QUESTION 2: [40 points]

a) Consider the operator

$$T(a) = \exp(-iaP/\hbar) \tag{0.9}$$

Where a is a constant and P is the momentum operator Show that

$$T^{\dagger}(a)\; X\; T(a) = x + a \tag{0.10}$$

$[X, P] = i\hbar$ use $e^{+A} B e^{-A} = \exp Ad_A\, B$

$$e^{+iaP/\hbar} X e^{-iaP/\hbar} = (1 + ad_{iaP/\hbar})X$$

$$= X + i a/\hbar\, [P, X]$$

$$= X + a$$

b) Show that $T(a)$ is unitary and show that T(a) has eigenvalues of of the form $e^{i\phi}$ where ϕ is real (You can assume that P is hermitian).

$T(a)^{+} T(a) = e^{iaP/\hbar} e^{-iaP/\hbar} = 1$ since P commuts with itself

same for $T(a) T(a)^{+}$ $\Rightarrow$ T unitary

$\langle\psi| T(a) |\psi\rangle = \langle\psi|\lambda|\psi\rangle = \lambda$

$\langle\psi| T(a) |\psi\rangle^{*} = \langle\psi| T(a)^{+} |\psi\rangle = \lambda^{*}$

on the other hand

$|\psi\rangle = \lambda\, T(a)^{+} |\psi\rangle \Rightarrow \lambda^{*} = \frac{1}{\lambda}$ $\lambda\lambda^{*} = 1$

$\Rightarrow \lambda = e^{i\phi}$

c) Consider the Hamiltonian which is periodic under shifts by a post quant.

$$H = \frac{P^2}{2m} + \sum_{n=-\infty}^{\infty} V(\cancel{X} - na) \tag{0.11}$$

Here you can assume that $V(x)$ goes exponentially fast to zero as $|x| \to \infty$ (This assumption makes the sum over n convergent). You can also assume that $V(x)$ can be expanded in a power series.

Prove that $T(a)$ commutes with H.

$$V(x) = \sum_n c_n x^n$$

$$T^\dagger X T = X + a$$

$$T^\dagger X^n T = T^\dagger X T T^\dagger \cdots T T^\dagger X T$$

$$= (X+a)^n$$

$$\Rightarrow T^\dagger V(X) T = \sum_n c_n (X+a)^n$$

$$= V(X+a)$$

$$T^\dagger \sum_{n=-\infty}^{\infty} V(X-na) T = \sum_{n=\infty}^{\infty} V(X+a-na)$$

$$= \sum_n V(x-(\underset{n'}{n-1})a)$$

$$= \sum_{n'} V(x-n'a) = \sum V(X-na)$$

$$T^\dagger P^2 T = P^2$$

$$T^\dagger H T = H \quad \text{or} \quad HT = TH \Leftrightarrow [T, H] = 0$$

d) It follows from the results in part c) that the Hamiltonian H and $T(a)$ can be diagonalized simultaneously. You can assume that there are eigenstates $\mid E,k\rangle$ which satisfy

$$H \mid E,k\rangle = E \mid E,k\rangle \tag{0.12}$$
$$T(a) \mid E,k\rangle = e^{-ika} \mid E,k\rangle \tag{0.13}$$

For the wave functions in position space define the following combination

$$u_k(x) = \langle x \mid E,k\rangle e^{-ikx} \tag{0.14}$$

Show that $u_k(x)$ is a periodic function with period a, i.e.

$$u_k(x+a) = u_k(x) \tag{0.15}$$

This is the Bloch's theorem for periodic potentials (i.e. an energy eigenstate can be written as a Bloch wave times a periodic function).

$$u_k(x+a) = \langle x+a \mid E k\rangle e^{-ik(x+a)}$$
$$= (\langle E k \mid x+a\rangle)^* e^{-ik(x+a)}$$
$$= \langle E,k \mid T \mid x\rangle^* e^{-ik(x+a)}$$
$$= \langle x \mid T^\dagger \mid E,k\rangle e^{-ikx} e^{-ika}$$
$$= \langle x \mid E,k\rangle \cdot \cancel{e^{ika}} e^{-ikx} \cancel{e^{-ika}}$$
$$= u_k(x)$$

Use $\quad T^\dagger(a) T(a) \mid E k\rangle = e^{-ika} T^\dagger \mid E k\rangle$

$$\to T^\dagger \mid E,k\rangle = e^{ika} \mid E,k\rangle$$