14. *Electricity and Magnetism* (Spring 2006)

A plane, transverse electromagnetic wave of frequency ω propagates through a scalar medium whose complex dielectric coefficient is given by

$$\epsilon(\omega) = 1 - \frac{a}{\omega(\omega + ib)}$$

↑ not conducting medium!

where a and b are positive real constants.

(a) What is the electrical conductivity of the medium?

(b) What is the ratio of the magnitude of the material current density to the displacement current density in the medium?

(c) Find the spatial damping coefficient of this wave (i.e., the imaginary part of k) in the limit of small b.

(d) Find the phase-shift between the electric and magnetic fields in the limit of small b.

(e) Does this $\epsilon(\omega)$ satisfy the required symmetry relation for general dielectric coefficients? Why?

(a) $\dfrac{\hat{\varepsilon}(\omega)}{\varepsilon_0} = 1 - \dfrac{a(\omega - ib)}{\omega(\omega^2+b^2)}$ $\Rightarrow$ $\vec{D}(\omega) = \varepsilon_0 \vec{E}(\omega) + \vec{P}(\omega)$ ← $\varepsilon_0 \chi(\omega)\vec{E}(\omega)$

← can't distinguish conduction from pol current for Harmonic sources

but $\vec{J}(\omega) = \hat{\sigma}(\omega)\vec{E} = \dfrac{\partial \vec{P}}{\partial t} = -i\omega\vec{P}(\omega) = -i\omega[\varepsilon_0\hat{\chi}(\omega)\vec{E}(\omega)]$

$= 1 - \dfrac{a}{\omega^2+b^2} + \dfrac{iba}{\omega(\omega^2+b^2)}$

$\vec{D}(\omega) = \varepsilon_0(1 + \hat{\chi}(\omega))\vec{E}(\omega)$

$\hat{\varepsilon}/\varepsilon_0 = 1 + i\left(\dfrac{ia}{\omega^2+b^2} + \dfrac{ab}{\omega(\omega^2+b^2)}\right)$

$= \varepsilon_0\left(1 + \dfrac{i\hat{\sigma}(\omega)}{\omega\varepsilon_0}\right)\vec{E}(\omega)$

$\boxed{\sigma(\omega) = \omega\varepsilon_0\left(\dfrac{ab/\omega + ia}{\omega^2+b^2}\right)}$

$= \hat{\varepsilon}(\omega)\vec{E}(\omega)$

↖ $\sigma(\omega) = \sigma' + i\sigma''$

(b) $\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\hat{\varepsilon}(\omega)\vec{E}) \neq 0$

$\vec{\nabla}\cdot\vec{E} \neq 0$; $\vec{J}_f = 0$

$(\sigma - i\omega\varepsilon_0)$

$\vec{\nabla}\times\vec{H} = \cancel{\vec{J}_f} + \dfrac{\partial\vec{D}}{\partial t}$ can take derivative of $\vec{D}$ or note the following ← $\vec{J}(\omega)$ from (a)

↖ $\vec{J}_D = \dfrac{\partial\vec{D}}{\partial t} = \varepsilon_0\partial_t\vec{E} + \partial_t\vec{P}$

$= -i\omega\varepsilon_0\vec{E} + \hat{\sigma}(\omega)\vec{E}$

$= -i\omega\varepsilon_0\left(1 - \dfrac{\hat{\sigma}(\omega)}{i\omega\varepsilon_0}\right)\vec{E}$

$= -i\omega\,\hat{\varepsilon}(\omega)\vec{E}$

$\sigma^*\sigma = \left(\dfrac{\omega\varepsilon_0}{\omega^2+b^2}\right)^2\left((b/\omega)^2 + a^2\right) = \sigma'^2 + \sigma''^2$

$\hat{\varepsilon}^*\hat{\varepsilon} = \varepsilon_0^2\left(1 + \dfrac{i\hat{\sigma}}{\omega\varepsilon_0}\right)\left(1 - \dfrac{i\hat{\sigma}^*}{\omega\varepsilon_0}\right)$

$= \varepsilon_0^2\left(1 - \dfrac{i\hat{\sigma}^*}{\omega\varepsilon_0} + \dfrac{i\hat{\sigma}}{\omega\varepsilon_0} + \dfrac{\hat{\sigma}\hat{\sigma}^*}{(\omega\varepsilon_0)^2}\right)$

$= \varepsilon_0^2\left\{1 + \dfrac{i}{\omega\varepsilon_0}(\cancel{\sigma'} + i\sigma'') - \dfrac{i}{\omega\varepsilon_0}(\cancel{\sigma'} - i\sigma'') + \dfrac{(\sigma'^2+\sigma''^2)}{(\omega\varepsilon_0)^2}\right\}$

$= \varepsilon_0^2\left\{1 - \dfrac{2\sigma''}{\omega\varepsilon_0} + \dfrac{(\sigma'^2+\sigma''^2)}{(\omega\varepsilon_0)^2}\right\}$

$\dfrac{|\vec{J}(\omega)|}{|\vec{J}_D(\omega)|} = \dfrac{\sqrt{\hat{\sigma}^*\cdot\hat{\sigma}}}{\sqrt{(-i\omega\hat{\varepsilon})^*(-i\omega\hat{\varepsilon})}} = \dfrac{\sqrt{\sigma'^2+\sigma''^2}}{\sqrt{\omega^2\hat{\varepsilon}^*\hat{\varepsilon}}} = \boxed{\sqrt{\dfrac{\sigma'^2+\sigma''^2}{\omega^2\varepsilon_0^2\left\{1 - \dfrac{2\sigma''}{\omega\varepsilon_0} + \dfrac{\sigma'^2+\sigma''^2}{(\omega\varepsilon_0)^2}\right\}}}}$

w/ $\sigma' = \dfrac{\omega\varepsilon_0}{\omega^2+b^2}\left(\dfrac{b}{\omega}\right)$

$\sigma'' = \dfrac{\omega\varepsilon_0}{\omega^2+b^2}(a)$

(c) find k'' for $\hat{k}(\omega) = k' + ik''$ in small b limit (will have a non-zero k'' b/c $\lim_{b\to 0}\{\varepsilon'', \sigma''\} \neq 0$)

$\hat{k}(\omega) = \dfrac{\omega}{c}\hat{n}(\omega) = \dfrac{\omega}{c}c\sqrt{\hat{\varepsilon}(\omega)\hat{\mu}(\omega)}$ (assumes $\vec{k}' \parallel \vec{k}''$)

The problem doesn't state if μ is complex, let's assume that $\hat{\mu}(\omega) = \mu$

$\hat{k}(\omega) = \omega\sqrt{\hat{\varepsilon}(\omega)\mu}$ $\Rightarrow$ $\hat{k}^2 = k'^2 - k''^2 + 2ik'k'' = \omega^2\mu\,\hat{\varepsilon}(\omega) = \omega^2\mu(\varepsilon' + i\varepsilon'')$

matching complex coefficients

$\mathrm{Re}[\hat{k}^2] = k'^2 - k''^2 = \omega^2\mu\varepsilon'$ and $2k'k'' = \omega^2\mu\varepsilon'' = \mathrm{Im}[\hat{k}^2]$

$k'^2 - k''^2 = \omega^2\mu\varepsilon'$ and $2k'k'' = \omega^2\mu\varepsilon''$

$4k'^2(k'^2 - \omega^2\mu\varepsilon') = \omega^4\mu^2\varepsilon''^2 \Rightarrow k'^4 - \omega^2\mu\varepsilon' k'^2 - \frac{1}{4}\omega^4\mu^2\varepsilon''^2 = 0$

$k'^2 = \left[\omega^2\mu\varepsilon' \pm \sqrt{\omega^4\mu^2\varepsilon'^2 + \omega^4\mu^2\varepsilon''^2}\right]/2$

Forward propagating sol'n and real

$k' = \pm\omega\sqrt{\mu\varepsilon'}\left(\dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{1 + \dfrac{\varepsilon''^2}{\varepsilon'^2}}\right)^{1/2}$

for small b, $\boxed{\begin{aligned}&\varepsilon'' \simeq \frac{ba}{\omega^3}\left(1 - b^2/\omega^2\right)\varepsilon_0 \simeq \frac{ba}{\omega^3}\varepsilon_0 \\ &\varepsilon' \simeq (1 - a/\omega^2)\varepsilon_0\end{aligned}}$

$k''^2 = k'^2 - \omega^2\mu\varepsilon' = \omega^2\mu\varepsilon'\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \varepsilon''^2/\varepsilon'^2}\right) - \omega^2\mu\varepsilon'$

$k''^2 = \omega^2\mu\varepsilon'/2\left(-1 + \sqrt{1 + \varepsilon''^2/\varepsilon'^2}\right) \simeq \dfrac{\omega^2\mu\varepsilon'}{2}\left(-1 + 1 + \dfrac{1}{2}\dfrac{\varepsilon''^2}{\varepsilon'^2}\right) \simeq \dfrac{\omega^2\mu\varepsilon''^2}{4\varepsilon'}$ $\Rightarrow$ $\boxed{k'' \simeq \dfrac{\omega\varepsilon''}{2}\sqrt{\dfrac{\mu}{\varepsilon'}}}$

(d) Since the plane waves are traverse w/in the medium, they'll be in phase

(e) The symmetry relation for $\sigma(\omega)$ is $\hat{\sigma}(-\omega) = \hat{\sigma}^*(\omega)$ s.t. $\sigma'(\omega) = \sigma'(-\omega)$ even, $\sigma''(\omega) = -\sigma''(-\omega)$ odd

Thus $\hat{\varepsilon}(\omega) = \varepsilon_0 + \dfrac{i}{\omega}(\sigma' + i\sigma'')$

$= \varepsilon_0 - \dfrac{\sigma''}{\omega} + i\sigma'/\omega$

The symmetry rel for $\hat{\varepsilon}(\omega)$ is that:

let $\omega \to -\omega$: $\varepsilon''(-\omega) = -\dfrac{\sigma'(-\omega)}{\omega} = -\dfrac{\sigma'(\omega)}{\omega}$ and $\varepsilon'(-\omega) = \varepsilon_0 + \dfrac{\sigma''(-\omega)}{\omega} = \varepsilon_0 - \dfrac{\sigma''(\omega)}{\omega} = \varepsilon'(\omega)$ must be even

$\varepsilon''(\omega)$ must be odd

$\boxed{\text{From (a), } \varepsilon' \text{ is even and } \varepsilon'' \text{ is odd. Satisfies condition!}}$