FA 2005 SM

Monday, June 29, 2020 12:37 PM

kdbarajas@live.com

6. *Statistical Mechanics and Thermodynamics* (Fall 2005)

"Cold" stars (that is, stars that have exhausted their nuclear fuel) are stabilized against gravitational collapse by the degeneracy pressure of the electrons, or, at higher densities, neutrons. To model this effect, consider a spherical star of mass M, mass density ρ, radius R, and volume V, consisting of neutrons of mass m_n.

(a) Calculate the gravitational potential of the star, that is, the gravitational potential of a uniform massive sphere of radius R.

Answer:

$$V_G = -\frac{(2\pi)^2\rho^2 G}{15}R^5$$

where G is Newton's gravitational constant.

(b) Obtain the corresponding gravitational pressure P_G.

(c) View the neutrons as a cold, ideal neutron gas. Compute the energy and the degeneracy pressure of the fermion gas at $T = 0$ assuming

(i) the gas is nonrelativistic ($p \ll m$).

(ii) the gas is ultrarelativistic ($p \gg m$).

Is there an equilibrium radius for the star in

(i) the nonrelativistic case?

(ii) the ultrarelativistic case?

If there is no equilibrium radius, what is the critical particle number $N = N_c$ above which gravitational collapse is unavoidable?

known:

(a)
- uniform massive sph $\Rightarrow \rho(r) = \text{const.}$
- $V_G = -\frac{G m_1 m_2}{r}$
- $m_1 = Nm_n = M(r) = \frac{4}{3}\pi r^3 \underset{\text{const.}}{\rho} \Rightarrow M(R) = \rho\left(\frac{4}{3}\pi R^3\right)$ as desired

If we consider the spatially extended body of the star w/ mass density ρ, we can take an integral over infinitesimal mass shells of width $dm_2 \equiv dm_n = \rho\, dV$

$$\Rightarrow dV_G = -\frac{G\left(\frac{4}{3}\pi r^3 \rho\right)(\rho\, dV)}{r}$$ ← no ang dep, $dV = 4\pi r^2 dr$

$$dV_G = -\left(\frac{16\pi^2}{3}\right) G\rho^2 r^4 dr$$

$$\Delta V_G = -\left(\frac{16\pi^2}{3}\right) G\rho^2 \int_0^R r^4 dr = -\frac{4(2\pi)^2}{3} G\rho^2 \frac{R^5}{5}$$

$$\boxed{\Delta V_G = -\frac{(4\pi)^2 \rho^2 G}{15} R^5}$$

* I think typo in answer

as a check, $\rho = M(R)/\frac{4}{3}\pi R^3$

and $V_G = -\frac{3}{5}\frac{GM^2}{R}$ as expected from classical mech.

(b) Obtain P_G:

Method 1: Thermodynamic Identity

$$dE = TdS - PdV$$

$$\left(\frac{\partial E_G}{\partial V}\right)_S = -P_G$$

uniform, massive sphere $\Rightarrow$ treat uniform star as adiabatic w/ no heat gained/lost

$$V_G = E_G \Rightarrow \left\{V = \frac{4}{3}\pi R^3\right\}$$

$$V_G = -\frac{(4\pi)^2\rho^2 G}{15}R^5 = -\frac{(4\pi)^2\rho^2 G}{15}\left(\frac{3}{4\pi}V\right)^{5/3}$$

$$= -\frac{3}{5}\rho^2 G\left(\frac{4\pi}{3}\right)^{2-5/3} V^{5/3} = -\frac{3}{5}G(M/V)^2\left(\frac{4\pi}{3}\right)^{1/3}V^{5/3}$$

$$= -\frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^2 V^{-1/3}$$

$$P_G = -\left[-\frac{\cancel{3}}{5}\left(\frac{4\pi}{3}\right)^{1/3}GM^2\right]\left(-\frac{1}{\cancel{3}}\right)V^{-4/3} = -\frac{1}{5}GM^2\left(\frac{4\pi}{3}\right)^{1/3}V^{-4/3}\left(\frac{V^2}{V^2}\right)$$

$$= -\frac{1}{5}G\rho^2\left(\frac{4\pi}{3}\right)^{1/3}V^{2/3} = \frac{1}{5}G\rho^2\left(\frac{4\pi}{3}\right)^{1/3}\left(\frac{4}{3}\pi R^3\right)^{2/3}$$

$$= -\frac{1}{5}G\rho^2\left(\frac{4\pi}{3}\right)R^2 = \boxed{-\frac{4\pi}{15}G\rho^2R^2}$$

Method 2: Pressure from Force

agrees w/ method 1

$$P_G = \frac{dF_G}{dA} = \frac{d}{dA}\left(-\frac{dV_G}{dR}\right) = \left\{-\frac{d}{dA}\left(\frac{dV_G}{dR}\right)\right\} = -\frac{dV_G}{dV}$$

$$\frac{dV_G}{dR} = -\frac{\cancel{(4\pi)^2}M^2G}{15\left(\frac{4}{3}\cancel{\pi}\right)^2}\frac{d}{dR}\left(\frac{R^5}{(R^3)^2}\right)$$

$$= \cancel{-}\frac{9}{15}M^2G\left(\cancel{-}\frac{1}{R^2}\right)$$

$$P_G = -\frac{d}{dA}\left(\frac{9}{15}M^2G\frac{1}{R^2}\right)$$

$A = 4\pi R^2$

$$= -\frac{9}{15}M^2G(4\pi)\left(\frac{1}{A^2}\right)$$

$$= -\frac{9}{15}M^2G(4\pi)\left(\frac{1}{(4\pi R^2)^2}\right)$$

$$= -\frac{9}{15}G\rho^2\left(\frac{4}{3}\pi R^3\right)^2 4\pi\left(\frac{1}{(4\pi R^2)^2}\right)$$

$R \cdot R^2$

$$= -\frac{\cancel{9}}{15}G\rho^2\left(\frac{4\pi}{\cancel{9}}\right)\left(4\pi \cancel{R^2}\right)^{\cancel{2}} R^2 \frac{1}{\cancel{(4\pi R^2)^2}}$$

$$\boxed{P_G = -\frac{4\pi}{15}G\rho^2R^2}$$

(c) known
- neutron — cold, ideal gas
- fermion gas @ $T=0$

Compute energy and degenerate pressure:

(i) non-relativistic ($p \ll m$)

We begin by finding the fermi energy (assumes cold gas) w/in 3D space filled w N fermions (neutrons) s.t. all lowest energy states are filled (and $E > E_F$ states are empty)

Method 1: Stat Mech dat Shit

$$E = T + V \Rightarrow T = E_F \Rightarrow E_F = E - V$$

$$E = \sqrt{(mc^2)^2 + (pc)^2}, \quad V = mc^2 \text{ (rest energy)}$$

Taking nonrel limit $p \ll m$,

$$E_F \simeq mc^2\left(\cancel{1} + \frac{1}{2}\frac{(pc)^2}{(mc^2)^2}\right) - \cancel{mc^2} \simeq \frac{p_F^2}{2m}$$

$\Rightarrow$ To find the Fermi momentum, we need the number density of particles (allowed occupation @ $T=0$)

$$N = g_s \int dn^* \, n_F(p) \Rightarrow n = \frac{N}{V} = \frac{g_s}{(2\pi\hbar)^3}\int d^3p \; n_F(p)$$

↖ degeneracy factor

*If you forget factor, $\sum_n \to \int dn$, $p_n = \hbar k_n = \hbar\left(\frac{2\pi n}{L}\right)$

Method 2: Approximate as square well

$\psi \sim \sin(k_F x) = 0$ @ boundary $x = 0, L$

$k_F = \frac{\pi n_F}{L}$, $p = \hbar k$

Approximate as square well $E_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$

$$= \frac{\pi^2\hbar^2}{2mL^2} r_n^2$$ ← radius in n-space

Since $n_i \in \mathbb{Z}^+$ we can find the total energy of fermions w/in n-space sphere by taking $\frac{1}{8}$ of total energy of sphere

only positive n values

$$E_{tot} = g_s\left(\frac{1}{8}\right)\int_0^{r_n} \frac{\pi^2\hbar^2}{2mL^2} r_n'^2 (4\pi r_n'^2)\, dr_n'$$

density of particles (allowed occupation @ $T=0$)

$$N = g_s \int dn^* \, n_F(p) \Rightarrow n = \frac{N}{V} = \frac{g_s}{(2\pi\hbar)^3} \int d^3p \; n_F(p)$$

↑ degeneracy factor

*If you forget factor,

$$\sum_n \to \int dn$$

$$p_n = \hbar k_n = \hbar \left(\frac{2\pi n}{L}\right)$$

$$\therefore \; n = \frac{Lp}{2\pi\hbar} \Rightarrow dn = \frac{dx\,dp}{2\pi\hbar}$$

Since fully-degenerate occupation up to fermi level,

$$\lim_{\substack{E_F \simeq \mu(T=0) \\ E < E_F \\ T \to 0}} n_F(p) = \frac{1}{1 + e^{(E-\mu)/kT}} \simeq 1 \; ; \quad g_s = 2s+1 = 2 \;\; \text{(degeneracy)}$$

↑ Fermi-Dirac distr.

$$n = \frac{2}{h^3} \frac{4\pi p^3}{3} \Big|_0^{p_F} \Rightarrow \left\{ p_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3} \right\}$$

(Fermi momentum of sphere of available degenerate occupation)

Thus, the Fermi energy is $\left(h = \frac{\hbar}{2\pi}\right)$

$$E_F = \frac{1}{2m}\left(\frac{3h^3 n}{8\pi}\right)^{2/3} = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

Finally, we can find the total energy!

$$E_{tot}^{(\text{nonrel})} = \langle \varepsilon \rangle = g_s \int \overbrace{\frac{d^3x \, d^3p}{(2\pi\hbar)^3}}^{dn(p)} \varepsilon(p) \, n_F(p)$$

- nonrel. $\varepsilon(p) = \frac{p^2}{2m}$
- Fermi gas $n_F(p) \to 1$ for cold, degenerate ($T=0$)

$$= \frac{g_s V}{h^3} 4\pi \int_0^{p_F} dp \, p^2 \left(\frac{p^2}{2m}\right) = \frac{g_s 4\pi V}{2m h^3} \int_0^{p_F} dp \, p^4$$

($g_s = 2$)

$$= \frac{8\pi V}{2m h^3} \frac{p_F^5}{5} = \frac{8\pi V}{10 m h^3}\left(\frac{3h^3 n}{8\pi}\right)^{5/3} = \frac{\cancel{8}\pi V}{10 m h^3} \frac{h^5}{8^{2/3}} \left(\frac{3N}{\pi V}\right)^{5/3}$$

($E_F = p_F^2/2m$)

$$= \frac{h^2 \pi V}{10 m (2^3)^{2/3}} \left(\frac{3N}{\pi V}\right)^{5/3} = \frac{(\cancel{2}\pi\hbar)^2 \pi}{10 m \cdot \cancel{2^2} \, V^{2/3}}\left(\frac{3N}{\pi}\right)^{5/3} \Rightarrow \boxed{E_{tot} = \frac{\pi^3 \hbar^2}{10 m V^{2/3}}\left(\frac{3N}{\pi}\right)^{5/3}}$$

only positive n values

$$E_{tot} = g_s \left(\frac{1}{8}\right) \int_0^{r_n} \frac{\pi^2 \hbar^2}{2mL^2} r_n'^2 \left(4\pi r_n'^2\right) dr_n'$$

↑ Fermion have two spin states (e.g. neutron) and thus 2-fold degenerate

$$E_{tot} = \frac{\pi^3 \hbar^2}{2mL^2} \int_0^{r_n} r_n'^4 \, dr_n'$$

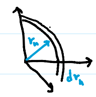
$$= \frac{\pi^3 \hbar^2}{10 m L^2} r_n^5$$

Next, we want to relate r_n to a quantity of interest

w/in the n-sphere, the number of available states (which also is # of particles supported b/c cold Fermions)

$$N = (2)\left(\frac{1}{8}\right) \int d^3 r_n$$

$$N = (2)\left(\frac{1}{8}\right) \frac{4}{3} \pi r_n^3$$



$$\Rightarrow r_n = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$\boxed{E_{tot} = \frac{\pi^3 \hbar^2}{10 m L^2}\left(\frac{3N}{\pi}\right)^{5/3} = \frac{\pi^3 \hbar^2}{10 m V^{2/3}}\left(\frac{3N}{\pi}\right)^{5/3}}$$

↑ $L^2 = (L^3)^{2/3} = V^{2/3}$

Finally, we find the degeneracy pressure using thermo. identity

$$P_{tot} = -\left(\frac{dE_{tot}}{dV}\right)_{S,N}$$

$$= -\left(\frac{\pi^3 \hbar^2}{10m}\right)\left(-\tfrac{2}{3}\right)\left(\frac{3N}{\pi V}\right)^{5/3}$$

$$\boxed{P_{tot} = \frac{\pi^3 \hbar^2}{15 m}\left(\frac{3N}{\pi V}\right)^{5/3}}$$

Now, finding equilibrium radius since for cold neutron star both balance out $\Rightarrow \; 0 = P_{tot} + P_G$

$$\frac{\pi^{\cancel{3}\,2} \hbar^2}{\cancel{15}\, m_n}\left(\frac{3N}{\pi V}\right)^{5/3} = \frac{4\cancel{\pi}}{\cancel{15}} G \rho^2 R^2$$

using $M = N m_n$, $V = \frac{4}{3}\pi R^3$, $\rho = M/V$

$$R^2 = \frac{\pi^2 \hbar^2}{4 G m_n} \frac{V^2}{M^2}\left(\frac{3N}{\pi V}\right)^{5/3} = \frac{\pi^2 \hbar^2}{4G} \frac{1}{M^2}\left(\frac{3M}{\pi m_n}\right)^{5/3} V^{1/3}$$

$$= \frac{\pi^2 \hbar^2}{4 G m_n}\left(\frac{3}{\pi m_n}\right)^{5/3} M^{-1/3} \left(\frac{4}{3}\pi\right)^{1/3} R$$

$$R = \frac{\hbar^2}{G m_n^{8/3}} \left(\frac{\pi}{2}\right)^2 \left(\frac{3}{\pi}\right)^{5/3} \left(\frac{4}{3}\pi\right)^{1/3} M^{-1/3}$$

$$= \; . \; . \; . \; \left(\pi^{2/3} \, 9^{2/3} \, 4^{-2/3}\right) M^{-1/3}$$

$$\boxed{R_{eq} = \frac{\hbar^2}{G m^{8/3}} \left(\frac{9\pi}{4}\right)^{2/3} M^{-1/3}}$$

(ii) ultrarelativistic ($p \gg m$)

$$E = \sqrt{(mc^2)^2 + (pc)^2} \simeq pc\left(1 + \frac{1}{2}\frac{(mc^2)^2}{(pc)^2}\right) \simeq pc$$

$$E_F = E - E_{rest} = E - mc^2 \simeq p_F c - mc^2 \simeq p_F c$$

Method 1:

as before, the Fermi Momentum is $p_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3}$ (see (i) method 1)

$$\therefore \; E_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3} \cdot c \quad \text{and} \quad E_{tot} = \langle \varepsilon \rangle = g_s \int \frac{d^3x \, d^3p}{(2\pi\hbar)^3} \varepsilon(p) \, n_F(p)$$

Method 2: Square well approximation

Repeating as before $\varepsilon_F = p_F c = (\hbar k_F) c = \hbar\left(\frac{\pi n_F}{L}\right) c = \frac{\pi \hbar c \, r_n}{L}$

as before, the Fermi Momentum is $p_F = \left(\frac{\quad}{8\pi} \right)$ (see (1) method 1)

$$\therefore \quad E_F = \left(\frac{3\hbar^3 n}{8\pi} \right)^{1/3} \cdot c \quad \text{and} \quad E_{tot} = \langle \varepsilon \rangle = g_s \int \frac{d^3x\, d^3p}{(2\pi\hbar)^3}\, \varepsilon(p)\, n_F(p)$$

- ultrarel. $\varepsilon(p) = pc \Rightarrow d\varepsilon(p) = dp\, c$
- Fermi gas $n_F(p) \to 1$ for cold, degenerate $(T=0)$

$$E_{tot}^{(ultrarel)} = (2) \cdot \left(\frac{V}{\hbar^3} \right) (4\pi) \int_0^{\varepsilon_F} \left(\frac{d\varepsilon}{c} \right) \left(\frac{\varepsilon}{c} \right)^2 \varepsilon \; (1)$$

$$= \frac{8\pi V}{(\hbar c)^3} \frac{\varepsilon_F^4}{4} = \frac{4\pi V}{(\hbar c)^3} \left(\frac{3\hbar^3 n}{8\pi} \right)^{4/3} c^4 = \frac{4\pi V \hbar c}{2^4} \left(\frac{3n}{\pi} \right)^{4/3}$$

$$= \frac{\pi V \hbar c}{4} \left(\frac{3n}{\pi} \right)^{4/3} = \boxed{\frac{\hbar c}{8 V^{1/3}} \left(\frac{3N}{\pi} \right)^{4/3} = E_{tot}^{(ultra)}}$$

Repeating as before $\varepsilon_F = p_F c = (\hbar k_F) c = \hbar \left(\frac{\pi n_F}{L} \right) c = \frac{\pi \hbar c\, r_n}{L}$

$$\therefore \quad E_{tot} = g_s \left(\frac{1}{8} \right) \int_0^{r_n} dr_n' \, (4\pi r_n'^2) \left(\frac{\pi \hbar c\, r_n}{L} \right)$$

same as before

$$= \frac{\pi^2 \hbar c}{L} \left(\frac{r_n^4}{4} \right) = \frac{\pi^2 \hbar c}{4L} \left(\frac{3N}{\pi} \right)^{4/3}$$

notice that the approximation is off by a factor compared to method 1 @ p>>m

$$\boxed{E_{tot}^{(ultra)} = \left(\frac{\pi}{2} \right)^2 \frac{\hbar c}{V^{1/3}} \left(\frac{3N}{\pi} \right)^{4/3}}$$

again, we find the degeneracy pressure using thermo. identity

$$P_{tot} = - \left(\frac{dE_{tot}}{dV} \right)_{S,N}$$

$$= - \frac{\hbar c}{8} \left(\frac{3N}{\pi} \right)^{4/3} \left(-\frac{1}{3} \frac{1}{V^{4/3}} \right)$$

$$\boxed{P_{tot}^{(ultra)} = + \frac{\hbar c}{24} \left(\frac{3N}{\pi V} \right)^{4/3}}$$

Finally, can we find the equilibrium radius?

$$0 = P_G + P_{tot}^{(ultra)} \Rightarrow \quad \frac{4\pi}{15} G \rho^2 R^2 = \frac{\hbar c}{24} \left(\frac{3N}{\pi V} \right)^{4/3}$$

using as before $M = N m_n$ and $\rho = M/V$

$$\frac{4\pi}{15} G M^2 \frac{R^2}{V^2} = \frac{\hbar c}{24} \left(\frac{3N}{\pi} \right)^{4/3} V^{-4/3}$$

$$V^{4/3} \frac{R^2}{V^2} = \frac{15 \hbar c}{4\pi \cdot 24} \left(\frac{3N}{\pi} \right)^{4/3} \frac{1}{GM^2}$$

$$\frac{R^2}{V^{2/3}} = \cdots \Rightarrow \frac{R^2}{\left(\frac{4}{3}\pi R^3 \right)^{2/3}} = \frac{\cancel{R^2}}{\left(\frac{4}{3}\pi \right)^{2/3} \cancel{R^2}} = \frac{15 \hbar c}{4\pi \cdot 24} \left(\frac{3N}{\pi} \right)^{4/3} \frac{1}{GM^2}$$

No equilibrium radius for ultra-rel Fermion gas!

Solving for critical particle number N_c,

$$GM^2 \frac{4\pi \cdot 24}{15 \hbar c} \left(\frac{4}{3}\pi \right)^{-2/3} = \left(\frac{3N}{\pi} \right)^{4/3}$$

$$GM^2 \frac{24}{15} \frac{(4\pi)^{1/3} \cdot 3^{2/3}}{\hbar c} \left(\frac{\pi}{3} \right)^{4/3} = N_c^{4/3}$$

$$\boxed{N_c = \left(\frac{24}{15} \left(\frac{2}{3} \right)^{2/3} \pi^{5/3} \frac{GM^2}{\hbar c} \right)^{3/4}}$$

unitless