeneracy 1 because there are 4 states and 4 energies.

5. *Quantum Mechanics* (Spring 2003)

Consider a particle of mass m which moves in the potential

$$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ ax & \text{for } x > 0 \end{cases}$$

Estimate the ground state energy.

see Griffiths Example 8.3

Perhaps you could use the variational method for this problem, but it lends itself more to the WKB approximation.

Recall the formula for the WKB approximation in a well with one infinite wall: $\int_0^{x_2} \sqrt{2m(E-V(x))}\,dx = (n-\frac{1}{4})\pi\hbar \quad (n \in \mathbb{Z}^+)$

where x_2 is the classical turning point: $E = V(x_2) \Rightarrow x_2 = \frac{E}{a}$

$$\Rightarrow \int_0^{E/a} \sqrt{2m(E_n - ax)}\,dx = (n-\tfrac{1}{4})\pi\hbar$$

Let $u = 2m(E_n - ax) \Rightarrow du = -2ma\,dx$

$$\int_{x=0}^{x=E/a} u^{1/2} \frac{du}{-2ma} = \frac{1}{-2ma}\left(\frac{2}{3}u^{3/2}\right)\Big|_{x=0}^{x=E/a}$$

$$= \frac{1}{-3ma}\left(2m(E_n - ax)\right)^{3/2}\Big|_{x=0}^{x=E/a} = \frac{1}{3ma}(2mE)^{3/2}$$

$$\Rightarrow \frac{1}{3ma}(2mE_n)^{3/2} = (n-\tfrac{1}{4})\pi\hbar$$

$$\Rightarrow (2mE_n)^{3/2} = 3ma(n-\tfrac{1}{4})\pi\hbar$$

$$\Rightarrow E_n = \frac{1}{2m}\left[3ma(n-\tfrac{1}{4})\pi\hbar\right]^{2/3}$$

So the ground state energy is approximately

$$E_1 = \frac{1}{2m}\left(\frac{9}{4}ma\pi\hbar\right)^{2/3}$$

8. *Electricity and Magnetism* (Spring 2003)

A cylindrical capacitor of length L is composed of an inner cylindrical conductor of radius r and a concentric outer conducting cylindrical shell of radius R.

(a) What is the capacitance of this arrangement (you may ignore fringing fields at the ends)?

(b) The two conductors are held at a constant potential difference, V, using a battery. A cylindrical shell of dielectric material of length L and which just fits in between the conductors (inner radius $\sim r$ and outer radius $\sim R$) is inserted so that half is inside of the capacitor (i.e. $L/2$ of the length of the capacitor is now filled with dielectric). What is the force on the dielectric in this position (magnitude and direction)?

a. $Q = CV$, so we calculate the voltage for a given total charge. Note that a charge Q on the capacitor means each plate has magnitude of charge Q.

We place a Gaussian cylinder around the inner cylinder with radius ρ. $\int_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow 2\pi\rho\,\Delta z\, E = \frac{\frac{Q}{L}\Delta z}{\epsilon_0}$

$$\Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 \rho L}\hat{r}$$

$$V = -\int_r^R \vec{E}(\vec{r})\cdot d\vec{l} = -\int_r^R \frac{Q}{2\pi\epsilon_0 L}\frac{1}{\rho}\,d\rho = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{R}{r}\right)$$

$Q = CV \Rightarrow C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{R}{r}\right)}$ since we use the magnitude of V

b. We find the stored energy as a function of z from $U = \frac{1}{2}CV^2$ and then find $\vec{F}$ by $\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial z}\hat{z}$.

If the material has dielectric constant ϵ, then the capacitance for the filled part follows the same derivation as in part (a), but ϵ_0 is replaced with ϵ. Assume the dielectric comes in from below.

$$U(z) = \frac{1}{2}C'(z)V^2 + \frac{1}{2}C(L-z)V^2$$

$$= \frac{1}{2}\frac{2\pi\epsilon z}{\ln\left(\frac{R}{r}\right)}V^2 + \frac{1}{2}\frac{2\pi\epsilon_0(L-z)}{\ln\left(\frac{R}{r}\right)}V^2$$

$$= \frac{\pi V^2}{\ln\left(\frac{R}{r}\right)}\left(\epsilon z + \epsilon_0(L-z)\right)$$

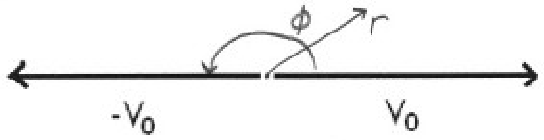
$$\Rightarrow \vec{F} = -\frac{\partial U}{\partial z}\hat{z} = -\frac{\pi V^2}{\ln\left(\frac{R}{r}\right)}(\epsilon - \epsilon_0)\hat{z}$$

And $\epsilon = (1+\chi_e)\epsilon_0 \Rightarrow \vec{F} = -\frac{\pi\chi_e\epsilon_0}{\ln\left(\frac{R}{r}\right)}V^2\hat{z}$

which is pushing the dielectric back out.

9. *Electricity and Magnetism* (Spring 2004)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



See Jackson Section 2.11

We solve Laplace's equation in cylindrical coordinates

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

There is no z dependence by symmetry so we use separation of variables and seek solutions of the form $\Phi(r,\phi) = R(r)Q(\phi)$.
(Or you could recall that the solution is $\Phi(r,\phi) = (A + B\ln(r))(C + D\phi)$ when r ranges from 0 to ∞).

$$\nabla^2\Phi = 0 \Rightarrow \frac{Q}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R}{r^2}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = 0$$

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda \quad \text{and} \quad \frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -\lambda$$

by independence of variables

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = \lambda R \quad \text{and} \quad \frac{\partial^2 Q}{\partial \phi^2} = -\lambda Q$$

$$\Rightarrow \begin{cases} R(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}} & \text{and } Q(\phi) = C\sin(\sqrt{\lambda}\phi) + D\cos(\sqrt{\lambda}\phi) \quad (\lambda \neq 0) \\ R(r) = A' + B'\ln(r) & \text{and } Q(\phi) = C' + D'\phi \quad (\lambda = 0) \end{cases}$$

The conditions that $|\Phi(r=\infty)| < \infty$ and $|\Phi(r=0)| < \infty$ imply $A = B = B' = 0$, so the $\lambda \neq 0$ case is excluded.

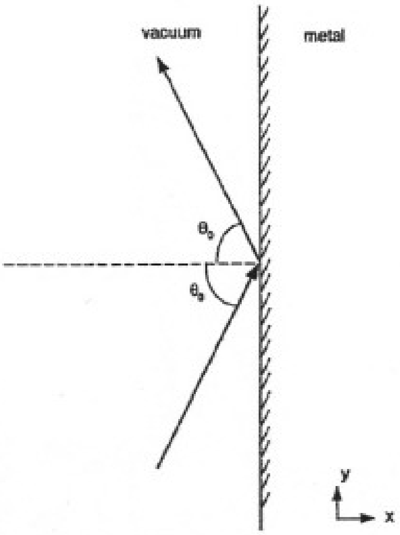
$$\Rightarrow \Phi(r,\phi) = C' + D'\phi$$

$$\Phi(\phi=0) = V_0 \Rightarrow C' = V_0 \quad \text{and} \quad \Phi(\phi=\pi) = -V_0 \Rightarrow D' = -\frac{2V_0}{\pi}$$

Therefore $\Phi(r,\phi) = V_0\left(1 - \frac{2}{\pi}\phi\right)$

10. *Electricity and Magnetism* (Spring 2003)

X-Ray Mirror: X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. As shown below, the metal occupies the region $x > 0$. The X-rays propagate in the x–y plane (the plane of the picture) and their polarization is in the z direction, coming out of the page. Assume that the metal contains n free electrons per unit volume and is non-magnetic. Derive an expression for the critical angle θ_0.



The critical angle comes from Snell's Law when $\theta_2 = \frac{\pi}{2}$

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \rightarrow n_2$$

$$\Rightarrow \sin(\theta_1) = \frac{n_2}{n_1} \Rightarrow \theta_1 = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

$$\Rightarrow \theta_c = \sin^{-1}(n_2) \quad \text{since } n_1 = 1 \text{ in vacuum}$$

The index of refraction of the metal is calculated from the plasma frequency in the high frequency approximation since X-rays are high frequency.

$$V = \frac{c}{n} \Rightarrow n = \frac{c}{v} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\frac{\epsilon}{\epsilon_0}} \quad \text{since the metal is non-magnetic}$$

$$\Rightarrow n \cong \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad \text{where } \omega_p^2 = \frac{ne^2}{\epsilon_0 m}$$

$$\Rightarrow \theta_c \cong \sin^{-1}(n_2) \cong \sin^{-1}\left(\sqrt{1 - \frac{ne^2}{\epsilon_0 m \omega^2}}\right)$$

11. *Electricity and Magnetism* (Spring 2003)

Secret Circuit: A two-terminal "black box" is given to you. Inside the box a circuit is attached to the terminals which is known to contain a lossless inductor L, a lossless capacitor C, and a resistor R. When a 1.5 Volt battery is connected across the terminals, a current of 1.5 milliamperes flows. When an AC voltage of 1.0 Volt (RMS) at a frequency of 60 Hz is connected, a current of 0.01 amperes (RMS) flows. As the AC frequency is increased while the applied voltage is maintained constant, the current is found to go through a maximum exceeding 100 amperes at $\nu = 1000$ Hz. What is the circuit inside the box? What are the values of R, L, and C?

The requirements on the circuit are

- Resistor of resistance $R = \frac{V}{I} = \frac{1.5V}{1.5mA} = 1000\Omega$ is in it
- Inductor L is not in parallel or it would short circuit
- Capacitor C is not in series or it would block D.C.
- Inductor and Capacitor are in series so there is a resonant frequency at which the impedance goes to zero

So the possible circuits are

Now we solve for C and L by applying the conditions

$$Z(\omega = 2\pi \cdot 60\,Hz) = \frac{V}{I} = \frac{1.0V}{0.01A} = 100\Omega$$

$$Z(\omega = 2\pi \cdot 1000\,Hz) = \frac{V}{I} = \frac{1.0V}{100A} \cong 0$$

For the first circuit, $\frac{1}{Z} = \frac{1}{R} + \frac{1}{Z_L + Z_C} = \frac{1}{R} + \frac{1}{i\omega L + \frac{1}{i\omega C}} = \frac{1}{R} + \frac{i\omega C}{1 - \omega^2 LC}$

So there is a frequency of zero impedance at $\omega^2 = \frac{1}{LC} \Rightarrow \omega = \sqrt{\frac{1}{LC}}$

$$\Rightarrow LC = \frac{1}{\omega^2} = \frac{1}{4\pi^2 \cdot 10^6\,Hz^2}$$

Now using the first condition, $\frac{1}{100\Omega} = \left| \frac{1}{1000\Omega} + \frac{i(2\pi \cdot 60Hz)C}{1 - (2\pi \cdot 60Hz)^2 LC} \right|$

$$\Rightarrow \frac{1}{100\Omega} \cong \left| \frac{1}{1000\Omega} + i(2\pi \cdot 60Hz)C \right|$$

$$\Rightarrow 10^{-4}\Omega^{-2} \cong 10^{-6}\Omega^{-2} + (2\pi \cdot 60Hz)^2 C^2.$$

$$\Rightarrow 10^{-4}\Omega^{-2} \cong (2\pi \cdot 60Hz)^2 C^2$$

$$\Rightarrow 10^{-2}\Omega^{-1} \cong 2\pi \cdot 60\,Hz\,C$$

$$\Rightarrow C \cong \frac{1}{12\pi}\,mF$$

$$\Rightarrow L = \frac{1}{4\pi^2 \cdot 10^6 Hz^2} \frac{1}{C} = \frac{12\pi \cdot 10^3}{4\pi^2 \cdot 10^6} H = \frac{3}{\pi}\,mH$$

12. *Electricity and Magnetism* (Spring 2003)

(a) Show that the field inside a sphere of uniformly magnetized material ($\mathbf{M} = M\,\hat{\mathbf{z}}$) is:

$$\mathbf{B} = \frac{2}{3}\mu_0 M \hat{\mathbf{z}}$$

(b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_o\hat{\mathbf{z}}$. Using the above result, find the magnetic field inside the sphere.

a. $\vec{\nabla}\times\vec{H} = \rho_f = 0 \Rightarrow \vec{H}$ is curl free $\Rightarrow \vec{H} = -\vec{\nabla}\Phi_m$ for some scalar field Φ_M

$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{\nabla}\Phi_M = -\nabla^2\Phi_m$ and $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}(\frac{1}{\mu_0}\vec{B}-\vec{M}) = -\vec{\nabla}\cdot\vec{M} \Rightarrow \nabla^2\Phi_M = \vec{\nabla}\cdot\vec{M}$

Since the magnetization is uniform, only the boundary contributes

$$\Phi_M = \frac{1}{4\pi}\int_S \frac{K_b(\vec{x}')}{|\vec{x}-\vec{x}'|}da' = \frac{1}{4\pi}\int_S \frac{\vec{M}(\vec{x}')\cdot\hat{n}}{|\vec{x}-\vec{x}'|}da'$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{M\cos(\theta')}{|\vec{x}-\vec{x}'|}a^2\sin(\theta')\,d\theta' d\phi'$$

$$= \frac{Ma^2}{4\pi}\int \frac{\cos(\theta')}{|\vec{x}-\vec{x}'|}d\Omega'$$

Now we use the expansion $\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)$

$$\Phi_M = \frac{Ma^2}{4\pi}\int 4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')Y_{\ell m}(\theta,\phi)\cos(\theta')\,d\Omega'$$

$$= Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\int Y^*_{\ell m}(\theta',\phi')\left(\sqrt{\tfrac{4\pi}{3}}Y_{10}(\theta',\phi')\right)d\Omega'$$

since $Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos(\theta)$. Now $\int Y^*_{\ell' m'}(\theta,\phi)Y_{\ell m}(\theta,\phi)\,d\Omega = \delta_{\ell'\ell}\delta_{m'm}$,

$$\Phi_M = Ma^2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}}\,\delta_{\ell 1}\delta_{m0}$$

$$= Ma^2\left(\frac{1}{3}\frac{r_<}{r_>^2}\right)Y_{10}(\theta,\phi)\sqrt{\tfrac{4\pi}{3}} = \frac{1}{3}Ma^2\frac{r_<}{r_>^2}\cos(\theta)$$

And $r_>, r_<$ are the greater and lesser between r and a, so inside $r<a$:

$$\Phi_M = \frac{1}{3}Ma^2\frac{r}{a^2}\cos(\theta) = \frac{1}{3}Mr\cos(\theta) = \frac{1}{3}Mz$$

Therefore $\vec{H} = -\vec{\nabla}\Phi_M = -\frac{1}{3}M\hat{z} = -\frac{1}{3}\vec{M}$

and $\vec{H} = \frac{1}{\mu_0}\vec{B}-\vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) = \frac{2}{3}\mu_0\vec{M} = \frac{2}{3}\mu_0 M\hat{z}$

b. Note that $\vec{B} = \vec{B}_0 + \vec{B}_{sphere}$

$$\vec{M} = \chi_m\vec{H} = \chi_m\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}(\vec{B}_0+\vec{B}_{sphere})-\vec{M}\right)$$

$$= \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M}-\vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

$$\Rightarrow \left(1+\frac{\chi_m}{3}\right)\vec{M} = \frac{\chi_m}{\mu_0}\vec{B}_0 \Rightarrow \vec{M} = \frac{\chi_m}{\mu_0}\vec{B}_0\left(1+\frac{\chi_m}{3}\right)^{-1}$$

Therefore $\vec{B} = \vec{B}_0 + \vec{B}_{sphere} = \vec{B}_0 + \frac{2}{3}\mu_0\vec{M} = \vec{B}_0 + \frac{2}{3}\frac{\chi_m}{1+\frac{\chi_m}{3}}\vec{B}_0$

$$= \frac{1+\frac{\chi_m}{3}}{1+\frac{\chi_m}{3}}\vec{B}_0 + \frac{\frac{2\chi_m}{3}}{1+\frac{\chi_m}{3}}\vec{B}_0 = \frac{1+\chi_m}{1+\frac{\chi_m}{3}}\vec{B}_0$$

13. *Statistical Mechanics and Thermodynamics* (Spring 2003)

Consider a d-dimensional material in which the important excitation are non-conserved bosons, and assume that the dispersion relation for these bosons is $\omega = ak^3$, where k is the wave vector's amplitude and a is a constant. The low temperature specific heat goes as T^q. What is the value of the power, q? Note: The dimensionality, d, of the material is not necessarily equal to three.

$C_v = \left(\frac{dE}{dT}\right)_v$ so we calculate $E = \int_0^\infty \epsilon\, f(\epsilon)\, \rho(\epsilon)\, d\epsilon$

where $f(\epsilon) = \dfrac{1}{e^{\beta(\epsilon-\mu)}-1} = \dfrac{1}{e^{\beta\epsilon}-1}$ since $\mu = 0$ for non-conserved particles

Now we find the energy density of states.

$$\rho(\epsilon)d\epsilon = \rho(\vec{n})\, d^d n \propto n^{d-1} dn$$

$$\epsilon \propto \omega = ak^3 = a\left(\frac{n\pi}{L}\right)^3 \Rightarrow n^3 = \frac{L^3}{a\pi^3}\epsilon \Rightarrow n = \frac{L}{\pi}\left(\frac{\epsilon}{a}\right)^{1/3}$$

$$\Rightarrow dn = \frac{1}{3}\frac{L}{\pi} a^{-1/3} \epsilon^{-2/3} d\epsilon$$

$$\Rightarrow \rho(\epsilon)d\epsilon \propto \left(\frac{L}{\pi}\left(\frac{\epsilon}{a}\right)^{1/3}\right)^{d-1}\left(\frac{1}{3}\frac{L}{\pi}a^{-1/3}\epsilon^{-2/3}d\epsilon\right) \propto \epsilon^{d/3-1} d\epsilon$$

Therefore $E \propto \int_0^\infty \epsilon \dfrac{1}{e^{\beta\epsilon}-1} \epsilon^{d/3-1} d\epsilon$

$$= \int_0^\infty \frac{\epsilon^{d/3}}{e^{\beta\epsilon}-1} d\epsilon$$

Let $x = \beta\epsilon$, $dx = \beta\, d\epsilon$

$$\Rightarrow E \propto \left(\frac{1}{\beta}\right)^{d/3+1} \int_0^\infty \frac{x^{d/3}}{e^x - 1} dx$$

$$\Rightarrow E \propto T^{d/3+1}$$

Therefore $C_v = \left(\frac{dE}{dT}\right)_v \propto T^{d/3}$

QM S'03 #1

$V(x) = \frac{1}{2} m\omega^2 (x - \epsilon(t))^2$; $\epsilon(t) = \epsilon\, e^{-t^2/\tau^2}$ $\epsilon \ll 1$

At $t = -\infty$ the particle is in the ground state ($n = 0$) what are the possible states the particle can be in at $t = \infty$. Work to lowest order of ϵ.

$$V(x) = \frac{1}{2} m\omega^2 (x^2 - 2x\epsilon(t) + \epsilon^2(t)) = \frac{1}{2} m\omega^2 x^2 - m\omega^2 x\epsilon(t) + \underbrace{\cancel{\tfrac{1}{2} m\omega^2 \epsilon^2(t)}}_{\text{disregard}}$$

$$c(t) = \frac{-i}{\hbar} \int_{-\infty}^{\infty} \langle H' \rangle e^{i\omega_0 t} \quad ; \quad \omega_0 = \frac{E_b - E_a}{\hbar}$$

Now $H' = -m\omega^2 x\, \epsilon\, e^{-t^2/\tau^2}$ and $\langle H' \rangle = \langle n | H' | 0 \rangle$

$$\Rightarrow \quad -m\omega^2 \epsilon\, e^{-t^2/\tau^2} \langle n | X | 0 \rangle \quad ; \quad \text{but} \quad X = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$\Rightarrow -m\omega^2 \epsilon\, e^{-t^2/\tau^2} \left(\sqrt{\frac{\hbar}{2m\omega}} \left[\cancelto{0}{\langle n | a | 0 \rangle} + \underbrace{\langle n | a^\dagger | 0 \rangle}_{\substack{= \sqrt{1+0} \text{ for } n=1 \\ = 0 \text{ for } n \neq 1}} \right] \right)$$

$$= -m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}}\, e^{-t^2/\tau^2} \quad \text{and} \quad \omega_0 = \frac{E_1 - E_0}{\hbar} = \frac{(1+\frac{1}{2})\hbar\omega - (0+\frac{1}{2})\hbar\omega}{\hbar} = \frac{3}{2}\omega - \frac{1}{2}\omega = \omega$$

so $$c_{0\to 1}(t) = \frac{-i}{\hbar} \left(-m\omega^2 \epsilon \sqrt{\frac{\hbar}{2m\omega}} \right) \int_{-\infty}^{\infty} e^{-t^2/\tau^2} e^{i\omega t}\, dt$$

we need to complete the square:

$$\int_{-\infty}^{\infty} e^{-(t^2/\tau^2 - i\omega t)}\, dt \qquad y \equiv \sqrt{a}\, t + \frac{b}{2\sqrt{a}} \Rightarrow y^2 = a t^2 + bt + \frac{b^2}{4a}$$

so $a = \frac{1}{\tau^2}$; $b = -i\omega$ $\Rightarrow$ $y = \frac{1}{\tau} t - \frac{i\omega\tau}{2}$ $\quad dy = \frac{1}{\tau}\, dt$

so $\int_{-\infty}^{\infty} e^{-(a\tau^2 - i\omega x)}\, dx = \frac{1}{\tau^{-1}} \int_{-\infty}^{\infty} e^{-(y^2 - \frac{b^2}{4a})}\, dy = \frac{e^{b^2/4a}}{\tau^{-1}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{2\int_0^{\infty} e^{-y^2} dy = \sqrt{\pi}} = \frac{\sqrt{\pi}}{\tau^{-1}} e^{b^2/4a}$

hence $c_{0\to 1}(t) = \frac{i\, m\, \omega^2 \epsilon}{\hbar \tau^{-1}} \sqrt{\frac{\hbar \pi}{2 m \omega}}\; e^{b^2/4a}$

and the probability is: $|c_{0\to 1}|^2 = \frac{m^{\cancel{2}}\omega^{\cancel{4}\,3}\epsilon^2}{\hbar^{\cancel{2}}\tau^{-2}} \frac{\cancel{\hbar}\pi}{2\cancel{m}\cancel{\omega}}\; e^{b^2/2a}$

$= \frac{m\omega^3\epsilon^2\pi}{2\hbar\tau^{-2}} e^{(-i\omega)^2/2(\frac{1}{\tau^2})} = \frac{m\omega^3\epsilon^2\pi}{2\hbar\tau^{-2}} e^{-\frac{\omega^2\tau^2}{2}} = \boxed{\frac{m\omega^3\epsilon^2\tau^2\pi}{2\hbar} e^{-\frac{\omega^2\tau^2}{2}}}$

Spring 2003 #1 $\quad \hbar, c = 1$

$$V(x) = \tfrac{1}{2} m\omega^2 (x - \epsilon(t))^2 \qquad \epsilon(t) = \epsilon\, e^{-t^2/\tau^2} \qquad \epsilon \ll 1$$

$$= \tfrac{1}{2} m\omega^2 x^2 \left(1 - \frac{\epsilon(t)}{x}\right)^2$$

$$= \frac{1}{2} m\omega^2 x^2 \left(1 - 2\frac{\epsilon(t)}{x} + \cancel{\frac{\epsilon^2(t)}{x^2}}\right)$$ lowest order in ϵ

$$= \underbrace{\frac{1}{2} m\omega^2 x^2}_{H_0} \underbrace{- m\omega^2 x\, \epsilon(t)}_{H'}$$

$$H' = -m\omega^2 x\, \epsilon\, e^{-t^2/\tau^2}$$

$\langle n|$

$$|\psi(t)\rangle = c(t)\, e^{-iE_n t} |\psi_n\rangle$$

$$\langle n|\psi(t)\rangle = c_n(t)\, e^{-iE_n t}$$

$$c_n(t) = -i \int_{-\infty}^{\infty} \langle n|H'|0\rangle \cdot e^{i\omega_{n0} t'}\, dt' \qquad \omega_{n,0} = E_n - E_0$$

first order

$$\omega_{1,0} = \omega(1 + \tfrac{1}{2}) - \omega(\tfrac{1}{2}) = \omega$$

$$= +i m\omega^2 \epsilon \langle n|x|0\rangle \int_{-\infty}^{\infty} e^{-t'^2/\tau^2} e^{i\omega_{n0} t'}\, dt'$$

but $x = \frac{1}{\sqrt{2m\omega}}(a + a^+)$ so $\langle n|\cancel{a} + a^+|0\rangle$ $= 0$ unless $n = 1$

since $\frac{\langle n|a^+|0\rangle}{\sqrt{2m\omega}} = \frac{1}{\sqrt{2m\omega}}$

$$c_n = \frac{i m\omega^2 \epsilon}{\sqrt{2m\omega}} \int_{-\infty}^{\infty} e^{-t^2/\tau^2 + i\omega_{1,0} t}\, dt$$

$$\int_{-\infty}^{\infty} e^{-(at^2 + bt + c)}\, dt = \sqrt{\frac{\pi}{a}}\, e^{\left(\frac{b^2 - 4ac}{4a}\right)} \Rightarrow c_{0\to 1} = \frac{i m\omega^2 \epsilon\, \tau}{\sqrt{2m\omega}} \sqrt{\pi}\, e^{-\frac{\omega^2 \tau^2}{4}}$$

$$a = \frac{1}{\tau^2} \quad b = i\omega_{1,0} \quad c = 0$$

$$|C_{0\to 1}(t)|^2 = \frac{m^2\omega^4\varepsilon^2\tau^2\pi}{2m\omega}\, e^{-\frac{\omega^2\tau^2}{2}}$$

$$= \frac{m\omega^3\varepsilon^2\tau^2\pi}{2}\, e^{-\frac{\omega^2\tau^2}{2}}$$

In one-dimension, a particle is subject to a harmonic oscillator potential with a time dependent origin,

$$V(x) = \frac{1}{2} m\omega^2 [x - \epsilon(t)]^2$$

where

$$\epsilon(t) = \epsilon e^{-t^2/\tau^2} \qquad \epsilon << 1$$

Suppose the particle is in the ground state at $t = -\infty$. What states can the particle be in at $t = +\infty$, and what are the probabilities for each? Work to lowest order in ϵ.

So,

$$V(x) \cong \underbrace{\frac{1}{2} m\omega^2 x^2}_{= V(x)} \underbrace{- m\omega^2 x \epsilon(t)}_{= V'(x)} \qquad \leftarrow \text{to lowest order in } \epsilon$$

From Zettili eq. 10.41, we have that the transition probability is given by ($\hbar = 1$)

$$P_{if}(t) = \left| -i \int_0^t \langle \psi_f | V'(t') | \psi_i \rangle e^{i\omega_{fi} t'} dt' \right|^2$$

For our case, $t \to \infty$ and $|\psi_f\rangle = |n\rangle$, $|\psi_i\rangle = |0\rangle$, $\omega_{fi} = E_n - E_0$

$$= \omega(n + \tfrac{1}{2}) - \frac{\omega}{2}$$

$$= \omega n$$

So, we have

$$P_{n0}(t) = \left| \int_{-\infty}^{\infty} \langle n | (-m\omega^2 x \epsilon(t')) | 0 \rangle e^{i\omega n t'} dt' \right|^2 , \quad \epsilon(t) = \epsilon e^{-t^2/\tau^2}, \quad x = \frac{1}{\sqrt{2m\omega}} (a + a^\dagger)$$

$$= \frac{m^2 \omega^4}{2m\omega} \epsilon^2 \left| \int_{-\infty}^{\infty} \langle n | (a + a^\dagger) | 0 \rangle e^{-t'^2/\tau^2} e^{i\omega n t'} dt' \right|^2$$

where $\langle n | (a + a^\dagger) | 0 \rangle = \cancel{\langle n | a | 0 \rangle}^{0} + \langle n | a^\dagger | 0 \rangle = \sqrt{0+1}\, \delta_{n,1} = \delta_{n,1}$

So, we have $P_{n0}(t) = 0 \quad \forall n \neq 1$ and

$$P_{10}(t) = \left| \int_{-\infty}^{\infty} e^{-\left(\frac{1}{\tau^2} t'^2 - i\omega t'\right)} dt' \right|^2 \frac{m\omega^3}{2} \epsilon^2$$

note: $\int_{-\infty}^{\infty} e^{-(ax^2 + bx + c)} dx = \sqrt{\frac{\pi}{a}} \, e^{\frac{b^2 - 4ac}{4a}}$

so,

$$P_{10}(t) = \frac{m\omega^3}{2} \epsilon^2 \left| \sqrt{\frac{\pi}{1/\tau^2}} \, e^{(-\omega^2 - 0)\left(\frac{1}{4(\frac{1}{\tau^2})}\right)} \right|^2$$

$$= \frac{m\omega^3}{2} \epsilon^2 \tau^2 \pi \left| e^{-\frac{\omega^2 \tau^2}{4}} \right|^2$$

$$\therefore \boxed{P_{10}(t) = \frac{m\omega^3 \tau^2 \pi \epsilon^2}{2} e^{-\frac{\omega^2 \tau^2}{2}}}$$

all other $P_{n0}(t) = 0 \quad \forall n \neq 1$

QM S'03 #2

$H = J\sigma_1^z\sigma_2^z + h(\sigma_1^x + \sigma_2^x)$ For two $s=\frac{1}{2}$ spins.

What is the ground state energy?

$$H|\uparrow\uparrow\rangle = J|\uparrow\uparrow\rangle + h(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$$
$$H|\downarrow\downarrow\rangle = J|\downarrow\downarrow\rangle + h(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$H|\uparrow\downarrow\rangle = -J|\uparrow\downarrow\rangle + h(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle)$$
$$H|\downarrow\uparrow\rangle = -J|\downarrow\uparrow\rangle + h(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$$

From inspection we can get two eigenfunctions:

$$H(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) = J|\uparrow\uparrow\rangle + h(\cancel{|\downarrow\uparrow\rangle} + \cancel{|\uparrow\downarrow\rangle}) - J|\downarrow\downarrow\rangle - h(\cancel{|\uparrow\downarrow\rangle} + \cancel{|\downarrow\uparrow\rangle})$$
$$= J(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$$

$$H(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = -J|\uparrow\downarrow\rangle + h(\cancel{|\downarrow\downarrow\rangle} + \cancel{|\uparrow\uparrow\rangle}) + J|\downarrow\uparrow\rangle - h(\cancel{|\uparrow\uparrow\rangle} + \cancel{|\downarrow\downarrow\rangle})$$
$$= -J(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

As for the other two eigenfunctions:

$$H(\alpha[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + \beta[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]) = \lambda(\alpha[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + \beta[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle])$$

$$\alpha[J|\uparrow\uparrow\rangle + h(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) + J|\downarrow\downarrow\rangle + h(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)]$$
$$+\beta[-J|\uparrow\downarrow\rangle + h(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) - J|\downarrow\uparrow\rangle + h(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)]$$

$$= (\alpha J[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + 2\beta h[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle])$$
$$+(-\beta J[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] + 2\alpha h[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle])$$

$$= (\alpha J + 2\beta h)[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + (2\alpha h - \beta J)[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] = \lambda\alpha[|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle] + \lambda\beta[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]$$

so

$$\begin{aligned} \alpha J + 2\beta h &= \alpha \lambda \\ 2\alpha h - J\beta &= \beta\lambda \end{aligned} \quad \Rightarrow \quad \begin{aligned} \alpha(J-\lambda) + 2\beta h &= 0 \\ 2\alpha h - \beta(J+\lambda) &= 0 \end{aligned}$$

which can be written in a 2x2 matrix:

$$\begin{bmatrix} (J-\lambda) & 2h \\ 2h & -(J+\lambda) \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

now we need to solve the characteristic equation:

$$-(J-\lambda)(J+\lambda) - 4h^2 = 0 \quad \Rightarrow \quad (J-\lambda)(J+\lambda) = -4h^2$$

$$J^2 - \lambda^2 = -4h^2$$

so $\quad \lambda = \pm\sqrt{J^2+4h^2}$

In the end we have the following energies:

$$+J, \ -J, \ +\sqrt{J^2+4h^2}, \ -\sqrt{J^2+4h^2}$$

if the ground state energy is the lowest energy then $-\sqrt{J^2+4h^2}$ should be that energy.

QM S'03 #3

A coherent state of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalues basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$.

We have

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

expand $|\alpha\rangle$ in the energy basis:

$$|\alpha\rangle = \sum c_n |n\rangle$$

now $\quad c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^n}{\sqrt{n!}} c_0$

to properly normalize the wavefunction

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \sum \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \quad \text{as } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots = \sum_n \frac{x^n}{n!}$$

so $\quad c_0 = e^{-\frac{|\alpha|^2}{2}}$

Hence

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Spring 2003 #3 (p 1 of 1)

A coherent state of a simple harmonic oscillator is an eigenstate of the annihilation operator, a. In terms of the energy eigenvalue basis, give an explicit expression for a coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$.

(see Abws # 3.22 part (b))

we can rewrite the coherent state as follows: (Zettili eq 2.163)

$$|\alpha\rangle = I|\alpha\rangle = \left(\sum_{n=1}^{\infty} |n\rangle\langle n|\right)|\alpha\rangle$$

$$= \sum_{n=1}^{\infty} |n\rangle \langle n|\alpha\rangle = \sum_{n=1}^{\infty} a_n |n\rangle$$

where the coefficient a_n represents the projection of $|\alpha\rangle$ onto $|n\rangle$. a_n can be written in terms of a_0 by (see Zettili eq 4.137)

$$a_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^n}{\sqrt{n!}} c_0$$

to determine c_0, we need to use the normalization condition

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \left(\sum_{n=1}^{\infty} \frac{|\alpha|^n}{\sqrt{n!}}\right)^2$$

note: $e^x = \sum_n \frac{x^n}{n!} \Rightarrow e^{x^2} = \sum_n \frac{x^{2n}}{n!}$

So,

$$|c_0|^2 = e^{-\alpha^2} \Rightarrow c_0 = e^{-\frac{\alpha^2}{2}}$$

Thus,

$$\boxed{|\alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

S'03 #4 Q.M.

$$H = -t\,(|1\uparrow,1\downarrow\rangle\langle 1\uparrow,2\downarrow| + h.c. + |2\uparrow,2\downarrow\rangle\langle 1\uparrow,2\downarrow| + h.c.)$$
$$+\,U\,(|1\uparrow,1\downarrow\rangle\langle 1\uparrow,1\downarrow| + |2\uparrow,2\downarrow\rangle\langle 2\uparrow,2\downarrow|)$$

What are the energies and degeneracies of the ground ~~st~~ and first excited states of the system to lowest order in t for $t \ll U$?

First re-label the above bra/kets:

$$|0\rangle \equiv |1\uparrow,2\downarrow\rangle;\quad |1\rangle \equiv |1\uparrow,1\downarrow\rangle;\quad |2\rangle \equiv |2\uparrow,2\downarrow\rangle$$

so the Hamiltonian now looks like:

$$H = U\,(|1\rangle\langle 1| + |2\rangle\langle 2|) - t\,(|1\rangle\langle 0| + |0\rangle\langle 1| + |2\rangle\langle 0| + |0\rangle\langle 2|)$$

with no perturbation ($t=0$):

$$H = U\,(|1\rangle\langle 1| + |2\rangle\langle 2|)$$

and the ground and first excited state are:

$$H|0\rangle = 0;\quad H|1\rangle = U|1\rangle;\quad H|2\rangle = U|2\rangle$$

so the ground state is none degenerate, while the first excited state is doubly degenerate:

Now let $t \neq 0$ and first the matrix corresponding to H

$\langle 0|H|0\rangle = 0 \qquad \langle 0|H|1\rangle = -t \qquad \langle 0|H|2\rangle = -t$

$\langle 1|H|0\rangle = -t \qquad \langle 1|H|1\rangle = U \qquad \langle 1|H|2\rangle = 0$

$\langle 2|H|0\rangle = -t \qquad \langle 2|H|1\rangle = 0 \qquad \langle 2|H|2\rangle = U$

so $\quad H = \begin{bmatrix} 0 & -t & -t \\ -t & U & 0 \\ -t & 0 & U \end{bmatrix}$ which can be diagonalized

$$\det \begin{vmatrix} -\lambda & -t & -t \\ -t & U-\lambda & 0 \\ -t & 0 & U-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda(U-\lambda)^2 - [+t^2(U-\lambda)] + [-t^2(U-\lambda)] = 0$

so

$$\lambda(U-\lambda)^2 + 2t^2(U-\lambda) = 0 \Rightarrow (U-\lambda)[\lambda(U-\lambda) + 2t^2] = 0$$

one eigenvalue is $\lambda = U$; as for the other: $\lambda(U-\lambda) + 2t^2 = 0$

$$\Rightarrow U\lambda - \lambda^2 + 2t^2 = 0 \Rightarrow \underbrace{\lambda^2 - U\lambda}_{(\lambda - \frac{U}{2})^2 - \frac{U^2}{4}} - 2t^2 = 0 \Rightarrow (\lambda - \tfrac{U}{2})^2 - \tfrac{U^2}{4} - 2t^2 = 0$$

$$(\lambda - \tfrac{U}{2})^2 = \tfrac{U^2}{4} + 2t^2$$

so $\lambda = \frac{U}{2} \pm \left(\frac{U^2}{4} + 2t^2\right)^{1/2}$ are the other two eigenvalues.

Expanding the above in terms of $t \ll U$

$$\lambda = \frac{U}{2} \pm \frac{U}{2}\left(1 + \frac{8t^2}{U^2}\right)^{1/2} \approx \frac{U}{2} \pm \frac{U}{2}\left(1 + \frac{4t^2}{U^2} + \dots\right)$$

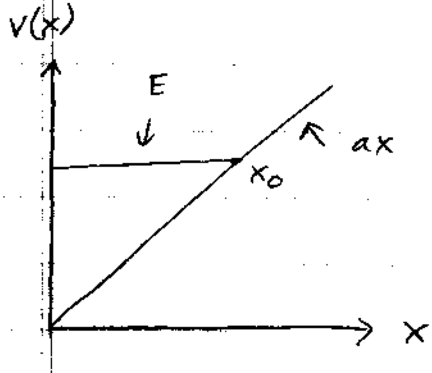
to lowest order in t: $\lambda = \frac{U}{2} \pm \frac{U}{2}\left(1 + \frac{4t^2}{U^2}\right) = \begin{matrix} U + \frac{2t^2}{U} \\ 0 - \frac{2t^2}{U} \end{matrix}$

So the energy levels now look like:

$t = 0$

$t=0$		
		$U + \frac{2t^2}{U}$
$|1\rangle, |2\rangle$	U	U
$|0\rangle$	0	$-\frac{2t^2}{U}$

QM S'03 #5



$$V(x) = \begin{cases} ax & x > 0 \\ \infty & x < 0 \end{cases}$$

Estimate the ground state energy.

Use the WKB approximation:

$$\int_0^{x_0} p\, dx = \left(n - \tfrac{1}{4}\right)\pi\hbar \quad \text{Griffiths } 8.47 \text{ p. } 289 \text{ (note: } n = 1, 2, 3 \ldots)$$

classical turning pt.

$$p = \sqrt{2m[E - V(x)]} = \sqrt{2m[E - ax]}\,; \quad x_0:\ E = ax_0 \Rightarrow x_0 = \frac{E}{a}$$

$$\Rightarrow \int_0^{E/a} (2m[E - ax])^{1/2} dx = \frac{-1}{2ma}\int_{2mE}^{0} u^{1/2} du = \frac{1}{2ma}\int_0^{2mE} u^{1/2} du$$

$u = 2m[E - ax]$

$du = -2ma\,dx \Rightarrow dx = \frac{-du}{2ma}$

$$= \frac{1}{\cancel{2}ma}\left(\frac{\cancel{2}}{3}\right) u^{3/2}\Big|_0^{2mE} = \frac{(2mE)^{3/2}}{3ma}$$

But this is equal to:

$$\frac{(2mE)^{3/2}}{3ma} = \left(n - \tfrac{1}{4}\right)\pi\hbar \Rightarrow 2mE = \left[3ma\left(n - \tfrac{1}{4}\right)\pi\hbar\right]^{2/3}$$

$$\Rightarrow \quad E = \frac{\left[3ma\pi\hbar(n - 1/4)\right]^{2/3}}{2m}$$

$$E_1 = \frac{\left[3ma\pi\hbar(3/4)\right]^{2/3}}{2m} = \frac{\cancel{9}}{2m}\left[\frac{9ma\pi\hbar}{4}\right]^{2/3}$$

Problem #6 Spring 2003

$U(n) = n e$

a) N particles N lattice sites

n_0 = # sites w/ 0 atoms

n_1 = # sites w/ 1 atom

n_2 = # sites w/ 2 atoms

$$\Omega = \frac{N!}{n_0!\, n_1!\, n_2!}$$

$n = n_0 = n_2$ always

$$\Omega = \frac{N!}{(n!)^2\, n_1!} \qquad 2n + n_1 = N$$

$$\boxed{\Omega = \frac{N!}{(n!)^2 (N-2n)!}}$$

b) $S = k \ln \Omega$

$\Delta E = T \Delta S$

$E_{min} = 0 \; , \; n = 0$

$S_{min} = 0 = k \ln \left(\frac{N!}{N!} \right) = 0$

$E_{max} = \frac{N}{2} e \; , \; n = \frac{N}{2}$

$S_{max} = k \ln \left(\frac{N!}{\left(\frac{N}{2}! \right)^2 (N-N)!} \right)$

$= k \ln \left(\frac{N!}{(N/2!)^2} \right) = k \left[\ln[N!] - 2 \ln\left[\left(\frac{N}{2}\right)!\right] \right]$

assume N large

sterlings formula

$\ln N! = N \ln N - N$

$S_{max} = k \left[N \ln N - \cancel{N} - N \ln \frac{N}{2} + \cancel{N} \right]$

$= k \left[\cancel{N \ln N} - \cancel{N \ln N} + N \ln 2 \right]$

$S_{max} = KN \ln 2$

$\frac{\cancel{N}}{2} e = TK\cancel{N} \ln 2$

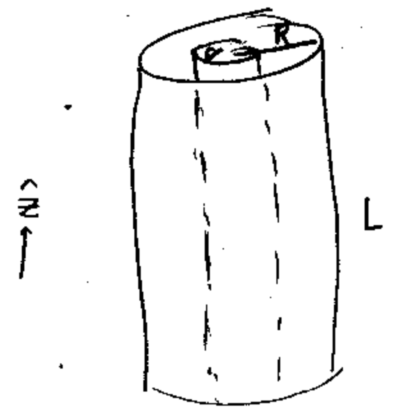
$$\boxed{T = \frac{e}{2K \ln 2}}$$

c) $\Xi = \sum_{n=0}^{2} e^{\beta \varepsilon n \mu}$ $\quad Z = \left(1 + e^{\beta\mu} + e^{2\beta\mu}\right) Z$

$$\Xi = \left(1 + e^{\beta\mu} + e^{2\beta\mu}\right)\left(2 + e^{\beta e}\right)$$

$$\Xi = \left(2 + 2e^{\beta\mu} + 2e^{2\beta\mu} + e^{\beta e} + e^{\beta(e+\mu)} + e^{\beta(e+2\mu)}\right)$$

Spring 2003 # 8



Lets put a charge Q on inner cylinder

$$\oint E \cdot da = 2\pi s L E = \frac{Q}{\epsilon_0}$$

$$\therefore E = \frac{Q}{2\pi s L \epsilon_0} \hat{s}$$

$$V = -\int_r^R E \cdot dl = \frac{-Q}{2\pi L \epsilon_0} \int_r^R \frac{1}{s} ds = \frac{-Q}{2\pi L \epsilon_0} \ln\left(\frac{R}{r}\right) = V(b) - V(a)$$

But $V(b) < V(a) \Rightarrow V(a) - V(b) = \frac{Q}{2\pi L \epsilon_0} \ln\left(\frac{R}{r}\right)$

a) $$C = \frac{Q}{V} = \frac{2\pi \epsilon_0 L}{\ln\left(\frac{R}{r}\right)}$$

b). Now half of gap is filled with a dielectric and have a potential difference V.

Empty part $V = \frac{\lambda}{2\pi \epsilon_0} \cdot \ln\left(\frac{R}{r}\right)$

Filled part $D = \frac{\lambda'}{2\pi s} \Rightarrow E = \frac{\lambda'}{2\pi \epsilon s} \Rightarrow V = \frac{\lambda'}{2\pi \epsilon} \ln\left(\frac{R}{r}\right)$

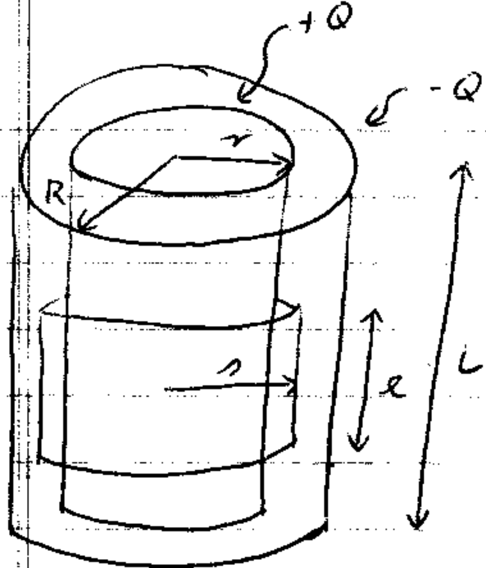
holding charge constant $\rightarrow \frac{\lambda}{\epsilon_0} = \frac{\lambda'}{\epsilon}$ $\lambda' = \frac{\epsilon}{\epsilon_0} \lambda = \epsilon_r \lambda$

$$Q = \lambda'(h) + \lambda(L - h) = \epsilon_r \lambda h - \lambda h + \lambda L = \lambda[\underbrace{(\epsilon_r - 1)}_{\chi_e} h + L]$$

$$C = \frac{Q}{V} = \frac{\lambda(\chi_e h + L)}{2\lambda \ln(R/r)} \cdot 4\pi \epsilon_0 = 2\pi \epsilon_0 \frac{(\chi_e h + L)}{\ln(b/a)}$$

$$F = \frac{1}{2}V^2 \frac{dC}{dh} = \frac{1}{2}V^2 \frac{2\pi\epsilon_0 \chi_e}{\ln(R/r)} \hat{z}$$

EM S'03 # 8



a) $C = ?$

$$C = \frac{Q}{V}; \quad V = -\int_a^b \vec{E}\cdot d\vec{l}$$

$$\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\epsilon_0$$

$$Q_{enc} = \sigma 2\pi r l \; ; \; \sigma = \frac{Q}{2\pi r L}$$

$$= \frac{Q\cancel{2\pi r} l}{\cancel{2\pi r} L} = \frac{Ql}{L}$$

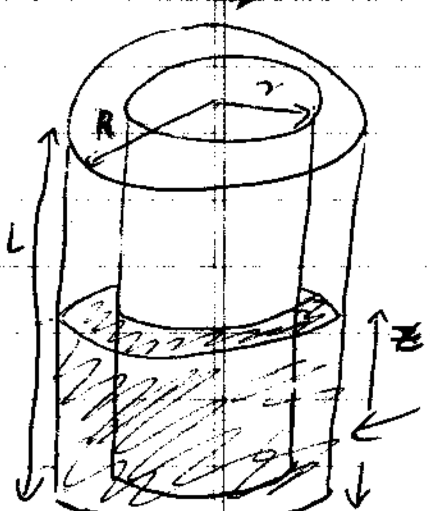
$$da = s\, d\phi\, dz \Rightarrow \int_0^{2\pi}\int_0^{l} E\, s\, dz\, d\phi = \frac{Ql}{\epsilon_0 L}$$

$$\Rightarrow E\, 2\pi s \cancel{l} = \frac{Q\cancel{l}}{\epsilon_0 L} \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 L s}\hat{s}$$

$$V = -\int_R^r \vec{E}\cdot \overset{\swarrow ds\hat{s}}{d\vec{s}} = -\frac{Q}{2\pi\epsilon_0 L}\int_R^r \frac{ds}{s} = \frac{-Q}{2\pi\epsilon_0 L}\ln(r/R) = \frac{Q}{2\pi\epsilon_0 L}\ln(R/r)$$

$$\therefore \; C = \frac{Q}{V} = \frac{\cancel{Q}}{\frac{\cancel{Q}\ln(R/r)}{2\pi\epsilon_0 L}} = \boxed{\frac{2\pi\epsilon_0 L}{\ln(R/r)}}$$

b)



Force on the dielectric : $F = -\frac{dW}{dz}$

$W = \frac{1}{2}CV^2$ as V is kept constant

$$= \frac{1}{2}(C_{\epsilon_0} + C_{\epsilon})V^2 = \frac{V^2}{\cancel{2}}\left(\frac{\cancel{2}\pi\epsilon_0(L-z)}{\ln(R/r)} + \frac{\cancel{2}\pi\epsilon z}{\ln(R/r)}\right)$$

dielectric ϵ

$$= \frac{V^2\pi}{\ln(R/r)}\left[\epsilon_0 L \underbrace{- \epsilon_0 z + \epsilon z}_{}\right] = \frac{V^2\pi\epsilon_0}{\ln(R/r)}\left[L + \chi_e z\right]$$

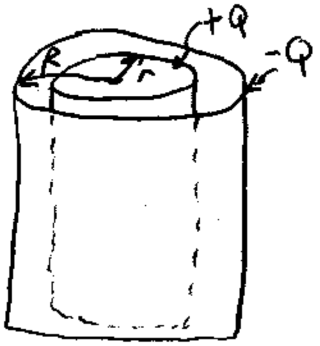
$$z(\epsilon - \epsilon_0) = z\epsilon_0(\underbrace{\epsilon_r - 1}_{\chi_e})$$

$$\boxed{F = -\frac{dW}{dx} = \frac{-V^2\pi\epsilon_0}{\ln(R/r)}\chi_e}$$

Spring 2003 # 8 (p 1 of 2)

A cylindrical capacitor of length L is composed of an inner cylindrical conductor of radius r and a concentric outer conducting cylindrical shell of radius R.

(a) what is the capacitance of the arrangement?



the capacitance is given by $C = \frac{Q}{V}$, (1)

where

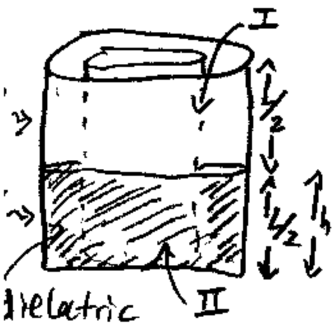
$$V = -\int_R^r \vec{E}\cdot d\vec{l} \quad , \quad \oint \vec{E}\cdot d\vec{a} = 4\pi Q \;\Rightarrow |\vec{E}| = \frac{4\pi Q}{2\pi s L} = \frac{2Q}{sL}$$

$$\Rightarrow V = -\int_R^r \frac{2Q}{sL}\,ds = -\frac{2Q}{L}\ln\frac{r}{R} = \frac{2Q}{L}\ln\frac{R}{r} \qquad (2)$$

Thus,

$$C = \frac{Q}{V} = \frac{Q}{\frac{2Q}{L}\ln\left(\frac{R}{r}\right)} = \boxed{\frac{L}{2\ln(R/r)}}$$

(b) The two conductors are held at constant potential difference, V, using a battery. A cylindrical shell of dielectric length L and which just fits between the conductors is inserted so that half is inside the conductor. what is the force on the dielectric in this position (see Fall 1997 #2)



the will be a force on the dielectric since the capacitance changes of the form

$$F = \frac{V^2}{2}\frac{dC}{dh} \qquad \text{(see Griffiths eq 4.64)}$$

since $C = \frac{Q}{V}$, we need to find Q and V for this arrangement.

(i) $Q = \lambda(L-h) + \lambda' h$ (2) ← need to keep general to allow dielectric to move!

(ii) From eq (2), we have that (with $Q \to \lambda\frac{L}{2}$ and $L \to L/2$)

$$V_I = 2\lambda \ln\frac{R}{r}$$

and

$$V_{II} = \frac{2\lambda'}{\epsilon}\ln\frac{R}{r} \qquad \leftarrow \vec{E} = \frac{\vec{D}}{\epsilon}$$

Since $V_I = V_{II}$, we have

$$2\lambda \ln\frac{R}{r} = \frac{2\lambda' \ln\frac{R}{r}}{\epsilon} \quad \Rightarrow \quad \lambda' = \epsilon\lambda \qquad (4)$$

inserting this result into eq (3) yields

$$Q = \lambda(L - h + \epsilon h) = \lambda\left[L + h(\epsilon - 1)\right]$$

note: $\epsilon - 1 = 4\pi\chi_e$

so,

$$Q = \lambda\left(L + h4\pi\chi_e\right)$$

Thus,

$$C = \frac{Q}{V_I} = \frac{\lambda(L + h4\pi\chi_e)}{2\lambda\ln(R/r)} = \frac{L + h4\pi\chi_e}{2\ln(R/r)}$$

and

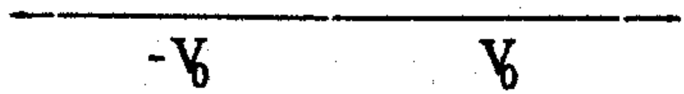
$$F = \frac{V^2}{2}\frac{dC}{dh} = \frac{V^2}{2}\frac{4\pi\chi_e}{2\ln(R/r)}$$

$$\Rightarrow \boxed{F = \frac{V^2\pi\chi_e}{\ln(R/r)}}$$

dielectric will rise until $\frac{dC}{dh} = 0$

9. ***Electricity and Magnetism***

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



$\phi(r, \phi = 0) = V_0$

$\phi(r, \phi = \pi) = -V_0$

$\phi(r,\phi) = [a_0 + b_0 \ln r][C_0 + d_0 \phi]$

$\phi(r,\phi) = [a r^{\alpha} + b r^{-\alpha}][C \cos\alpha\phi + d \sin\alpha\phi]$

for $\alpha > 0$

$\alpha = 0$, is appropiate since the angular spread is less than 2π

C morales Lec notes page 120)

$\phi(0) = V_0 = [a_0 + \cancel{b_0 \ln r}][C_0] \quad V_0 = a_0 C_0 \quad b_0 = 0$

$\phi(\pi) = -V_0 = [a_0][C_0 + d_0\pi] = a_0 C_0 + a_0 d_0 \pi$

$= V_0 + a_0 d_0 \pi = -V$

$a_0 d_0 \pi = -2V$

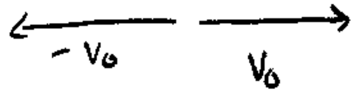
$a_0 d_0 = \frac{-2V}{\pi}$

$\phi(r,\phi) = a_0 C_0 + a_0 d_0 \phi$

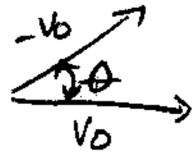
$\boxed{= V_0 - \frac{2V\phi}{\pi}}$ 9

Spring 2003 # 9 (p 1 of 1)

Consider the infinite two-dimensional conducting plane depicted in the figure. The right half is maintained at electrostatic potential V_0 while the left half is maintained at potential $-V_0$. What is the potential above the plane?



we can approach this problem the same way we would this problem



with $\theta = \pi$

Since the angle is restricted (That is, θ does not range to 2π), the general solution to the potential is

$$\Phi(r,\theta) = (a_0 + b_0 \ln r)(c_0 + d_0 \theta)$$

Now, apply boundary conditions.

- $\Phi(r,\theta=0) = V_0 = (a_0 + b_0 \ln r)\, c_0$

  the only way to satisfy that the rhs equals a constant is for $b_0 = 0$.

  $$\Rightarrow V_0 = a_0 c_0$$

- $\Phi(r,\theta=\pi) = -V_0 = a_0 (c_0 + d_0 \pi) = V_0 + a_0 d_0 \pi$

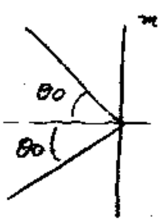
  $$\Rightarrow a_0 d_0 = \frac{-2V_0}{\pi}$$

Thus,

$$\Phi(r,\theta) = a_0 c_0 + a_0 d_0 \theta = V_0 - \frac{2V_0}{\pi}\theta = \boxed{V_0\left(1 - \frac{2\theta}{\pi}\right)}$$

→ verify that this potential satisfies boundary conditions!

S'03 E.M. #10



at θ_0 or greater the incident x-ray beam is completely reflected. Derive an expression for θ_0.

Regardless of frequency Snell's Law holds:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

now for $\theta_1 = \theta_0 \Rightarrow \sin\theta_0 = \frac{n_2}{n_1}\sin\theta_2$ with $\theta_2 = 90^\circ$

$$= \frac{n_2}{n_1}$$

Assuming $n_1 = 1$:

$$\theta_0 = \sin^{-1}(n_2)$$

now n_2 is given the plasma frequency:

$$\epsilon_r = \frac{\epsilon(\omega)}{\epsilon_0} = 1 + \frac{\omega_p^2}{\omega^2} \; ; \; \omega_p^2 = \frac{ne^2}{\epsilon_0 m_e}$$

so $\frac{\epsilon(\omega)}{\epsilon_0} = 1 \mp \frac{ne^2}{\epsilon_0 m_e \omega^2}$

but $n_2 = \sqrt{\frac{\epsilon}{\epsilon_0}\frac{\mu}{\mu_0}}$, but we are told $\mu = \mu_0$

so

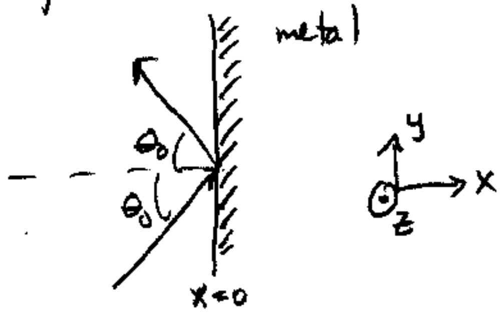
$$n_2 = \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1 \mp \frac{ne^2}{\epsilon_0 m_e \omega^2}}$$

hence $\theta_0 = \sin^{-1}\left(\sqrt{1 - \frac{ne^2}{\epsilon_0 m_e \omega^2}}\right)$

Spring 2003 # 10 (p 1 of 2)

X-ray Mirror : X-rays which strike the metal surface at an angle of incidence to the normal greater than a critical angle θ_0 are totally reflected. As shown below, the metal occupies the region x>0. The x-rays propagate in the x-y plane and their polarization is in the z-direction, coming out of the page. Assume that the metal contains n free electrons per unit volume and is non-magnetic. Derive an expression for the critical angle θ_0.

(see Lim Yung-Kuo #4029 part(c))



Snell's law is applicaple to this poblem (note: snell's law is independent of frequency). That is,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

for critical angle, we have $\theta_1 = \theta_0$ & $\theta_2 = 90°$. Assume $n_1 = 1$. So, we have

$$\sin\theta_0 = n_2$$

$$\Rightarrow \quad \theta_0 = \sin^{-1}(n_2) \qquad (1)$$

What is n_2?

The equation of motion for an electron in a field of X-rays is (from Lorentz force law)

$$m\ddot{x} = -eE_0 e^{-i\omega t} = -eE \qquad (2)$$

the solution to this D.E. is

$$x = x_0 e^{-i\omega t} \quad \Rightarrow \quad \ddot{x} = -\omega^2 x$$

substituting this result into eq (2) yields

$$m\omega^2 x = eE \qquad (3)$$

← see Jackson 2nd ed Section 4.6 (Molecular Polarizabil[ity]

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The dipole moment is given by

$$p = -ex \overset{\text{eq (3)}}{=} \frac{-e^2}{m\omega^2} E$$

Then the induced polarization is

$$\underline{P} = -\frac{ne^2}{m\omega^2} E = \chi_e E \qquad \left(\text{Jackson eq 4.36, 2nd ed}\right)$$

where n is the density of electrons.
So,

$$\chi_e = \frac{-ne^2}{m\omega^2}$$

note! from Jackson eq 10.79 2nd ed. $\omega_p^2 = \frac{ne^2}{m}$

Then,

$$\chi_e = -\frac{\omega_p^2}{\omega^2} \qquad (4)$$

And finally the index refraction of a metal is given by

$$n = \sqrt{1 + \chi_e}$$

So,

$$n_2 = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad (5)$$

Substituting eq (5) into eq (1) yields the expression for the critical angle

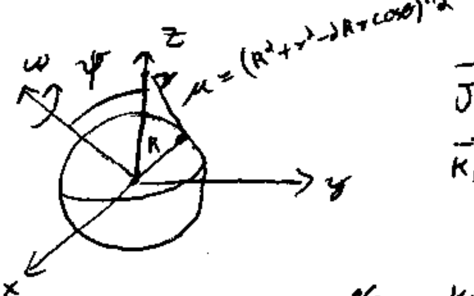
$$\boxed{\theta_0 = \sin^{-1}\left[1 - \frac{\omega_p^2}{\omega^2}\right]^{1/2}}$$

S'03 E.M. #12

a) Show that the field inside of a sphere of uniformly magnetized material ($\vec{M} = M\hat{z}$) is:

$$\vec{B} = \frac{2}{3}\mu_0 M\hat{z}$$

Two ways of doing it:



$$\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$$
$$\vec{K}_b = \vec{M}\times\hat{n} = \vec{M}\times\hat{r} = M\hat{\phi}$$

Now K_b can be thought of as being due to a surface charge σ on a rotating (ω) sphere.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')}{\imath}\,da' = \frac{\mu_0\sigma}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\vec{v}(\vec{r}\,')\; R^2\sin\theta\, d\theta\, d\phi}{\sqrt{R^2+r^2-2Rr\cos\theta}}$$

now $\vec{v}(\vec{r}\,') = \vec{\omega}\times\vec{r}\,'$; $\vec{r}\,' = R\sin\theta\cos\phi\,\hat{x} + R\sin\theta\sin\phi\,\hat{y} + R\cos\theta\,\hat{\phi}$

$$\vec{\omega} = \omega\sin\psi\,\hat{x} + 0\,\hat{y} + \omega\cos\psi\,\hat{z}$$

hence; $\vec{\omega}\times\vec{r}\,' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta\cos\phi & R\sin\theta\sin\phi & R\cos\theta \end{vmatrix}$

$= \hat{x}\left(-R\omega\sin\theta\cos\phi\cos\psi\right) - \hat{y}\left(R\omega\cos\theta\sin\psi - R\omega\sin\theta\cos\phi\cos\psi\right)$
$+ \hat{z}\left(R\omega\sin\theta\sin\phi\sin\psi\right)$

now $\int_0^{2\pi}\sin\phi\, d\phi = 0 = \int_0^{2\pi}\cos\phi\, d\phi$, hence all of the above terms with either $\sin\phi$ or $\cos\phi$ in them will vanish; we are left with

$$\vec{\omega}\times\vec{r}\,' = -R\omega\cos\theta\sin\psi\,\hat{y}$$

So

$$\vec{A} = -\frac{\mu_0 \sigma R^3 \omega \sin\psi}{4\pi} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}\ \hat{y}$$

$$= \underbrace{-\frac{\mu_0 \sigma R^3 \omega \sin\psi}{2}}_{\equiv C} \int_0^{\pi} \frac{\sin\theta\cos\theta\, d\theta}{\sqrt{R^2+r^2-2Rr\cos\theta}}\ \hat{y}$$

let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta$

$$= C\int_1^{-1} \frac{u\,du}{\sqrt{R^2+r^2-2Rru}}\ \hat{y}$$

This integral is just

$$\int \frac{x\,dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2}(ax+b)^{1/2}$$

with $a = -2Rr$; $b = R^2+r^2$

$$= C\left[\frac{2(-2Rru - 2(R^2+r^2))}{3(4R^2r^2)}\left(-2Rru+R^2+r^2\right)^{1/2}\right]_{+1}^{-1}\ \hat{y}$$

$$= C\left[-\frac{(R^2+r^2+Rru)}{3R^2r^2}\left(R^2+r^2-2Rru\right)^{1/2}\right]_{+1}^{-1}$$

at the cost of a minus sign we can swith the limits

$$= \frac{+C}{3R^2r^2}\Big[(R^2+r^2+Rr)\underbrace{(R^2+r^2-2Rr)^{1/2}}_{\substack{R-r \quad r<R \\ r-R \quad r>R}} - (R^2+r^2-Rr)\underbrace{(R^2+r^2+2Rr)^{1/2}}_{(R+r)}\Big]$$

we are interrested in the case $r<R$:

$$= +\frac{C}{3R^2r^2}\left[(R^2+r^2+Rr)(R-r) - (R^2+r^2-Rr)(R+r)\right]$$

$$= +\frac{C}{3R^2r^2}\left[\cancel{R^3} + \cancel{Rr^2} + \cancel{R^2r} - \cancel{R^2r} - r^3 - \cancel{Rr^2} - \cancel{R^3} - \cancel{Rr^2} + \cancel{R^2r} - \cancel{R^2r} - r^3 + \cancel{Rr^2}\right]$$

$$= \frac{+C}{3R^2r^2}\left[-2r^3\right] = \frac{-2}{3}\frac{C}{R^2}\, r\ \hat{y}$$

or as $-r\omega\sin\psi = \vec{\omega}\times\vec{r}$

So $\vec{A}(\vec{r}) = -\frac{\cancel{2}}{3}\frac{\mu_0\sigma R^{\cancel{3}\,1}\omega\sin\psi\, r}{\cancel{2}\cancel{R^2}}\ \hat{y} = \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$

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Now aligning $\vec{\omega}$ to coincide with the $\hat{z}$ axis:

$$\vec{A}(\vec{r},\vec{\theta},\vec{\phi}) = \underbrace{\frac{\mu_0 R\omega\sigma}{3}}_{c'} r\sin\theta\hat{\phi}$$

To get $\vec{B}$:

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta A_\phi - \frac{\partial}{\partial\phi}\cancelto{0}{A_\theta}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\cancelto{0}{A_r} - \frac{\partial}{\partial r} rA_\phi\right]\hat{\theta}$$

$$= \frac{1}{\cancel{r}\sin\theta} c'\cancel{r}\frac{\partial}{\partial\theta}\sin^2\theta\ \hat{r} + -\frac{c'\sin\theta}{r}\frac{\partial}{\partial r} r^2\ \hat{\theta}$$

$$= c'\left[\frac{1}{\cancel{\sin\theta}} 2\cancel{\sin\theta}\cos\theta\hat{r} - \frac{\sin\theta}{\cancel{r}} 2\cancel{r}\ \hat{\theta}\right]$$

$$= 2c'\left[\cos\theta\hat{r} - \sin\theta\hat{\theta}\right]$$

now $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$

$\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$

so

$$\left.\begin{aligned} \cos\theta\hat{r} &= \cancel{\cos\theta\sin\theta\cos\phi\hat{x}} + \cancel{\cos\theta\sin\theta\sin\phi\hat{y}} + \cos^2\theta\hat{z} \\ -\sin\theta\hat{\theta} &= \cancel{-\cos\theta\sin\theta\cos\phi\hat{x}} - \cancel{\cos\theta\sin\theta\sin\phi\hat{y}} + \sin^2\theta\hat{z} \end{aligned}\right\} \hat{z}$$

So $$\boxed{\vec{B} = 2c'\hat{z} = \frac{2\mu_0}{3}\underbrace{R\omega\sigma}_{=K_b = M}\hat{z} = \frac{2}{3}\mu_0 M\hat{z}}$$

The second method is via boundary conditions:

As $\vec{\nabla} \times \vec{H} = 0$, this means $\vec{H} = -\overrightarrow{\nabla W}$ just like in the electrical case. So one B.C. is

$$W_{above} = W_{below}$$

the second B.C. comes from

$$H_{above}^{\perp} - H_{below}^{\perp} = -(M_1^{\perp} - M_2^{\perp})$$

$M_{above} = 0$ $\quad$ $M_{below} = M\cos\theta$

which becomes $\quad \dfrac{\partial W_{above}}{\partial r} - \dfrac{\partial W_{below}}{\partial r} = \vec{M} \cdot \hat{n} = (M_1^{\perp} - M_2^{\perp})$

now we can use the usual Legendre polynomial stuff:

$$W_{above} = \sum_{\ell} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta); \quad W_{below} = \sum_{\ell} A_\ell r^\ell P_\ell(\cos\theta)$$

now from $\quad W_{above} = W_{below} \Rightarrow \dfrac{B_\ell}{r^{\ell+1}} = A_\ell r^\ell \Rightarrow B_\ell = A_\ell r^{2\ell+1}$

from the second B.C.

$$-(\ell+1) \sum_{\ell} \frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) - \sum_{\ell} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -M\cos\theta$$

only works for $\ell = 1$, so

$$2\frac{B_1}{R^3} + A_1 = M \quad \text{but} \quad B_1 = A_1 R^3$$

so $\quad 2A_1 + A_1 = M \rightarrow A_1 = \dfrac{M}{3} \Rightarrow B_1 = \dfrac{M}{3} R^3$

hence

$$\underset{(below)}{W_{inside}} = \frac{M}{3} r\cos\theta$$

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and

$$\vec{H} = -\vec{\nabla W} = -\left[\frac{\partial}{\partial r} W \hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta} W \hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} W \hat{\phi} \right]$$

$$= -\left[\frac{M}{3}\cos\theta \hat{r} - \frac{M}{3}\sin\theta \hat{\theta} + 0 \right]$$

now $\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$

hence

$$\left.\begin{aligned} \cos\theta\,\hat{r} &= \cancel{\cos\theta\sin\theta\cos\phi\,\hat{x}} + \cancel{\cos\theta\sin\theta\sin\phi\,\hat{y}} + \cos^2\theta\,\hat{z} \\ -\sin\theta\,\hat{\theta} &= \cancel{-\cos\theta\sin\theta\cos\phi\,\hat{x}} - \cancel{\cos\theta\sin\theta\sin\phi\,\hat{y}} + \sin^2\theta\,\hat{z} \end{aligned}\right\} = \hat{z}$$

so $\vec{H} = -\frac{M}{3}\hat{z}$

but $\vec{H} = \frac{1}{\mu_0}\vec{B} - \underbrace{\vec{M}}_{M\hat{z}} \Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M})$

$$= \mu_0\left(-\frac{M}{3} + M\right)\hat{z}$$

$$= \boxed{\frac{2}{3}\mu_0 M \hat{z}}$$

b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_0\hat{z}$. Using the above result, find the magnetic field inside the sphere.

$$\vec{M} = \chi_m \vec{H} = \chi_m\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\overbrace{\left[\vec{B}_0 + \frac{2}{3}\mu_0\vec{M}\right]}^{=\vec{B}} - \vec{M}\right)$$

$$= \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M} - \vec{M}\right) = \chi_m\left(\frac{1}{\mu_0}\vec{B}_0 - \frac{1}{3}\vec{M}\right)$$

so $\vec{M}\left(1 + \frac{\chi_m}{3}\right) = \frac{\chi_m}{\mu_0}\vec{B}_0 \Rightarrow \vec{M} = \dfrac{\chi_m/\mu_0\,\vec{B}_0}{\left(1 + \frac{\chi_m}{3}\right)}$

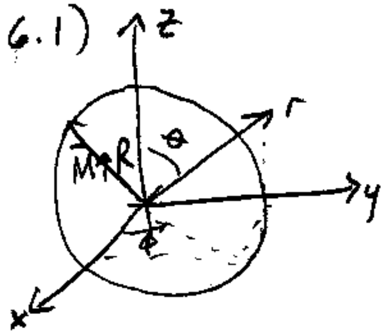
Now $\vec{B} = \vec{B}_0 + \vec{B}_{sphere} = \vec{B}_0 + \frac{2}{3}\mu_0 \dfrac{\chi_m/\mu_0}{(1+\chi_m/3)} B_0 = \left(1 + \dfrac{2\chi_m}{(3+\chi_m)}\right)\vec{B}_0 = \boxed{\left(\dfrac{1+\chi_m}{1+\chi_m/3}\right)\vec{B}_0}$

Spring 2003 # 12 (p 1 of 9)

(a) show that the field inside a sphere of uniformly magnetized material ($\vec{M} = M\hat{z}$) is

$$\vec{B} = \frac{2}{3}\mu_0 M \hat{z}$$

(see Griffiths' example 6.1)



the current density is given by

$$j_m = \nabla \times \vec{m} = 0$$

and the surface current is

$$|\vec{K}| = |\vec{M} \times \hat{n}| = M|\hat{z} \times \hat{r}| = M\sin\theta$$

note that a rotating spherical shell of uniform surface charge has:

$$K = \sigma v = \sigma \omega R \sin\theta$$

So, the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell with $M \to \sigma R \omega$. So, let's find the field of a spinning spherical shell (see Griffiths' example 5.11). From Griffiths' example 5.11, we have that the vector potential is given by (eq 5.67)

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\phi} & r < R \\ \dfrac{\mu_0 R^4 \omega \sigma}{3r^2}\sin\theta\,\hat{\phi} & r > R \end{cases}$$

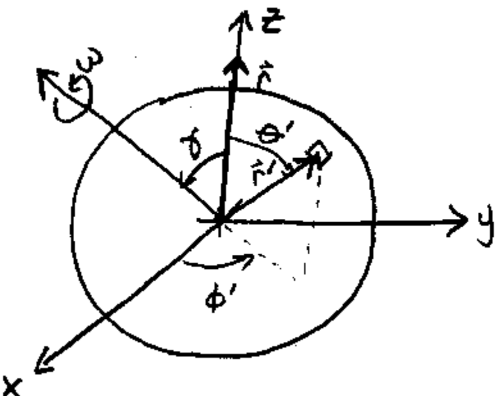
How are these derived? see Jackson problem 5.13. See next 8 pages for 2 different ways to do this.

Jackson 5.13 A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and the magnetic-flux density both inside and outside the sphere.

From Griffiths' eqn. 5.64, we know that the vector potential can be written in terms of the surface current as follows

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{\vec{K}(\vec{r}')}{|\vec{r}-\vec{r}'|} da'$$

taking the advice in Griffiths' example 5.11, let's orient the coordinates such that $\vec{r}$ lies along the z-axis and the axis of rotation lies in the xz plane. That is, (Griffiths' Figure 5.46)



(i) First, let's consider what $\vec{K}(\vec{r}')$ is. From Griffiths' eqn 5.23, we know

$$\vec{K} = \sigma \vec{v}$$

where $\vec{v} = \vec{\omega} \times \vec{r}'$ is the velocity of a point $\vec{r}'$ in a rotating rigid body (see Griffiths' example 5.11). Now, since we oriented the coordinate system such that the axis of rotation is in the xz plane, we have

$$\vec{\omega} = \omega \sin\gamma \,\hat{x} + \omega\cos\gamma\,\hat{z}$$

we also know that in spherical coordinates, $\vec{r}'$ is given by

$$\vec{r}' = r\sin\theta'\cos\phi'\,\hat{x} + r\sin\theta'\sin\phi'\,\hat{y} + r\cos\theta'\,\hat{z}$$

, where $r=a$ for our case

so,

$$\vec{K} = \sigma(\vec{\omega}\times\vec{r}') = \sigma \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\gamma & 0 & \omega\cos\gamma \\ a\sin\theta'\cos\phi' & a\sin\theta'\sin\phi' & a\cos\theta' \end{vmatrix}$$

$$= \sigma\Big[\hat{x}(-\omega a\sin\theta'\sin\phi'\cos\gamma) + \hat{y}(\omega a\sin\theta'\cos\phi'\cos\gamma - \omega a\sin\gamma\cos\theta') + \hat{z}(a\omega\sin\gamma\sin\theta'\sin\phi')\Big]$$

(ii) Now, note

$$|\vec{r}-\vec{r}'| = \sqrt{|\vec{r}-\vec{r}'|^2} = \left[(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')\right]^{1/2} = \left[r^2+(r')^2-2rr'\cos\theta'\right]^{1/2}\Big|_{r'=a}$$

so, we have

$$|\vec{r}-\vec{r}'| = \left[r^2+a^2-2ra\cos\theta'\right]^{1/2}$$

(iii) now consider da'

For our case, da' is given by

$$da' = a^2\sin\theta'\,d\theta'\,d\phi'$$

Putting the results from parts (i), (ii), and (iii) into our expression for $\vec{A}(\vec{r})$, we get

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi}d\phi'\int_0^{\pi}\sin\theta'\,d\theta'\left[\frac{-\sin\theta'\sin\phi'\cos\gamma\,\hat{x}+(\sin\theta'\cos\phi'\cos\gamma-\sin\gamma\cos\theta')\hat{y}+\sin\gamma\sin\theta'\sin\phi'\,\hat{z}}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}\right]$$

Now, consider the integration over ϕ'. Note that

$$\int_0^{2\pi}d\phi'\sin\phi' = -\left[\cos\phi'\right]_0^{2\pi} = -\left[1-1\right] = 0$$

and

$$\int_0^{2\pi}d\phi'\cos\phi' = \left[\sin\phi'\right]_0^{2\pi} = 0-0 = 0$$

with this in mind, the integral over the $\hat{x}$ and $\hat{z}$ directions vanish as well as the first term in the $\hat{y}$ direction. So, the only non-zero term that survives is

$$\vec{A}(\vec{r}) = \frac{\sigma\omega a^3}{c}\int_0^{2\pi}d\phi'\int_0^{\pi}\sin\theta'\,d\theta'\left[\frac{-\sin\gamma\cos\theta'}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}\right]$$

$$= \frac{2\pi\sigma\omega a^3}{c}\sin\gamma\int_0^{\pi}d\theta'\frac{(-\sin\theta'\cos\theta')}{\left[r^2+a^2-2ra\cos\theta'\right]^{1/2}}$$

In order to solve this integral, we need to make the following substitution

$$u = \cos\theta' \Rightarrow du = -\sin\theta'\,d\theta'$$

So, we have

$$\vec{A}(\vec{r}) = \frac{2\pi\sigma\omega a^3}{c}\sin\gamma \int_{1}^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}}$$

note: from Schaum's Outlines Mathematical Handbook of Formulas and Tables, eqn. 17.2.2, we know

$$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b}$$

for our case, $x \to u$
$b \to r^2+a^2$
$a \to -2ra$

So, we have

$$\int_{+1}^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}} = \left[\frac{2\left[(-2ra)u - 2r^2-2a^2\right]}{3(-2ra)^2}\sqrt{-2rau+r^2+a^2}\right]_{1}^{-1}$$

$$= \frac{2}{12r^2a^2}\left[2ra-2r^2-2a^2\sqrt{2ra+r^2+a^2} - (-2ra-2r^2-2a^2)\sqrt{-2ra+r^2+a^2}\right]$$

$$= \frac{4}{12r^2a^2}\left[-(r^2+a^2-ra)\sqrt{(r+a)^2} + (r^2+a^2+ra)\sqrt{(r-a)^2}\right]$$

$$= \frac{1}{3r^2a^2}\left[-(r^2+a^2-ra)(r+a) + (r^2+a^2+ra)\,|r-a|\right]$$

here we have left $r-a$ in absolute value signs because the result of $r-a$ must remain positive in order for the anser to be real. This is the point where the solution will vary if $r<a$ or $r>a$. So, if

(i) $r<a$

$$\int_{+1}^{-1} \frac{u\,du}{\left[r^2+a^2-2rau\right]^{1/2}} = \frac{1}{3r^2a^2}\left[-(r^2+a^2-ra)(r+a) + (r^2+a^2+ra)(a-r)\right]$$

$$= \frac{1}{3r^2a^2}\left[-(r^3+ra^2-r^2a+r^2a+a^3-ra^2) + \right.$$

$$\left. + r^2a + a^3 + ra^2 - r^3 - ra^2 - r^2a\right]$$

$$\Rightarrow \int_{+1}^{-1} \frac{u\,du}{[r^2+a^2-2rau]^{1/2}} = \frac{1}{3r^2a^2}\left[-2r^3\right] = -\frac{2r}{3a^2} \qquad (1)$$

(ii) $r > a$

$$\int_{+1}^{-1} \frac{u\,du}{[r^2+a^2-2rau]^{1/2}} = \frac{1}{3r^2a^2}\left[-(r^2+a^2-ra)(r+a) + (r^2+a^2+ra)(r-a)\right]$$

$$= \frac{1}{3r^2a^2}\left[\underbrace{-r^3 - a^3 + r^3}_{0} + ra^2 + \underbrace{r^2a - r^2a}_{0} - a^3 - ra^2\right]$$

$$= \frac{1}{3r^2a^2}\left[-2a^3\right] = \frac{-2a}{3r^2} \qquad (2)$$

Substituting the results of equations (1) & (2) into our expression for $\vec{A}(\vec{r})$, we get

$$A(r) = \frac{2\pi\sigma\omega a^3}{c}\sin\gamma \begin{cases} -\dfrac{2r}{3a^2} & r < a \\[2ex] -\dfrac{2a}{3r^2} & r > a \end{cases}$$

At this point, we can note that $\vec{\omega}\times\vec{r} = -\omega r\sin\gamma$ (see Figure on first page of the problem). By making this substitution in $\vec{A}(\vec{r})$, then reorienting our coordinates such that $\vec{\omega}$ is aligned with the z-axis, we get that $\gamma \to \theta$ in our expression for A above (see Griffiths' example 5.11). Thus, we have the answer in what Griffiths calls natural coordinates,

$$\vec{A}(r,\theta) = \frac{2\pi\sigma\omega a^3}{c}\sin\theta \begin{cases} \dfrac{2r}{3a^2}\hat{\phi} & r \leq a \\[2ex] \dfrac{2a}{3r^2}\hat{\phi} & r \geq a \end{cases}$$

$$\therefore \boxed{\vec{A}(r,\theta) = \frac{4\pi\sigma\omega}{3c}\sin\theta \begin{cases} ra\,\hat{\phi} & r \leq a \\[2ex] \dfrac{a^4}{r^2}\hat{\phi} & r \geq a \end{cases}}$$

Now, we want to find the magnetic-flux density which as we saw in problem 5.6, is just the magnetic field.

In general, we have

$$\vec{B} = \nabla \times \vec{A}_\phi = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi)\,\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\,\hat{\theta}$$

So, for $r<a$

$$\vec{B} = \frac{4\pi\sigma\omega a}{3c}\left[\frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}(r^2)\,\hat{\theta}\right]$$

$$= \frac{4\pi\sigma\omega a}{3c}\left[\frac{2\sin\theta\cos\theta}{\sin\theta}\,\hat{r} - \frac{2r\sin\theta}{r}\,\hat{\theta}\right]$$

$$= \frac{8\pi\sigma\omega a}{3c}\left[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right]$$

note: $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

Thus, $\boxed{\vec{B} = \frac{8\pi\sigma\omega a}{3c}\hat{z}}$ $r<a$ ✓

for $r>a$

$$\vec{B} = \frac{4\pi\sigma\omega a^4}{3c}\left[\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\,\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\theta}\right]$$

$$= \frac{4\pi\sigma\omega a^4}{3c}\left[\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\,\hat{r} + \frac{\sin\theta}{r^3}\,\hat{\theta}\right]$$

$$\Rightarrow \boxed{\vec{B} = \frac{4\pi\sigma\omega a^4}{3cr^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]}$$ $r>a$ ✓

+3/3

Jackson 5.13

A sphere of radius a carries a uniform surface-charge distribution σ. The sphere is rotated about a diameter with constant angular velocity ω. Find the vector potential and magnetic-flux density both inside and outside the spehre.

Write the current density in spherical coordinates:

$$\vec{J}(\vec{r}\,') = J_\phi \hat{\phi}\,' = [\sigma\,\omega\,r'\,\sin(\theta')\,\delta(r'-a)]\,\hat{\phi}\,'$$

Choosing the Coulomb gauge (thus can set the divergence of some scalar function to zero, see Jackson discussion p.181,) we write the vector potential using Jackson eq. 5.32. But from the azimuthal symmetry present in this case, we can evaluate the potential at $\phi = 0$ and the result will still be generally applicalbe. If we then proceed with the calculations using $\phi = 0$, we'd see that the only non-zero contribution to the potential is from the "y" component of the current density. Thus we have (in SI units):

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{J_\phi \cos(\phi')\hat{y}}{|\vec{r}-\vec{r}\,'|}\,(r')^2 dr'\,d\Omega'$$

We can immediately perform the r' integral using the delta function using the explicit expression for the current density. The result is:

$$\vec{A} = \hat{y}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\!\!\int \frac{\sin(\theta')\;\cos(\phi')\;d\Omega'}{|\vec{r}-a\hat{r}\,'|}$$

Now recall our discussion in the previous problem that the unit vector $\hat{y}$ for an azimuthally symmetric problem evaluated at $\phi = 0$ is generally equivalent ot $\hat{\phi}$. So we have:

$$\vec{A} = \hat{\phi}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\!\!\int \frac{\sin(\theta')\;\cos(\phi')\;d\Omega'}{|\vec{r}-a\hat{r}\,'|}$$

Now expand the inverse relative distance using spherical harmonics (Jackson eq. 3.70.) The vector potential expression becomes:

$$\vec{A} = \hat{\phi}\frac{\mu_0\sigma\omega a^3}{4\pi}\int\!\!\int \left[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}}\right)\,Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi=0)\right]\sin(\theta')\cos(\phi')\,d\Omega'$$

where $r_<(r_>)$ is the smaller (larger) between r (field point) and a (source poiint). Now notice that

$$\sin(\theta')\cos(\phi') = \frac{1}{2}\sin(\theta')\left[e^{i\phi'} + e^{-i\phi'}\right] = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\left[-Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')\right]$$

where we've made use of Jackson eq. 3.54 and the definition for the spherical harmonics. With the above, we can perform the θ' and ϕ' integrals easily using the orthogonality property of the spherical harmonics. After a little manipulation, and using the fact that

$$Y_{11}(\theta,0) = -\sqrt{\frac{3}{8\pi}}\sin\theta = -Y_{1,-1}(\theta,0)$$

we arrive at the following result:

$$\vec{A} = \hat{\phi}\,\frac{\mu_0\sigma\omega a^3}{3}\left(\frac{r_<}{r_>^2}\right)\sin\theta$$

Hence the vector potentials both inside and outside the sphere are given by the following:

$$\vec{A}_{in} = \hat{\phi}\,\frac{\mu_0\sigma\omega a}{3}\,r\,\sin\theta$$

$$\vec{A}_{out} = \hat{\phi}\,\frac{\mu_0\sigma\omega a^4}{3}\,\frac{\sin\theta}{r^2}$$

Converting the above into gaussian units, we find:

$$\vec{A}_{in} = \hat{\phi}\,\frac{4\pi\sigma\omega a}{3c}\,r\,\sin\theta$$

$$\vec{A}_{out} = \hat{\phi}\,\frac{4\pi\sigma\omega a^4}{3c}\,\frac{\sin\theta}{r^2}$$

The magnetic flux density then is just $\vec{B} = \vec{\nabla}\times\vec{A}$. Straightforward calculations reveal the following:

$$\vec{B}_{in} = \frac{8\pi}{3c}\sigma\omega a\hat{z}$$

let $\sigma a \omega \to M$, $\boxed{\vec{B}_{in} = \frac{8\pi}{3c} M\,\hat{z}}$ (, recall $\mu_0 \to \frac{4\pi}{c}$)

b) A sphere of material with linear magnetic susceptibility χ_m is placed in a region of uniform magnetic field $B_0 \hat{z}$. Using the above result, Find the magnetic field inside the sphere.

From Griffiths (eq. 6.29), we have (use MKS units ¨)

$$\vec{M} = \chi_M \vec{H} \quad , \quad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\Rightarrow \vec{M} = \frac{\chi_M}{\mu_0}\vec{B} - \chi_m \vec{M} \quad , \quad \vec{B} = \left(B_0 + \frac{2\mu_0}{3}M\right)\hat{z}$$

$$\Rightarrow \vec{M}(1+\chi_M) = \frac{\chi_M}{\mu_0}\vec{B}_0 + \frac{2}{3}\vec{M}\chi_M$$

$$\therefore \quad \vec{M} = \frac{\chi_M \vec{B}_0}{\mu_0\left(1+\frac{1}{3}\chi_M\right)}$$

Thus, the field inside the sphere is

$$\vec{B} = \vec{B}_0 + \frac{2\mu_0}{3}\left(\frac{\chi_M \vec{B}_0}{\mu_0(1+\frac{1}{3}\chi_M)}\right) = \vec{B}_0\left[1 + \frac{2\chi_M}{3+\chi_M}\right]$$

$$= \vec{B}_0\left[\frac{3+3\chi_M}{3+\chi_M}\right]$$

$$\therefore \quad \boxed{\vec{B} = \vec{B}_0\left[\frac{1+\chi_M}{1+(1/3)\chi_M}\right]}$$

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Consider a d-dimensional material in which the important excitations are non-conserved Bosons, and assume that the dispersion relation for those Bosons is

$$\omega = ak^3$$

where k is the wave vector's amplitude and a = constant. The low temperature specific heat goes as T^q. What is the value of the power, q?

Use Debye theory (see Reif problem 10.1)

From eq 10.1.20 in Reif, we have an expression for the heat capacity

$$C_V = k \int_0^\infty \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} (\beta\hbar\omega)^2 \sigma(\omega)\, d\omega \qquad (1)$$

where $\sigma(\omega)d\omega$ is the number of normal modes with angular frequency in the range between ω and $\omega + d\omega$, where in d-dimensions,

$$\sigma(\omega)d\omega = \frac{V}{(2\pi)^d} \frac{2\pi^{d/2} K^{d-1}}{\Gamma(d/2)} dK \text{, } V \text{ is the spatial volume element in d dims}$$

and $\omega = aK^3 \Rightarrow d\omega = 3aK^2 dK$

So, we have

$$K = \left(\frac{\omega}{a}\right)^{1/3} \Rightarrow K^{d-1} = \left(\frac{\omega}{a}\right)^{\frac{d-1}{3}}$$

Making these substitutions into $\sigma(\omega)$, we get

$$\sigma(\omega) = \frac{V}{(2\pi)^d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{\omega}{a}\right)^{\frac{d-1}{3}} \frac{dK}{d\omega} \equiv \gamma \left(\frac{\omega}{a}\right)^{\frac{d-1}{3}} \underbrace{\frac{dK}{d\omega}}_{=(3aK^2)^{-1}}$$

$$= \gamma\left(\frac{\omega}{a}\right)^{\frac{d-1}{3}} \frac{1}{3aK^2} = \gamma\left(\frac{\omega}{a}\right)^{\frac{d-1}{3}} \frac{a^{2/3}}{3a\omega^{2/3}}$$

$$= \frac{\gamma}{3a}\left(\frac{\omega}{a}\right)^{\left(\frac{d}{3}-1\right)}$$

Substituting the result into eq (1) yields

$$C_V = \frac{\gamma K}{3\, a^{\left(\frac{d}{3}-2\right)}} \int_0^{\omega_c} \frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2} (\beta\hbar\omega)^2 \; \omega^{\left(\frac{d}{3}-1\right)} \, d\omega$$

Now let $x \equiv \beta\hbar\omega \Rightarrow dx = \beta\hbar\, d\omega$. So, we have

$$C_V = \frac{\gamma K}{3 a^{\frac{d}{3}-2}} \int_0^{\frac{\hbar\omega_c}{KT}} \frac{x^2 e^x}{(e^x-1)^2} \left(\frac{x}{\beta\hbar}\right)^{\frac{d}{3}-1} dx$$

$$= \frac{\gamma K}{3 a^{d/3-2}} \left(\frac{KT}{\hbar}\right)^{\frac{d}{3}-1} \int_0^{\frac{\hbar\omega_c}{KT}} \frac{x^{\frac{d}{3}+1} e^x}{(e^x-1)^2} dx$$

in the limit of low temperature, $\frac{\hbar\omega_c}{KT} \to \infty$

So, we have

$$C_V = \frac{\gamma K^{d/3+1}}{3 a^{d/3-2} \hbar^{\frac{d}{3}-1}} T^{\frac{d}{3}-1} \int_0^{\infty} \frac{x^{\frac{d}{3}+1} e^x}{(e^x-1)^2} dx$$

since the integral no longer depends on temperature in this limit, we can immediately write down that

$$\boxed{C_V \propto T^{\frac{d}{3}-1}}$$

where d is the dimension. So, this dispersion relation ($\omega = ak^3$) implies that the heat capacity is independent of temperature in 3-d ... ?

14) A system$^{(1)}$ can exchange energy & volume with a large reservoir$_{(R)}$

a) show that $S_{tot} = S_{max}$ when $T_1 = T_R$

AND when $P_1 = P_R$.

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In general

$\Delta S_{tot} = \Delta S_1 + \Delta S_R$

$\Delta E_1 = T_1 \Delta S_1 - P_1 \Delta V_1$ & $\Delta E_R = T_R \Delta S_R - P_R \Delta V_R$

Here $\Delta E_1 = -\Delta E_R$ & $\Delta V_1 = -\Delta V_R$

$T_1 \Delta S_1 - P_1 \Delta V_1 = -T_R \Delta S_R + P_R \Delta V_R$

$T_1 \Delta S_1 + T_R \Delta S_R = P_1 \Delta V_1 + P_R \Delta V_R$

$T_1 \Delta S_1 + T_R \Delta S_R = (P_1 - P_2) \Delta V_1$

IF $P_1 = P_2$ then $\Delta S_1 = -\frac{T_R}{T_1} \Delta S_R$

and $\Delta S_{tot} = \left(1 - \frac{T_R}{T_1}\right) \Delta S_2$

and IF $T_1 = T_R$ then $\Delta S_{tot} = 0$ so $S_{tot} = S_{max}$

b) Expanding the entropy of the subsystem (1) to second order ...

$$dS_1 = \underbrace{\frac{1}{T_1}dE_1 - \frac{P_1}{T_1}dV_1}_{\text{first order}} + \underbrace{\left(\frac{\partial 1/T_1}{\partial E}\right)_V \frac{(\delta E)^2}{2} + \left(\frac{\partial P_1/T_1}{\partial V}\right)_E \frac{(\delta V)^2}{2}}_{\text{second order}} \cdots$$

we find

$$dS_{tot} = \frac{1}{T_1}dE_1 - \frac{P_1}{T_1}dV_1 + \frac{1}{T_R}dE_R - \frac{P_R}{T_R}dV_R$$

$$+ \left(\frac{\partial \frac{1}{T_1}}{\partial E}\right)_V \frac{(\delta E)^2}{2} + \left(\frac{\partial P_1/T_1}{\partial V}\right)_E \frac{(\delta V)^2}{2} \cdots$$

$$= \left(\frac{1}{T_1} - \frac{1}{T_R}\right)dE_1 - \left(\frac{P_1}{T_1} - \frac{P_R}{T_R}\right)dV_1$$

$$+ \left(\frac{\partial 1/T_1}{\partial E}\right)_V \frac{(\delta E)^2}{2} + \left(\frac{\partial P_1/T_1}{\partial V}\right)_E \frac{(\delta V)^2}{2}$$

where the second order expations for the reservoir are neglected because it is large. At equilibrium

$T_1 = T_R$, $P_1 = P_R$ (from part a)

So the first two terms vanish

$$\boxed{dS_{tot} = \left(\frac{\partial 1/T_1}{\partial E}\right)_V \frac{(\delta E)^2}{2} + \left(\frac{\partial P_1/T_1}{\partial V}\right)_E \frac{(\delta V)^2}{2}} \quad (I)$$

from $\delta E = \underbrace{\left(\frac{\partial E}{\partial T}\right)_V}_{C_V} \delta T + \underbrace{\left(\frac{\partial E}{\partial V}\right)_T}_{P} \delta V$

$$(\delta E)^2 = C_V^2 (\delta T)^2 + P^2(\delta V)^2 + 2PC_V(\delta T)(\delta V)$$

Now the first term in dS_{tot}:

$$\left(\frac{\partial\, ^{1}/_{T_1}}{\partial E}\right)_V (\delta E)^2 = -\frac{1}{T_1^2}\left(\frac{\partial T_1}{\partial E}\right)_V \frac{(\delta E)^2}{2}$$

$$= -\frac{1}{T_1^2 C_V}\frac{(\delta E)^2}{2}$$

$$\doteq -\frac{1}{2T_1^2 C_V}\left[C_V^2(\delta T)^2 + P^2(\delta V)^2\right] \qquad (\mathrm{II})$$

the $(\delta V)(\delta T)$ term vanishes due to the equilibrium condition $\Rightarrow$ $\langle \delta V\rangle = 0$, $\langle \delta T\rangle = 0$. These are independent variables and so $(\delta V)(\delta T) \approx 0$

Now the second term in ΔS_{tot}.

$$\left(\frac{\partial\, ^{P_1}/_{T_1}}{\partial V}\right)_E = \frac{1}{T_1}\left(\frac{\partial P_1}{\partial V}\right)_{T_1} - \frac{P_1}{T_1^2}\left(\frac{\partial T_1}{\partial V}\right)_{P_1}$$

$$= \frac{1}{T_1}\left(\frac{\partial P_1}{\partial V}\right)_{T_1} - \frac{P_1}{T_1^2}\left[\underbrace{\left(\frac{\partial T_1}{\partial E}\right)_V}_{1/C_V}\underbrace{\left(\frac{\partial E}{\partial V}\right)_{T_1}}_{-P_1}\right]$$

$$= \frac{1}{T_1}\left(\frac{\partial P_1}{\partial V}\right)_{T_1} + \frac{P_1^2}{T_1^2 C_V} \qquad (\mathrm{III})$$

Combining (II) & (III) into (I) ...

$$dS_{tot} = \underbrace{-\frac{C_v}{T_1^2}\frac{(\delta T)^2}{2} - \frac{P_1^2}{T_1^2 C_v}\frac{(\delta v)^2}{2}}_{(\mathrm{II})} + \underbrace{\left[\frac{1}{T_1}\left(\frac{\partial P_1}{\partial V}\right)_{T_1} + \frac{P_1^2}{T_1^2 C_v}\right]\frac{(\delta v)^2}{2}}_{\mathrm{III}}$$

$$\boxed{dS_{tot} = -\frac{C_v}{2T_1^2}(\delta T)^2 + \frac{1}{2T_1}\left(\frac{\partial P_1}{\partial V}\right)_{T_1}} \quad (\mathrm{IV})$$

from the second law $\rightarrow$ $\Delta S_{tot} \geq 0$

so $$\boxed{\left(\frac{\partial P}{\partial V}\right)_T (\delta v)^2 \geq \frac{C_v}{T}(\delta T)^2}$$