Q8 -- anisotropic medium

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8. *Electricity and Magnetism* (Fall 2005)

An anisotropic medium has a tensor conductivity given by

$$\overleftrightarrow{\sigma} = \begin{pmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}$$

where $\sigma_\parallel$ and $\sigma_\perp$ are real and independent of frequency. The symbol $\perp$ refers to the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ direction and the symbol $\parallel$ to the $\hat{\mathbf{z}}$ direction in a Cartesian coordinate system.

(a) Find the dispersion relation $k = k(\omega)$ for an electromagnetic wave with O-mode (ordinary mode) polarization with the k vector along $\hat{\mathbf{x}}$.

(b) Write an expression for the damping decrement $k_I = \text{Im } k$ in the limit of high frequency.

(c) If the amplitude of the electric field is E_0 at $x = 0$, find the time-avaraged power per unit volume delivered to this medium at the location $x > 0$. (No need to write down k_I explicitly.)

(a) Find dispersion relation $k(\omega)$

- Ordinary mode $\leftrightarrow$ linearly polarized
- $\vec{k} = k\hat{x}$ (ordinary mode, $\vec{E}\cdot\hat{k} = 0$ and $\perp$ to optical axis) $\Rightarrow \vec{E} = \hat{y} E_0 e^{i(\underbrace{\vec{k}\cdot\vec{r}}_{kx} - \omega t)}$
- We are only told that the medium is anisotropic in the conductivity tensor, so we can assume that μ and ε are constant

First starting w/ ME in medium

$\vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = 0$

$\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\varepsilon\vec{E}) = \varepsilon\vec{\nabla}\cdot\vec{E} = 0$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\mu\vec{H})$$

$$= -\mu\frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial\vec{D}}{\partial t}\right)$$

$\vec{J}_f = \overleftrightarrow{\sigma}\cdot\vec{E}$

$$= -\mu\left(\frac{\partial\vec{J}_f}{\partial t} + \varepsilon\frac{\partial^2\vec{E}}{\partial t^2}\right)$$

Finding $\vec{\nabla}\times\vec{\nabla}\times\vec{E}$:

$\frac{\partial\vec{J}_f}{\partial t} = \overleftrightarrow{\sigma}\cdot\frac{\partial\vec{E}}{\partial t}$

$$= -\mu\left(\overleftrightarrow{\sigma} + \varepsilon\frac{\partial}{\partial t}\right)\frac{\partial\vec{E}}{\partial t} = -\mu\left(\overleftrightarrow{\sigma} + \varepsilon(-i\omega)\right)(-i\omega\vec{E})$$

(1) $\vec{\nabla}\times\vec{E} = \varepsilon_{ijk}(\nabla_i E_j)_k$

$j = y$ (only comp of $\vec{E}$)

$= \varepsilon_{xyz}(\nabla_x E_y)_z = (ik)E_y(x)\,\hat{z}$)

$$= -\mu\left(-i\omega\overleftrightarrow{\sigma} - \omega^2\varepsilon\mathbb{1}\right)\vec{E}$$

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$\nabla \times E = \epsilon_{ijk}(\nabla_i E_j)_k$

$j=y$ (only comp of $\vec{E}$)

$$= \epsilon_{xyz}(\nabla_x E_y)_z = (ik)E_y(x)\,\hat{z}$$

$$\nabla \times (\nabla \times E) = \epsilon_{ijk}\left[\nabla_i\left(ik\,E_y(x)\right)_j\right]_k$$

$j=z$

$$= \cancel{\epsilon_{xzy}}^{-1}\,(ik)\left[(ik)\,E_z(x)\right]_y$$

$$= k^2 \vec{E}$$

(2) or using BAC-CAB:

$$= \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{E}}^{\,0}) - (\vec{\nabla}^2)\vec{E}$$

$$= -(ik)(ik)\,\vec{E} = k^2\vec{E} \quad \text{(much easier)}$$

$$= -\mu\left(-i\omega\overleftrightarrow{\sigma} - \omega^2\epsilon\mathbb{1}\right)\vec{E}$$

$$\Rightarrow\quad k^2\vec{E} = \left[i\mu\omega\overleftrightarrow{\sigma} + \omega^2\mu\epsilon\,\mathbb{1}\right]\vec{E}$$

$$(k^2 - \omega^2\mu\epsilon)\vec{E} = i\mu\omega\overleftrightarrow{\sigma}\cdot\vec{E}$$

for O-mode polarization: $(k^2 - \omega^2\mu\epsilon)E_y = i\mu\omega\,\sigma_\perp E_y$

$$\boxed{k^2(\omega) = \mu\epsilon\omega^2\left(1 + \frac{i\sigma_\perp}{\epsilon\omega}\right)}$$

(b) $n = c\sqrt{\mu\epsilon}$

$$\lim_{\frac{\omega\epsilon}{\sigma_\perp}\gg 1} k(\omega) = \sqrt{\mu\epsilon}\,\omega\left(1 + \frac{1}{2}\frac{i\sigma_\perp}{\epsilon\omega}\right)$$

$$\boxed{k_2 = \mathrm{Im}(k) = \left(\frac{n}{c}\right)\frac{\sigma_\perp}{2\epsilon} \quad @\ \text{high freq.}}$$

(c) Find time-averaged power/volume @ $x>0$

Want power density delivered to volume for $x>0$

$$-\vec{\nabla}\cdot\vec{S} = \frac{\partial}{\partial t}(u_{em} + u_{mech})$$

$$-\vec{\nabla}\cdot\vec{S} - \frac{\partial u_{em}}{\partial t} = \bar{P} = \vec{E}\cdot\vec{J}$$

rate of energy density transport

u_{em}: EM field energy density;

power density delivered to medium $\left(\vec{E}\cdot\vec{J} = \frac{\partial u_{mech}}{\partial t}\right)$

$-\frac{\partial u_{em}}{\partial t}$: absorbed by medium from EM fields (positive would be taken to be stored in fields)

- $\vec{S} = \vec{E}\times\vec{H}$ (power per unit area; energy density flow)
- Time-averaging:

$$\langle\bar{P}(x>0)\rangle_t = \langle\vec{E}\cdot\vec{J}(x>0)\rangle_t = -\vec{\nabla}\cdot\langle\vec{S}\rangle_t - \left\langle\frac{\partial u_{em}}{\partial t}\right\rangle_t$$

power density $\bar{P} = P/V$

Method 1: Calc LHS $\langle\vec{E}\cdot\vec{J}(x>0)\rangle_t$

← power density $p = {}^{1}/_{V}$

Method 1: Calc LHS $\langle \vec{E}\cdot\vec{J}(x>0)\rangle_t$

$$\langle \vec{E}\cdot\vec{J}\rangle_t = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\ \vec{E}\cdot\vec{J} = \frac{1}{2}\,\mathrm{Re}\left[\vec{E}_0\cdot\vec{J}_0^*\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\left[E_0 e^{ikx}\cdot\left(\overleftrightarrow{\sigma}\cdot\vec{E}_0\right)^*\right]$$

$$\hat{E}_0 = \hat{E}_0^* = \hat{z}$$

$$= \frac{1}{2}E_0^2\,\mathrm{Re}\left[e^{-k_I x}\,\cancel{e^{in\omega/c}}\ \sigma_\perp\, e^{-k_I x}\,\cancel{e^{-in\omega/c}}\,\cancel{(\hat{E}_0\cdot\hat{E}_0^*)}\right]$$

$$= \frac{\sigma_\perp}{2}E_0^2 e^{-2k_I x} \Rightarrow \boxed{\langle \bar{P}\rangle_t = \left(\frac{c k_I}{n}\right)\varepsilon E_0^2 e^{-2k_I x}}$$

Method 2: Calc RHS

Using time-average theorem for complex vector fields

$$\langle \vec{S}\rangle_t = \frac{1}{2}\,\mathrm{Re}\left[\vec{E}_0\times\vec{H}_0^*\right] \qquad \vec{B} = \mu\vec{H} = \sqrt{\mu\varepsilon}\,\hat{k}\times\vec{E}$$

$$\vec{H} = \sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\vec{E}\times\left\{\sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}\right\}^*\right]$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,\mathrm{Re}\left[\hat{k}(\vec{E}\cdot\vec{E}^*) - \cancelto{0}{(\vec{E}\cdot\hat{k})}\,\vec{E}^*\right]$$

$$= \hat{x}\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,\mathrm{Re}\left[E_0^2 e^{ikx} e^{-ik^*x}\right] = \hat{x}\frac{1}{2}\sqrt{\varepsilon/\mu}\,E_0^2 e^{-2k_I x} \Rightarrow \vec{\nabla}\cdot\langle S\rangle_t = -k_I\sqrt{\varepsilon/\mu}\,E_0^2 e^{-2k_I x}$$

$$k = n\frac{\omega}{c} + ik_I$$

$$\left\langle\frac{\partial u_{em}}{\partial t}\right\rangle_t = \frac{1}{T}\int_0^T\left(\frac{\partial u_{em}}{\partial t}\right)dt = \left(\frac{\omega}{2\pi}\right)u_{em}\Big|_0^{2\pi/\omega} \quad w/\ \omega = 2\pi/T$$

$$u_{em} = \frac{1}{2}\varepsilon|\vec{E}|^2 + \frac{1}{2}\mu|\vec{H}|^2$$

$$= \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x} + \frac{1}{2}\mu\left|\sqrt{\varepsilon/\mu}\,\hat{k}\times\vec{E}\right|^2$$

$$= \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x} + \frac{1}{2}\varepsilon E_0^2 e^{-2k_I x}\left|\hat{k}\times\hat{z}\right|^2$$

$$u_{em} = \varepsilon E_0^2 e^{-2k_I x}$$

$$\left\langle\frac{\partial u_{em}}{\partial t}\right\rangle_t = \frac{\omega}{2\pi}\varepsilon E_0^2 e^{-2k_I x}\Big|_0^{2\pi/\omega} = 0$$

$$\langle \bar{P}(x>0)\rangle = -\vec{\nabla}\cdot\langle\vec{S}\rangle_t = k_I\sqrt{\frac{\varepsilon}{\mu}}E_0^2 e^{-2k_I x}$$

$$\boxed{\langle \bar{P}(x>0)\rangle = (c k_I/n)\,\varepsilon E_0^2 e^{-2k_I x}}$$