12. *Electricity and Magnetism* (Spring 2006)

Two point charges $+Q_0$ and $-Q_0$ are placed at opposite poles of a spherical balloon of initial radius R_0. The radius of the balloon is set to oscillate as follows: $R(t) = R_0 + \rho \sin \omega t$. Assume $\rho\omega \ll c$.

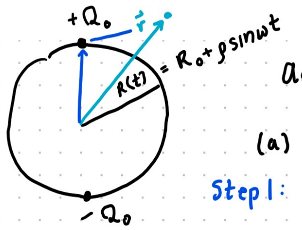
(a) Determine the total power radiated by the oscillating balloon, if any, in terms of Q_0, R_0, ρ, and ω. Show your work and explain your reasoning.

Note: If you are unable to write an expression for the total power radiated, explain how the total power radiated scales with each of the above variables.

(b) Suppose instead that charges are deposited on the balloon as described below. For each case, determine the ratio of the total power radiated by the oscillating balloon, if any, to the total power radiated in (a). Show your work and explain your reasoning.

(i) One point charge $+Q_0$ is placed at a given point on the balloon. The radius of the balloon is set to oscillate as above.

(ii) A total charge $+Q_0$ is deposited uniformly on the surface of the balloon. The radius of the balloon is set to oscillate as above.



Assume $\rho\omega \ll c$

(a) Total power radiated

Step 1: using the eqn of a dipole: $\vec p(\vec r) = \int_V d^3r' \rho(\vec r')\vec r'$ (relative to origin @ $\vec r_0 = 0$)

$\rho(\vec r) = +Q_0\delta(\vec r - R(t)\hat z) + -Q_0\delta(\vec r + R(t)\hat z)$

now, $\vec p(\vec r) = \int_V Q_0\{\delta(\vec r - R(t)\hat z) - \delta(\vec r + R(t)\hat z)\}\vec r\, d^3r$

$= Q_0\{R(t)\hat z - (-R(t)\hat z)\} = 2Q_0 R(t)\hat z$ (as expected)

Step 2: Time-Harmonic Dipole

Special care must be taken for averaging time-harmonic sources

$\vec p = 2Q_0 R(t)\hat z = 2Q_0(R_0 + \rho\, \mathrm{Re}[i e^{-i\omega t}])$

$= 2Q_0 \mathrm{Re}[R_0 + i\rho e^{-i\omega t}]$

from Zangwill 20.158

$P_{El} = \frac{1}{4\pi\varepsilon_0}\frac{2|\ddot{\vec p}_{ret}|^2}{3c^2}$ (ess Larmor when $\vec p = q\vec r$)

but, we time avg using $\langle \mathrm{Re}[A]\,\mathrm{Re}[B]\rangle = \frac{1}{2}\mathrm{Re}[a^* b]$ for $A = a(t)e^{-i\omega t}$; $B = b(t)e^{-i\omega t}$

$\langle P_{El}\rangle = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\langle|\ddot{\vec p}_{ret}|^2\rangle$ w/ $\vec p_{ret} = \mathrm{Re}[2Q_0(R_0 + i\rho e^{-i\omega t})]$

$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\langle \ddot{\vec p}_{ret}\cdot\ddot{\vec p}_{ret}\rangle$ $\quad \ddot{\vec p}_{ret} = \mathrm{Re}[2Q_0 i\rho(-i\omega)^2 e^{-i\omega t}]$

$= \mathrm{Re}[-i2Q_0\rho\omega^2 e^{-i\omega t}]$

$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\frac{1}{2}\mathrm{Re}[(-i2Q_0\rho\omega^2)(i2Q_0\rho\omega^2)]$

$= \frac{\mu_0}{12\pi c}(4Q_0^2\rho^2\omega^4) = \boxed{\frac{\mu_0 Q_0^2\rho^2\omega^4}{3\pi c}}$

(b) Det ratio of power

(i) Revisiting step 1 in (a), we simply remove the $-Q_0$ charge

$\vec p_2 \to Q_0 R(t)\hat z$ and $\vec p_2 \to \frac{1}{2}\vec p$ from (a)

$\langle P_Q\rangle = \frac{1}{4}\langle P_{El}\rangle$ since $P_{El} \propto |\ddot{\vec p}|^2$

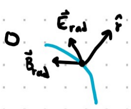
$\boxed{\langle P_Q\rangle / \langle P_{El}\rangle = 1/4}$

(ii) Q_0 uniformly on surface

Method 1: Since the shell expands radially, Gauss' Law says $\vec E = \frac{Q_0}{4\pi R(t)^2}\hat r$ (which we know isn't a radiation field)

But, $c\vec B_{rad} = \hat r \times \vec E_{rad} \sim \hat r \times \hat r = 0$

$\therefore$ $\vec S_{rad} = 0$ and $\frac{dP_{rad}}{d\Omega} = r^2\hat r\cdot\vec S_{rad} = 0$



$\boxed{\langle P_{shell}\rangle / \langle P_{El}\rangle = 0}$

Method 2: $\vec p = \int_V d^3r' \rho(\vec r')\vec r'$ $\Rightarrow$ $Q_0 = \int d^3r' Q_0\delta(\vec r' - R(t)\hat R)$

$\rho(\vec r) = Q_0\delta(\vec r - R(t)\hat R)$

$= \int_V d^3r' Q_0\delta(\vec r' - R(t)\hat R)\vec r'$

$= Q_0 R(t)\hat r$ $\quad \hat r \parallel \ddot{\vec p}_{ret}$

$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{2}\frac{\mu_0}{16\pi^2 c}|\hat r \times \ddot{\vec p}_{ret}|^2 = 0$