5. (Classical Mechanics)

(a) Find a canonical transformation $P = P(p,q)$, $Q = Q(p,q)$ that turns the Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$$

into

$$H_0 = i\omega_0 PQ$$

where P, Q are the new momentum and position.

(b) Next, consider the driven harmonic oscillator:

$$m\ddot{q} + kq = F\cos\Omega t$$

Using the transformation $p, q \to P, Q$ that you derived in (a), compute the transformed Hamiltonian and write down Hamilton's equations of motion for P, Q.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

(a) In order to get started on this problem, you need to think of the correct ansatz

$$P = Ap + Bq \quad \text{and} \quad Q = Cp + Dq \tag{152}$$

There are a few ways to come up with this ansatz. One way is to realize that, in order to turn the Hamiltonian in the new coordinate system $H_0' \equiv i\omega_0 PQ$ into the Hamiltonian in the old coordinate system $H_0 \equiv \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2$, P and Q will need to be linear in p and q.

Another way is to remember that in quantum mechanics, the Hamiltonian in terms of the raising and lowering operators $H_0 = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)$ looks a lot like the new Hamiltonian in this problem $H_0' = i\omega_0 PQ$, where P and Q play the roles of the raising and lowering operators. (The extra factor of $\frac{1}{2}$ is due to the noncommutativity of quantum operators and does not appear in classical mechanics.) The raising and lowering operators are linear in p and x, so, by analogy, we should expect P and Q in this problem to be linear in p and q.

Once we have the correct ansatz (152), we need to find the proper values of A, B, C, and D. The first condition to impose is the relationship between the Hamiltonians. We will suppose that our canonical transformation has no explicit time dependence, so the old and new Hamiltonians should be equal:

$$H_0(p, q) = H_0'(P, Q) \implies \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = i\omega_0 PQ \tag{153}$$

Plugging in the ansatz (152), we get the following:

$$\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = i\omega_0(Ap + Bq)(Cp + Dq)$$
$$\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = i\omega_0 AC\, p^2 + i\omega_0 BD\, q^2 + i\omega_0(AD + BC)\, pq$$

Collecting terms of the same form, we get

$$\frac{1}{2m} = i\omega_0 AC \tag{154}$$
$$\frac{1}{2}m\omega_0^2 = i\omega_0 BD \tag{155}$$
$$AD + BC = 0 \tag{156}$$

This is a set of three equations for four unknowns, so we can write three of the unknowns B, C, D in terms of the fourth A:

By (154), $A \neq 0$, so we can rearrange (154) to get

$$C = -\frac{i}{2m\omega_0 A} \tag{157}$$

By (155), $B \neq 0$, so we can rearrange (155) to get

$$D = -\frac{im\omega_0}{2B} \tag{158}$$

Plugging (157) and (158) into (156), we get

$$-\frac{im\omega_0}{2}\frac{A}{B} - \frac{i}{2m\omega_0}\frac{B}{A} = 0 \tag{159}$$

Simplifying, we can solve for B in terms of A:

$$\begin{aligned} m\omega_0 A^2 + \frac{1}{m\omega_0} B^2 &= 0 \\ B^2 &= -m^2\omega_0^2 A^2 \\ B &= \pm i m\omega_0 A \end{aligned} \tag{160}$$

Plugging (160) into (158), we can write D in terms of A:

$$D = \mp \frac{1}{2A} \tag{161}$$

Collecting our results, we have

$$B = \pm i m\omega_0 A; \qquad C = -\frac{i}{2m\omega_0 A}; \qquad D = \mp\frac{1}{2A} \tag{162}$$

You can check for yourself that any solution of this form satisfies the conditions (154), (155), and (156). These three conditions are equivalent to the single condition that the Hamiltonian H_0 is invariant under the transformation. Therefore, the set of transformations of the form (152) that keep the Hamiltonian invariant are

$$P = Ap + Bq = A\left(p \pm i m\omega_0 q\right) \quad \text{and} \quad Q = Cp + Dq = \frac{1}{2A}\left(-\frac{i}{m\omega_0}p \mp q\right) \tag{163}$$

Note that we still have the freedom to choose a value for A. For the $\pm$ and $\mp$ signs, we may also choose the "upper option" ($\pm \to +$ and $\mp \to -$) or the "lower option" ($\pm \to -$ and $\mp \to +$).

So far, we have just imposed the condition that the Hamiltonian is invariant under the coordinate transformation. But we have not yet imposed the condition that the coordinate transformation is canonical. We may impose this condition in one of two ways:

Method 1: Invariance of the Poisson bracket

A canonical transformation leaves the Poisson bracket invariant. For a system with one canonical coordinate q, the Poisson bracket is defined as

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{164}$$

and the canonical Poisson bracket for p and q is

$$\{q, p\} = 1 \tag{165}$$

In order for the Poisson bracket to be invariant under the coordinate transformation, we must therefore have

$$1 = \{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \tag{166}$$

Using our result (163), this boils down to

$$1 = AD - BC = A\left(\pm\frac{1}{2A}\right) - \left(\pm i m\omega_0 A\right)\left(-\frac{i}{2m\omega_0 A}\right) = \mp 1 \tag{167}$$

This equation is only valid if we choose the "lower option" ($\pm \to -$ and $\mp \to +$) in (163). Therefore, the canonical transformations that leave the Hamiltonian invariant are

$$P = A(p - i m\omega_0 q) \quad \text{and} \quad Q = \frac{1}{2A}\left(-\frac{i}{m\omega_0}p + q\right) \tag{168}$$

Any nonzero value of A will lead to a possible canonical transformation. We could, for example, set $A = 1$ to get

$$\boxed{P = p - im\omega_0 q \quad \text{and} \quad Q = \frac{1}{2}\left(-\frac{i}{m\omega_0}p + q\right) \qquad (A = 1)} \tag{169}$$

or, for more symmetry, we could set $A = -\frac{i}{\sqrt{2m\omega_0}}$ to get

$$\boxed{P = -\frac{i}{\sqrt{2m\omega_0}}p - \sqrt{\frac{m\omega_0}{2}}q \quad \text{and} \quad Q = \sqrt{\frac{1}{2m\omega_0}}p + i\sqrt{\frac{m\omega_0}{2}}q \qquad \left(A = -\frac{i}{\sqrt{2m\omega_0}}\right)} \tag{170}$$

Method 2: Generating function

Any transformation that is produced by a generating function is canonical. For simplicity, let's choose the generating function of the second kind $F_2(q, P)$, which defines the other coordinates as follows:

$$p = \frac{\partial F_2}{\partial q} \quad \text{and} \quad Q = \frac{\partial F_2}{\partial P} \tag{171}$$

We need to rewrite the equations in (163) so that one side of the equations is only in terms of q and P. For the first equation, this is straightforward:

$$\begin{aligned} P = Ap + Bq &\Longrightarrow Ap = P - Bq \\ &\Longrightarrow \frac{\partial F_2}{\partial q} = p = \frac{1}{A}(P - Bq) \end{aligned} \tag{172}$$

Taking the partial antiderivative, with respect to q, this gives us

$$F_2(q, P) = \frac{1}{A}\left(Pq - \frac{B}{2}q^2 + f\right) \tag{173}$$

where f, the constant of integration, depends only on P and not on q. We can reverse-engineer f by using the second equation in (171) to get

$$Q = \frac{\partial F_2}{\partial P} = \frac{1}{A}\left(q + \frac{df}{dP}\right) \tag{174}$$

But from (163) and (172), we also know that

$$Q = Cp + Dq = C\left(\frac{1}{A}(P - Bq)\right) + Dq = \frac{C}{A}P + \left(D - \frac{BC}{A}\right)q \tag{175}$$

Combining (174) and (175) and multiplying through by A, we get

$$q + \frac{df}{dP} = (AD - BC)\,q + CP \tag{176}$$

Using the fact that q and P are independent and that f only depends on P, this implies that

$$AD - BC = 1 \quad \text{and} \quad CP = \frac{df}{dP} \tag{177}$$

Plugging in our values for B, C, and D from (163) into the equation $AD - BC = 1$, we get

$$1 = AD - BC = A\left(\pm\frac{1}{2A}\right) - (\pm im\omega_0 A)\left(-\frac{i}{2m\omega_0 A}\right) = \mp 1 \tag{178}$$

This equation is only valid if we choose the "lower option" ($\pm \to -$ and $\mp \to +$) in (163). Therefore, the canonical transformations that leave the Hamiltonian invariant are

$$P = A(p - im\omega_0 q) \quad \text{and} \quad Q = \frac{1}{2A}\left(-\frac{i}{m\omega_0}p + q\right) \tag{179}$$

The equation $CP = \frac{df}{dP}$ implies that

$$f(P) = \frac{C}{2}P^2 + \text{ constant} \tag{180}$$

so, plugging this back into (173), we can find the generating function for the canonical transformation:

$$F_2(q, P) = \frac{1}{A}\left(Pq - \frac{B}{2}q^2 + \frac{C}{2}P^2 + \text{ constant}\right) \tag{181}$$

Plugging in our values for B, C, and D from (163) into this equation and choosing the "lower option" ($\pm \to -$ and $\mp \to +$), we get

$$F_2(q, P) = \frac{Pq}{A} + \frac{im\omega_0}{2}q^2 - \frac{i}{4m\omega_0 A^2}P^2 + \text{constant} \tag{182}$$

Any nonzero value of A, and any constant, will lead to a possible canonical transformation. We could, for example, set $A = 1$ and the constant to zero to get

$$\boxed{P = p - im\omega_0 q \quad \text{and} \quad Q = \frac{1}{2}\left(-\frac{i}{m\omega_0}p + q\right) \qquad (A = 1)} \tag{183}$$

and

$$\boxed{F_2(q, P) = Pq + \frac{im\omega_0}{2}q^2 - \frac{i}{4m\omega_0}P^2 \qquad (A = 1)} \tag{184}$$

or, for more symmetry, we could set $A = -\frac{i}{\sqrt{2m\omega_0}}$ and the constant to zero to get

$$\boxed{P = -\frac{i}{\sqrt{2m\omega_0}}p - \sqrt{\frac{m\omega_0}{2}}q \quad \text{and} \quad Q = \sqrt{\frac{1}{2m\omega_0}}p + i\sqrt{\frac{m\omega_0}{2}}q \qquad \left(A = -\frac{i}{\sqrt{2m\omega_0}}\right)} \tag{185}$$

and

$$\boxed{F_2(q, P) = i\sqrt{2m\omega_0}Pq + \frac{im\omega_0}{2}q^2 + \frac{i}{2}P^2 \qquad \left(A = -\frac{i}{\sqrt{2m\omega_0}}\right)} \tag{186}$$

(b) To start, we need to write the Hamiltonian for the driven harmonic oscillator in the coordinates (p, q). The problem gives us the Newton's second law equation

$$m\ddot{q} = -kq + F\cos\Omega t \tag{187}$$

which corresponds to a driving force

$$F(t) = F\cos\Omega t \tag{188}$$

The potential energy V associated with this force is given by the one-dimensional equation $F = -\frac{\partial V}{\partial q}$:

$$V(q, t) = -Fq\cos\Omega t + \text{ constant} \tag{189}$$

We may set the constant to zero. The Hamiltonian is equal to the sum of the kinetic and potential energies, so we can modify the original Hamiltonian H_0 to include the effects of the potential due to the driving force:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 - Fq\cos\Omega t = H_0 - Fq\cos\Omega t \quad \text{where } \omega_0 \equiv \sqrt{\frac{k}{m}} \tag{190}$$

To compute the transformed Hamiltonian (i.e. the Hamiltonian transformed into the new coordinates), we just use the definitions of P and Q in terms of p and q (either (168) or (179)). We don't need to worry about adding new terms to the Hamiltonian since the *coordinate transformation* is not time-dependent, even though the *Hamiltonian* is now time dependent. (If the coordinate transformation were time-dependent, we would have to add a new term to the Hamiltonian based on the partial time derivative of the generating function.)

The original Hamiltonian H_0 transforms into $H_0' = i\omega_0 PQ$ (this is the condition we used to find the coordinate transformation). The new piece of the Hamiltonian $-Fq\cos\Omega t$ depends on q, so we need to write q in terms of the new coordinates Q and P. From (168) or (179), we can derive the following:

$$\begin{array}{rcr} P = A(p - im\omega_0 q) & \implies & \frac{P}{A} = p - im\omega_0 q \\ Q = \frac{1}{2A}\left(-\frac{i}{m\omega_0}p + q\right) & \implies & 2iAm\omega_0 Q = p + im\omega_0 q \\ \hline & & \frac{P}{A} - 2iAm\omega_0 Q = -2im\omega_0 q \end{array}$$

This implies that

$$q = AQ + \frac{i}{2m\omega_0 A}P \tag{191}$$

so

$$\begin{aligned} H'(P,Q,t) &= H_0' - Fq\cos\Omega t \\ &= i\omega_0 PQ - F\left(AQ + \frac{i}{2m\omega_0 A}P\right)\cos\Omega t \end{aligned} \tag{192}$$

For the choice $A = 1$, we get

$$\boxed{H'(P,Q,t) = i\omega_0 PQ - F\left(Q + \frac{i}{2m\omega_0}P\right)\cos\Omega t \qquad (A=1)} \tag{193}$$

and for the choice $A = -\frac{i}{\sqrt{2m\omega_0}}$, we get

$$H'(P,Q,t) = i\omega_0 PQ - F\left(-\frac{i}{\sqrt{2m\omega_0}}Q - \frac{1}{\sqrt{2m\omega_0}}P\right)\cos\Omega t$$

or

$$\boxed{H'(P,Q,t) = i\omega_0 PQ + F\cdot\frac{P+iQ}{\sqrt{2m\omega_0}}\cos\Omega t \qquad \left(A = -\frac{i}{\sqrt{2m\omega_0}}\right)} \tag{194}$$

To find the Hamilton's equations of motion, use the formulas

$$\dot{Q} = \frac{\partial H'}{\partial P} \quad \text{and} \quad \dot{P} = -\frac{\partial H'}{\partial Q} \tag{195}$$

Applying them to (192), we get

$$\dot{Q} = i\omega_0 Q - \frac{i}{2m\omega_0 A}F\cos\Omega t \quad \text{and} \quad \dot{P} = -i\omega_0 P + AF\cos\Omega t \tag{196}$$

For the choice $A = 1$, we get

$$\boxed{\dot{Q} = i\omega_0 Q - \frac{i}{2m\omega_0} F \cos \Omega t \quad \text{and} \quad \dot{P} = -i\omega_0 P + F \cos \Omega t \qquad (A = 1)} \tag{197}$$

and for the choice $A = -\frac{i}{\sqrt{2m\omega_0}}$, we get

$$\boxed{\dot{Q} = i\omega_0 Q + \frac{1}{\sqrt{2m\omega_0}} F \cos \Omega t \quad \text{and} \quad \dot{P} = -i\omega_0 P - \frac{i}{\sqrt{2m\omega_0}} F \cos \Omega t \qquad \left(A = -\frac{i}{\sqrt{2m\omega_0}} \right)} \tag{198}$$