WRITTEN COMPREHENSIVE EXAMINATION FOR THE MASTER'S DEGREE AND QUALIFYING EXAMINATION FOR THE PH.D. DEGREE
DEPARTMENT OF PHYSICS

Wednesday, September 19, and Thursday, September 20, 2001

PART I - WEDNESDAY, SEPTEMBER 19

Important – please read carefully.

The exam (8 hours) is in two parts:

Part 1 Quantum Mechanics, Thermodynamics, Statistical Mechanics

September 19 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

PART 2 Electromagnetic Theory,Thermodynamics,Statistical Mechanics

September 20 7 Problems – **DO ALL PROBLEMS.**

9:00–1:00 This part will be collected at the end of four hours.
Each problem counts for 20 points; the total is 140.

Instructions

1) This is a closed book exam and calculators are not be used.

2) Work each problem on a separate sheet of paper. Use one side only.

3) Print your name and problem number on EACH **AND EVERY** page. (Note: Pages without names may not be counted.)

4) Return the problem page as the first page of your answers.

5) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

1. *Quantum Mechanics.*

Consider a simple harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

which is in its n^{th} energy eigenstate, $H|n\rangle = (n+1/2)\hbar\omega|n\rangle$.

(a) What is $\langle n|p^2|n\rangle$?

(b) What is $\langle n|x^2|n\rangle$?

2. *Quantum Mechanics.*

The Hamiltonian for a system consisting of three distinguishable spin one particles is

$$H = A\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1\right)$$

where $\mathbf{S}_i$ is the spin of the ith particle, and all the components of the spin of one particle commute with all the components of the spins of the other two.

(a) How many independent states are there?

(b) What are the eigenvalues of H?

(c) What are the degeneracies of each energy level?

3. *Quantum Mechanics.*

The electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$ are neutrino states produced and detected in experiments, but recent experiments suggest that they are not eigenstates of the total Hamiltonian. Rather, if the state is known to have momentum $\mathbf{p}$, it is some linear combination of the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ of the form

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \end{aligned}$$

where

$$H|\nu_1\rangle = \sqrt{p^2c^2 + m_1^2c^4}\,|\nu_1\rangle$$
$$H|\nu_2\rangle = \sqrt{p^2c^2 + m_2^2c^4}\,|\nu_2\rangle$$

for two possibly different masses m_1 and m_2, and some angle θ.

Neutrinos produced by nuclear reactions in the sun are definitely of the type $|\nu_e\rangle$. For each electron neutrino produced in the sun, what is the probability of detecting it as a ν_μ after it has traveled a distance L to the earth? Assume that $m_1c \ll p$ and $m_2c \ll p$, so that the neutrinos are moving at almost, or even exactly, the speed of light, so you can ignore corrections of the order $1 - v/c$ compared to terms of order 1. State your result in terms of p and L, and to first order in the difference $\Delta m^2 = m_1^2 - m_2^2$.

> The Sudbury Neutrino Observatory published a paper last June that claimed to observe μ-neutrinos from the sun, in sufficient numbers to explain the thirty year old solar neutrino deficit puzzle. The experiment also put new limits on the mass difference between neutrino eigenstates.

4. *Quantum Mechanics.*

A particle moves in a potential which looks like the harmonic oscillator for positive x, but is infinite for negative x, so that the wave function must vanish for $x \leq 0$:

$$V(x) = \frac{m\omega^2 x^2}{2} \qquad (x > 0) \qquad \text{and} \qquad V(x) = \infty \qquad (x \leq 0)$$

(a) Estimate the energy using the variational method with a trial function of the form

$$\psi(x) = Nxe^{-\mu x} \qquad (x > 0) \qquad \text{and} \qquad \psi(x) = 0 \qquad (x \leq 0)$$

(b) What is the exact energy of the ground state of this system?

Hint: You may need the integral

$$\int_0^\infty r^n e^{-r} dr = n!$$

5. *Quantum Mechanics.*

For any spherically symmetric potential $V(r)$, the radial wave function is a solution to the integral equation

$$R_l(r) = j_l(kr) - 2mik \int_0^\infty j_l(kr_<)h_l(kr_>)V(r')R_l(r')r'^2 dr'$$

where $j_l(\rho)$ are the spherical Bessel functions, $n_l(\rho)$ are the spherical Neumann functions, k is the radial wavevector, m is the momentum, $h_l(\rho) = j_l(\rho) + in_l(\rho)$, and the notation means

$$j_l(kr_<)h_l(kr_>) = j_l(kr)h_l(kr')\Theta(r' - r) + j_l(kr')h_l(kr)\Theta(r - r')$$

where $\Theta(x)$ is the step function.

(a) Write the formula for the partial-wave scattering amplitude, defined as

$$f_l(k) = \frac{e^{i\delta_l} \sin \delta_l}{k}$$

for a particle scattering off the potential $V(r)$, in terms of the radial wave function, the spherical Bessel functions, etc. δ_l are the phase shifts.

(b) Suppose $V(r)$ is an attractive delta-shell potential

$$V(r) = -V_o a\delta(r - a)$$

with $V_o > 0$. Find a closed expression for $f_l(k)$ *algebraically* for any l.

(c) What is the cross section at zero incident momentum (i.e. at "threshold")?

6. *Statistical Mechanics and Thermodynamics*

Suppose that the energy-versus-momentum relation for a collection of noninteracting, conserved Bosons were

$$E(\vec{p}) = \mathcal{A}|\vec{p}|^4$$

(a) Find the lowest (integer) spatial dimension d_{lc} for which this system of Bosons will undergo Bose-Einstein condensation.

(b) Making use of the identity

$$\int f(|\vec{p}|)d^dp = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int f(p)p^{d-1}dp$$

Determine the temperature at which Bose-Einstein condensation takes place for this set of Bosons in dimensions $d > d_{lc}$. Assume that their spin is equal to zero.

7. *Statistical Mechanics and Thermodynamics*

Consider a set of particles obeying Boltzmann statistics in which the total energy of a single particle is given by

$$E = \frac{p^2}{2m} + E_{\text{internal}}$$

where $\vec{p}$ is the particle's momentum in three dimensions, m is its mass and E_{internal} is the particle's "internal" energy. Here,

$$E_{\text{internal}} = 0, \epsilon, 2\epsilon, 3\epsilon, \ldots$$

where ϵ is a constant energy. The particles do not interact with each other.

(a) Write down an expression for the partition function of this system. From this expression obtain an expression for the system's Helmboltz free energy.

(b) What is the heat capacity at constant pressure of this system as a function of temperature? What is the limit of this expression at temperatures $T \gg \epsilon/k_B$?

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PART II - THURSDAY, SEPTEMBER 20

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PART 2 Electromagnetic Theory, Thermodynamics, Statistical Mechanics

September 20 7 Problems – **DO ALL PROBLEMS.**

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8) If a part of any question seems ambiguous to you, state clearly what your interpretations and answer the question accordingly.

8. *Electricity and Magnetism*

Consider an infinitely long, thin rod of charge density λ which lies along the y axis.

(a) What is the electric field due to the rod.

(b) Now suppose that the rod moves in the y-direction with velocity v. What are the electric and magnetic fields due to the rod? Do not assume that v is small compared to the speed of light c.

9. *Electricity and Magnetism*

Design an experiment to measure the energy density of electromagnetic radiation in the "FM radio" band (88–108 MHz). You may use some or all of the following (but are not limited to): antenna, oscilloscope, amplifier, filter, transmitter. Be specific about how you would turn the measured quantities into the final number. Exact numerical calculations for all the steps are not necessary, but give values valid to within an order-of-magnitude. What value would you expect to measure in Los Angeles?

10. *Electricity and Magnetism*

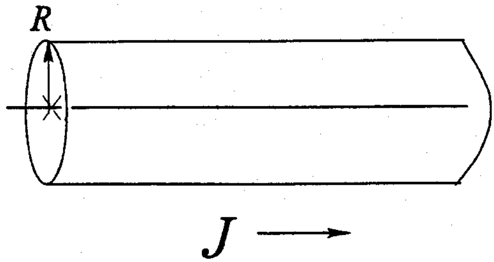
Two non-relativistic particles of mass m_1 and mass m_2 and charges q_1 and q_2, respectively collide and scatter in their center of mass frame from an initial velocity $\vec{v}$ to a final velocity $\vec{v}'$.

(a) The electric dipole radiation vanishes if $q_1/m_1 = q_2/m_2$. Give a simple physical explanation of why this is so.

(b) For general masses and charges, compute the energy spectrum (per unit frequency per unit solid angle) $d^2E/d\Omega d\omega$ in the dipole approximation, ignoring the back-reaction of the radiation.

11. *Electricity and Magnetism*

An infinite column of a non-viscous conducting fluid carries a constant current density $\vec{J}$ and is confined to a radius R by the magnetic field that it induces. (Ignore gravity and electrostatic charge build-up.)

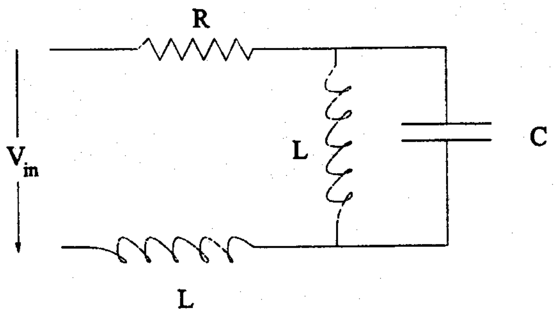


(a) What is the electro-magnetic force density experienced by the fluid?

(b) In a steady state situation what is the required pressure profile of the fluid as a function of the radial distance r? That is, explicitly compute $p(r)$.

12. *Electricity and Magnetism*

In the ac circuit shown below, the input voltage V_{in} and the circuit elements R, L, C are known quantities.



(a) Find the frequency $\omega = \omega_{res}$ at which the input impedance $Z_{in} = V_{in}/I_{in}$ is real.

(b) At this frequency, ω_{res}, what is the time-averaged power dissipated in the circuit, P_{dissip}?

(c) At $\omega = \omega_{res}$, what is the stored energy, U_{stored}?

(d) Find the quality factor $Q = \omega_{res} U_{stored}/P_{dissip}$?

13. *Statistical Mechanics and Thermodynamics*

An ideal gas, enclosed in an insulated (upright) cylinder with a piston at the top, is at equilibrium with conditions p_1, V_1, T_1. A weight is placed on the piston. After some oscillations, the motion subsides (note that this is *not* a reversible process) and the gas reaches a new equilibrium at conditions p_2, V_2, T_2.

(a) Find the temperature ratio T_1/T_2 in trms of the pressure ratio $\lambda = p_2/p_1$.

(b) Find the entropy change.

(c) If $\lambda = 1 + \epsilon$, with $\epsilon \ll 1$, show that the entropy change is of second order in ϵ.

Suppose $n(R)$ is the concentration of air molecules at the surface of the Earth, R is the radius of the Earth, M is the mass of the molecules (assume that the atmosphere is made up of a single species of molecule), and g is the acceleration due to the gravitational attraction of the Earth at its surface. Making the simplifying assumption of a constant temperature throughout the whole atmosphere, show that the total number of molecules in the atmosphere is

$$N = 4\pi n\, e^{-\frac{MgR}{k_B T}} \int_R^\infty dr\, r^2\, e^{\frac{MgR^2}{k_B T r}}$$

with r measured from the center of the Earth.

Fall 2001 #1

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad \hbar = 1 \quad \Rightarrow \quad H|n\rangle = \underbrace{(n+1/2)\omega}_{E_n}|n\rangle$$

$$x = \sqrt{\frac{1}{2m\omega}}\,(a+a^+) \qquad p = \frac{\sqrt{2\omega m}}{2i}\,(a-a^+)$$

a) $\langle n|p^2|n\rangle = -\frac{\omega m}{2}\langle n|(a-a^+)^2|n\rangle = -\frac{\omega m}{2}\langle n|aa + a^+a^+ - aa^+ - a^+a|n\rangle$

$H = \frac{\omega}{2}(a^+a + aa^+)$

$= \frac{\omega m}{2}\langle n|a^+a + aa^+|n\rangle = \frac{\omega m}{2}\frac{2}{\omega}\langle n|H|n\rangle = m\langle n|H|n\rangle$

$= m(n+1/2)\omega = m\omega(n+1/2)$

b) $\langle n|x^2|n\rangle = \frac{1}{2m\omega}\langle n|(a+a^+)^2|n\rangle = \frac{1}{2m\omega}\langle n|\cancel{aa} + aa^+ + a^+a + \cancel{a^+a^+}|n\rangle$

would make $\langle n|n-2\rangle$

$\langle n|n+2\rangle$

$H = \frac{1}{2}\omega(a^+a + aa^+)$

$= \frac{1}{2m\omega}\langle n|aa^+ + a^+a|n\rangle = \frac{2}{2m\omega^2}\langle n|H|n\rangle = \frac{1}{m\omega^2}(n+1/2)\omega$

$$= \frac{(n+1/2)}{m\omega}$$

F'01 #1; Q.M.

Consider a simple harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

which is in the nth energy eigenstate, $H|n\rangle = (n+1/2)\hbar\omega|n\rangle$.

(a) what is $\langle n|p^2|n\rangle$?

(b) what is $\langle n|x^2|n\rangle$?

(a) From the Virial theorem $\langle \frac{p^2}{2m}\rangle = \frac{\langle H\rangle}{2}$

so $\langle p^2\rangle = \cancel{2}m \frac{\langle H\rangle}{\cancel{2}} = m(n+1/2)\hbar\omega = \boxed{(n+1/2)\hbar\omega m}$

(b) From the Virial theorem $\langle \frac{1}{2} m\omega^2 x^2\rangle = \frac{\langle H\rangle}{2}$

so $\langle x^2\rangle = \frac{\cancel{2}}{m\omega^2} \frac{\langle H\rangle}{\cancel{2}} = (n+1/2)\frac{\hbar\cancel{\omega}}{m\omega^{\cancel{2}}} = \boxed{(n+1/2)\frac{\hbar}{m\omega}}$

Proof of above relations:

Virial theorem in 1-D: $2\langle T\rangle = \langle x\frac{\partial}{\partial x}V\rangle$

- $\langle T\rangle = \langle \frac{p^2}{2m}\rangle$; the kinetic energy term
- $\langle x\frac{\partial}{\partial x}V\rangle$: $V = \frac{1}{2}m\omega^2x^2 \Rightarrow \frac{\partial}{\partial x}V = \frac{1}{2}m\omega^2(2x) = m\omega^2 x$

so $\langle x\frac{\partial}{\partial x}V\rangle = \langle \underbrace{m\omega^2x^2}_{2\cdot\underbrace{\frac{1}{2}m\omega^2x^2}_{=V}}\rangle = 2\langle V\rangle$

hence $\cancel{2}\langle T\rangle = \cancel{2}\langle V\rangle \Rightarrow \langle T\rangle = \langle V\rangle$

now

$$\underset{\substack{\uparrow\\ \text{by definition}}}{\langle H\rangle =} \langle T\rangle + \langle V\rangle = 2\langle T\rangle \Rightarrow \langle T\rangle = \frac{\langle H\rangle}{2} \text{ and } \langle V\rangle = \frac{\langle H\rangle}{2}$$

Problem #1 Fall 2001

b) $E = \langle a_0 | H | a_0 \rangle$

$$= \frac{1}{2m} \langle a_0 | P^2 | a_0 \rangle + \frac{m\omega^2}{2} \langle a_0 | x^2 | a_0 \rangle$$

$$\hat{P}_x = -i \sqrt{\frac{m\omega\hbar}{2}} (\hat{a} - \hat{a}^{+})$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^{+})$$

$$P^2 = \frac{m\omega\hbar}{2} \quad \hat{a}\hat{a} - \hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a} + \hat{a}^{+}\hat{a}^{+}$$

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1) $H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2$

$$\langle H \rangle = E = \frac{1}{2m} \langle P^2 \rangle + \frac{1}{2} m\omega^2 \langle x \rangle$$

$$\langle H \rangle = n\hbar\omega + \frac{\hbar\omega}{2}$$

$P \propto (a - a^{+}) \quad P^2 \propto aa - aa^{+} - a^{+}a + a^{+}a^{+}$

$x \propto (a + a^{+}) \quad x^2 \propto aa + aa^{+} + a^{+}a + a^{+}a^{+}$

$$\langle n|P^2|n\rangle \propto \langle n|\sqrt{n}\sqrt{n-1}\,|n-2\rangle^{\,0} - \langle n|\sqrt{n+1}\sqrt{n+1}\,|n\rangle$$

$$-\langle n|\sqrt{n}\sqrt{n}\,|n\rangle + 0$$

$$\propto -(n+1)-n \quad \propto -2n-1$$

$$\langle n|x^2|n\rangle \propto \quad 2n+1$$

a) $\langle n|p^2|n\rangle = \dfrac{m\omega\hbar}{2\cdot 2M}(2n+1)$

$$= \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

b) $\langle n|x^2|n\rangle = \dfrac{\hbar}{2\cancel{m\omega}}\dfrac{m\omega^2}{2}(2n+1)$

$$= \frac{\hbar\omega}{2}\left(n+\frac{1}{2}\right)$$

The Hamiltonian for a system consisting of three distinguishable spin one particles is

$$H = A(\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1)$$

where $\vec{S}_i$ is the spin of the ith particle, and all the components of the spin of one particle commute with all the components of the spins of the other two.

(a) How many independent states are there?

$$\underset{\substack{\uparrow\\3}}{1}\otimes\underset{\substack{\uparrow\\3}}{1}\otimes\underset{\substack{\uparrow\\3}}{1} = 1\otimes(2\oplus1\oplus0) = 1\otimes2\oplus1\otimes1\oplus1\otimes0$$

3 states

so $3\cdot3\cdot3 = 27$ states

$$= \underbrace{3\oplus2\oplus1}_{2\otimes1}\oplus\underbrace{2\oplus1\oplus0}_{1\otimes1}\oplus\underset{\substack{\uparrow\\1\otimes0}}{1}$$

$$= \underset{\substack{\uparrow\\7}}{3}\oplus\underset{\substack{\uparrow\\5}}{2}\oplus\underset{\substack{\uparrow\\5}}{2}\oplus\underset{\uparrow}{1}\oplus\underset{\substack{\uparrow\\3}}{1}\oplus\underset{\uparrow}{1}+\underset{\substack{\uparrow\\1}}{0}$$

states

or 27 states (just a check)

(b) what are the eigenvalues of H?

$$\vec{S} = \vec{S}_1+\vec{S}_2+S_3 \to \vec{S}^2 = (\vec{S}_1+\vec{S}_2+\vec{S}_3)^2 = \vec{S}_1^2+\vec{S}_2^2+\vec{S}_3^2+2(\vec{S}_1\cdot\vec{S}_2+\vec{S}_1\cdot\vec{S}_3+\vec{S}_2\cdot\vec{S}_3)$$

or $\vec{S}_1\cdot\vec{S}_2 + \vec{S}_2\cdot\vec{S}_3 + \vec{S}_3\cdot\vec{S}_1 = \frac{1}{2}(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 - \vec{S}_3^2)$

now $\vec{S}_i^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$ but as we are dealing with spin 1 particles $s=1 \Rightarrow \vec{S}_i^2 = 2\hbar^2$

Hence

$$H = \frac{A}{2}(\vec{S}^2 - 6\hbar^2)$$

2/2

Now when total s is = 3 $\quad H = \frac{A}{2}\left(\hbar^2\, 3(3+1) - 6\hbar^2\right) = 3A\hbar^2$

$= 2 \quad H = \frac{A}{2}\left(\hbar^2\, 2(2+1) - 6\hbar^2\right) = 0$

$= 1 \quad H = \frac{A}{2}\left(\hbar^2\, 1(1+1) - 6\hbar^2\right) = -2A\hbar^2$

$= 0 \quad H = \frac{A}{2}\left(\hbar^2\, 0(0+1) - 6\hbar^2\right) = -3A\hbar^2$

(c) What are the degeneracies of each energy level?

$3A\hbar^2$	7× degeneracy
0	10×
$-2A\hbar^2$	9×
$-3A\hbar^2$	1×

Problem #2 Fall 2001

$$H = A\left(\hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_1\right)$$

a) $1 \otimes 1 \otimes 1$

$$\underbrace{\begin{array}{ccc} 111 & & 100 \\ 110 & & 010 \\ 101 & & 100 \\ 011 & & 000 \end{array}}_{\times 2} \quad + \quad \underbrace{\begin{array}{ccc} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{array}}_{\times 2} \quad + \quad \underbrace{\begin{array}{ccc} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{array}}_{\times 2}$$

not multiplied by 2

$\boxed{27 \text{ states}}$ = 3 particles with 3 states

$$= 3^3 = (\#\text{ of states})^{(\#\text{ of particles})}$$

b) $|s_1 m_1, s_2 m_2, s_3 m_3\rangle = |m_1, m_2, m_3\rangle$

$$\hat{S}_1 \cdot \hat{S}_2 = \hat{S}_{1z}\hat{S}_{2z} + \frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)$$

$$H = A\Big(\hat{S}_{1z}\hat{S}_{2z} + \frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right) + \hat{S}_{2z}\hat{S}_{3z} + \frac{1}{2}\left(\hat{S}_{2+}\hat{S}_{3-} + \hat{S}_{2-}S_{3+}\right)$$
$$+ \hat{S}_{3z}\hat{S}_{1z} + \frac{1}{2}\left(\hat{S}_{3+}\hat{S}_{1-} + \hat{S}_{3-}\hat{S}_{1+}\right)\Big)$$

$$\hat{S}^2 = (\hat{S}_1+\hat{S}_2+S_3)(\hat{S}_1+\hat{S}_2+\hat{S}_3)$$

$$= \hat{S}_1^2+\hat{S}_2^2+\hat{S}_3^2+2\hat{S}_1\cdot\hat{S}_2+2\hat{S}_1\cdot\hat{S}_3+2\hat{S}_2\cdot\hat{S}_3$$

$$H = \frac{A}{2}\left(\hat{S}^2-(\hat{S}_1^2+\hat{S}_2^2+\hat{S}_3^2)\right)$$

$$H|\rangle = \frac{A}{2}\left\{s(s+1)\hbar^2-\left[s_1(s_1+1)\hbar^2+s_2(s_2+1)\hbar^2+s_3(s_3+1)\hbar^2\right]\right\}|\rangle$$

$$E_3 = \frac{A}{2}\left\{12\hbar^2-6\hbar^2\right\} = 3A\hbar^2$$

$|1,1,1\rangle$, $|-1,-1,-1\rangle$

$$E_2 = \frac{A}{2}\left\{6\hbar^2-6\hbar^2\right\} = 0$$

$|1,0,1\rangle$, $|1,1,0\rangle$
$|0,1,1\rangle$, $|-1,0,-1\rangle$
$|-1,-1,0\rangle$, $|0,-1,-1\rangle$

$$E_1 = \frac{A}{2}\left\{2\hbar^2-6\hbar^2\right\} = -2A\hbar^2$$

$|0,0,1\rangle$, $|0,1,0\rangle$
$|1,0,0\rangle$, $|-1,0,0\rangle$
$|0,0,-1\rangle$, $|0,-1,0\rangle$
$|1,-1,1\rangle$, $|-1,1,1\rangle$
$|1,1,-1\rangle$, $|1,-1,-1\rangle$
$|-1,1,-1\rangle$, $|-1,-1,1\rangle$

$$E_0 = \frac{A}{2}\left\{0-6\hbar^2\right\} = -3A\hbar^2$$

$|0,0,0\rangle$, $|0,1,-1\rangle$
$|1,0,-1\rangle$, $|0,-1,+1\rangle$
$|-1,0,1\rangle$, $|1,-1,0\rangle$
$|-1,1,0\rangle$

c) $E_3 \longrightarrow$ 2-fold degeneracy

$E_2 \longrightarrow$ 6-fold degeneracy

$E_1 \longrightarrow$ 12-fold degeneracy

$E_0 \longrightarrow$ 7-fold degeneracy

Problem # 3 Fall 2001

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

$$H|\nu_1\rangle = E_1|\nu_1\rangle$$

$$H|\nu_2\rangle = E_2|\nu_2\rangle$$

$$|\psi_0\rangle = |\nu_e\rangle$$

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\nu_e\rangle$$

$$= \cos\theta\, e^{-iE_1t/\hbar}|\nu_1\rangle + \sin\theta\, e^{-iE_2t/\hbar}|\nu_2\rangle$$

$$\langle\nu_\mu|\psi(t)\rangle = -\cos\theta\sin\theta\, e^{-i\sqrt{p^2c^2+m_1^2c^4}\,t/\hbar}\langle\nu_1|\nu_1\rangle$$

$$+ \cos\theta\sin\theta\, e^{-i\sqrt{p^2c^2+m_2^2c^4}\,t/\hbar}\langle\nu_2|\nu_2\rangle$$

in the limit $ct = L$

$$\langle\nu_\mu|\psi(t)\rangle = \cos\theta\sin\theta\left(-e^{-\frac{iL}{\hbar}\sqrt{p^2+m_1^2c^2}} + e^{-\frac{iL}{\hbar}\sqrt{p^2+m_2^2c^2}}\right)$$

$$= \cos\theta\sin\theta\left(-e^{\frac{-iLp}{\hbar}\sqrt{1+\frac{m_1^2c^2}{p^2}}} + e^{\frac{-iLp}{\hbar}\sqrt{1+\frac{m_2^2c^2}{p^2}}}\right)$$

$$\sqrt{1+\frac{m_1^2c^2}{p^2}} \approx 1 + \frac{m_1^2c^2}{2p^2} + \cdots$$

$$\langle \nu_e | \psi \rangle = \cos\theta\sin\theta\, e^{\frac{-iLp}{\hbar}}\left(-e^{\frac{iLp}{\hbar}\frac{m_1^2c^2}{2p^2}} + e^{\frac{-iLp}{\hbar}\frac{m_2^2c^2}{2p^2}}\right)$$

$$P(t) = \cos^2\theta\sin^2\theta\left[-e^{\frac{-iLp}{\hbar}\frac{c^2}{2p^2}(m_1^2-m_2^2)} - e^{-\frac{iLp}{\hbar}\frac{c^2}{2p^2}(m_2^2-m_1^2)} + 2\right]$$

$$= +\cos^2\theta\sin^2\theta\left[-e^{\frac{iLp}{\hbar}\frac{c^2}{2p^2}\Delta m^2} - e^{-\frac{iLp}{\hbar}\frac{c^2}{2p^2}\Delta m^2} + 2\right]$$

$$= +\cos^2\theta\sin^2\theta\left[2 - 2\cos\left(\frac{Lc^2\Delta m^2}{2p\hbar}\right)\right]$$

$$\boxed{P(t) = +\sin^2(2\theta)\sin^2\left(\frac{Lc^2\Delta m^2}{4p\hbar}\right)}$$

Problem #4 Fall 2001 Comp

$$V(x) = \begin{cases} \frac{m\omega^2}{2}x^2 , & x > 0 \\ \infty , & x \leq 0 \end{cases}$$

a) $$\psi(x) = \begin{cases} Nxe^{-\mu x} , & x > 0 \\ 0 , & x \leq 0 \end{cases}$$

$$1 = |N|^2 \int_0^\infty x^2 e^{-2\mu x} dx = |N|^2 \frac{2}{(2\mu)^3}$$

$|N|^2 = 4\mu^3$ because $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

$E = \langle \psi | H | \psi \rangle \qquad H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad (x>0)$

for $x>0$

$$E = -\frac{\hbar^2}{2m} \underbrace{\langle \psi | \frac{d^2}{dx^2} | \psi \rangle}_{①} + \frac{m\omega^2}{2} \underbrace{\langle \psi | x^2 | \psi \rangle}_{②}$$

① $$\int_0^\infty x e^{-\mu x} \frac{d^2}{dx^2} \left(x e^{-\mu x} \right) dx$$

$$= \int_0^\infty x e^{-\mu x} \frac{d}{dx}\left\{ e^{-\mu x} - \mu x e^{-\mu x} \right\} dx$$

$$= \int_0^\infty x e^{-\mu x} \left\{ -\mu e^{-\mu x} + \mu^2 x e^{-\mu x} - \mu e^{-\mu x} \right\} dx$$

$$= \int_0^\infty \left\{ -2\mu x e^{-2\mu x} + \mu^2 x^2 e^{-2\mu x} \right\} dx$$

$$= -2\mu \left(\frac{1}{(2\mu)^2} \right) + \mu^2 \left(\frac{2}{(2\mu)^3} \right) = \frac{-1}{2\mu} + \frac{1}{4\mu} = \frac{-1}{4\mu}$$

(2) $$\int_0^\infty x^4 e^{-2\mu x} dx = \frac{24}{(2\mu)^5}$$

$$\Rightarrow E = 4\mu^3 \left\{ \frac{\hbar^2}{8m\mu} + \frac{3 \cdot 2^3}{2^5 \mu^5} \frac{m\omega^2}{2} \right\}$$

$$\boxed{E = \left\{ \frac{\hbar^2 \mu^2}{2m} + \frac{3m\omega^2}{2\mu^2} \right\} = \frac{1}{2} \left\{ \frac{\hbar^2 \mu^2}{m} + \frac{3m\omega^2}{\mu^2} \right\}}$$

Now to minimize

$$\frac{\partial E}{\partial \mu} = 0 = \frac{\hbar^2 \mu}{m} - \frac{3m\omega^2}{\mu^3} = 0$$

$$\mu^4 = \frac{3m^2\omega^2}{\hbar^2} \Rightarrow \mu^2 = \sqrt{3}\, \frac{m\omega}{\hbar}$$

$$\boxed{E = \frac{\sqrt{3}\,\hbar\omega}{2} + \frac{3\hbar\omega}{2\sqrt{3}} = \sqrt{3}\,\hbar\omega}$$

b) For the exact solution

$E = \hbar\omega\left(n + \frac{1}{2}\right)$ for n odd because their wavefunctions go to zero at $x = 0$ and can therefore meet boundary conditions

$$\boxed{E_0 = \frac{3\hbar\omega}{2}}$$

Fall... 2001 #5 also look at Abers hw # 8.4

$$R_\ell(r) = j_\ell(kr) - 2mik\int_0^\infty j_\ell(kr_<)h_\ell(kr_>)V(r')R_\ell(r')r'^2\,dr'$$

$$j_\ell(kr_<)h_\ell(kr_>) = j_\ell(kr)h_\ell(kr')\Theta(r'-r) + j_\ell(kr')h_\ell(kr)\Theta(r-r')$$

$\Theta(r-r')$: 1 $r>r'$, 0 $r<r'$

a) $f_\ell(k) = \dfrac{e^{i\delta_\ell}\sin\delta_\ell}{k} = -\int_0^\infty 2mV(r)\,j_\ell(kr)R_\ell(r)r^2\,dr$

Abers 8.100

b) Suppose $V(r) = -V_0\,a\,\delta(r-a)$ $\quad \lambda = 2mV_0a^2$

$\Rightarrow 2mV(r) = -\dfrac{\lambda}{a}\delta(r-a)$

$$R_\ell(r) = j_\ell(kr) + \frac{ik\lambda}{a}\int_0^r j_\ell(kr')h_\ell(kr)\delta(r'-a)R_\ell(r')\,dr'\,r'^2$$
$$+ \frac{ik\lambda}{a}\int_r^\infty j_\ell(kr)h_\ell(kr')\delta(r'-a)R_\ell(r')\,dr'\,r'^2$$

$$R_\ell(r) = j_\ell(kr) + ik\lambda a\,j_\ell(ka)R_\ell(a)h_\ell(kr)\,\Theta(r-a) + i\lambda ka\,j_\ell(kr)R_\ell(a)h_\ell(ka)\,\Theta(a-r)$$

or for

$$R_\ell(a) = j_\ell(ka) + ik\lambda a\,j_\ell(ka)R_\ell(a)h_\ell(ka)$$ (don't double count step functions)

$$R_\ell(a) = \frac{j_\ell(ka)}{1 - ik\lambda a\,j_\ell(ka)h_\ell(ka)}$$

Then

$$\frac{e^{i\delta_e}}{k}\sin\delta_e = -\int_0^\infty j_e(ka)\, 2m\underbrace{V(r)}_{-V_0 a\delta(r-a)} r^2 R_e(r)\, dr$$

$$= +j_e(ka)\, 2ma^3 \cdot V_0\, R_e(a) = a j_e(ka)\lambda R_e(a)$$

$$= \frac{\lambda a\, j_e(ak)^2}{1 - ika\lambda\, j_e(ka)\, h_e(ka)}$$

c) For threshold case $e=0$

$$\frac{e^{i\delta_e}\sin\delta_e}{k} \rightarrow \frac{a\lambda}{1-\lambda}$$

$$h_0(ka) = -i\frac{e^{ika}}{ka} \qquad j_0(ka) = \frac{\sin(ka)}{ka}$$

$$\rightarrow \frac{-i}{ka} \text{ for small } K \qquad j_0(ka) = 1 \text{ small } K$$

$$\rightarrow \frac{\lambda a}{1-\lambda}$$

$$\sigma_{threshold} = 4\pi\left|\frac{e^{i\delta_e}\sin\delta_e}{k}\right|^2$$

$$\sigma_{threshold} = \frac{4\pi a^2\lambda^2}{(\lambda-1)^2}$$

Problem 66. Scattering by a Delta-Shell Potential

Part a) Set $\lambda = 2mV_o a^2$ as in Problem 62. Then $U(r) = -\lambda\delta(r-a)/a$. The radial wave function satisfies

$$R_l(r) = j_l(kr) - ik\int_0^\infty j_l(kr_<)h_l(kr_>)U(r')R_l(r')r'^2dr'$$
$$= j_l(kr) + \frac{ik\lambda}{a}\int_o^r j_l(kr')h_l(kr)\delta(r'-a)R_l(r')r'^2dr' + \frac{ik\lambda}{a}\int_r^\infty j_l(kr)h_l(kr')\delta(r'-a)R_l(r')r'^2dr'$$
$$= j_l(kr) + ik\lambda aR_l(a)j_l(ka)h_l(kr)\Theta(r-a) + j_l(kr) + ik\lambda aR_l(a)j_l(kr)h_l(ka)\Theta(a-r) \tag{S17.16}$$

or

$$R_l(a) = j_l(ka) + ik\lambda aR_l(a)j_l(ka)h_l(ka) \tag{S17.17}$$

The solution is

$$R_l(a) = \frac{j_l(ka)}{1 - ik\lambda aj_l(ka)h_l(ka)} \tag{S17.18}$$

and the partial wave amplitudes are

$$\frac{e^{i\delta_l}\sin\delta_l}{k} = -\int_0^\infty j_l(kr)U(r)R_l(r)r^2dr = \lambda aj_l(ka)R_l(a) = \boxed{\frac{\lambda aj_l(ka)^2}{1 - ika\lambda j_l(ka)h_l(ka)}} \tag{S17.19}$$

Part b) In particular, for $l = 0$,

$$\frac{e^{i\delta_l}\sin\delta_l}{k} \to \frac{a\lambda}{1-\lambda} \tag{S17.20}$$

and the threshold cross section is

$$\boxed{\sigma \to \frac{4\pi a^2\lambda^2}{(\lambda-1)^2}} \tag{S17.21}$$

Part c) If the force is repulsive, $V_o < 0$ and therefore $\lambda < 0$. The scattering length is between 0 and a, so the threshold cross section is between 0 and $4\pi a^2$.

Problem # 6 Fall 2001

non-interacting conserved Bosons

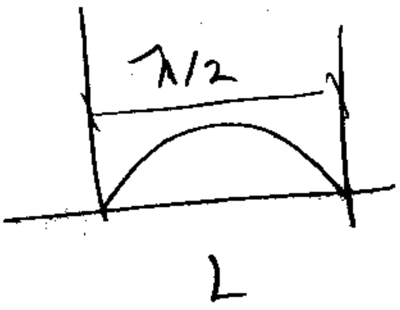
$$E(\vec{p}) = A|\vec{p}|^4 = \epsilon$$

a) $$N = \sum_\epsilon \frac{\bar{n}\, e^{-\beta\epsilon}}{\sum_\epsilon e^{-\beta\epsilon}}$$

for bosons $\bar{n} = \dfrac{1}{e^{\beta\epsilon}-1}$

$$N = \int \bar{n}(\epsilon) D(\epsilon)\, d\epsilon, \quad D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon}$$

$$\epsilon = A p^4 = A(\hbar K)^4$$
$$= A\left(\frac{\hbar n\pi}{L}\right)^4$$



$$\frac{L(\epsilon/A)^{1/4}}{\hbar\pi} = n = C\epsilon^{1/4}$$

$$\boxed{\frac{dn}{d\epsilon} = \frac{L}{\hbar\pi(A)^{1/4}}\frac{1}{4}\epsilon^{-3/4}}$$

same for all D.'s

$$\boxed{\frac{dn}{d\epsilon} = \frac{C}{4}\epsilon^{-3/4}}$$

$$\frac{n\lambda}{2} = L$$

$$\frac{2\pi}{\lambda} = \frac{n\pi}{L} = K$$

$$p = \hbar K = \frac{h K}{2\pi}$$

$$p = \frac{h}{\lambda}$$

3D $\longrightarrow$ $\frac{dN}{dn} = \frac{d}{dn}\left(\frac{4}{3}\pi n^3\right) = 4\pi n^2$

$$N = \frac{C}{8}\int \frac{\epsilon^{-3/4}}{e^{\beta\epsilon}-1} 4\pi n^2 \, d\epsilon$$

$$n^2 = \frac{L^2\sqrt{\epsilon}}{\hbar^2\pi^2} = C^2\sqrt{\epsilon}$$

$$N = \frac{4\pi C^3}{8}\int_0^\infty \frac{\epsilon^{-1/4}}{e^{\beta\epsilon}-1} d\epsilon$$

$$= \frac{4\pi C^3}{8}\int_0^\infty e^{-\beta\epsilon}\left(\frac{\epsilon^{-1/4}}{1-e^{-\beta\epsilon}}\right) d\epsilon$$

$$= \frac{4\pi C^3}{8}\int_0^\infty e^{-\beta\epsilon}\,\epsilon^{-1/4}\sum_{l=0}^\infty \left(e^{-\beta\epsilon}\right)^l d\epsilon$$

$$= \frac{4\pi C^3}{8}\int_0^\infty \epsilon^{-1/4}\sum_{l=1}^\infty \left(e^{-\beta\epsilon}\right)^l d\epsilon$$

$$= \frac{4\pi C^3}{8}\sum_{l=1}^\infty \int_0^\infty \epsilon^{-1/4} e^{-\beta\epsilon l} d\epsilon$$

$$= \frac{4\pi C^3}{8}\sum_{l=1}^\infty \frac{(-1/4)!}{(\beta l)^{3/4}}$$

or any Dimension d

$$N = C_0 \int_0^\infty \frac{\epsilon^{-3/4}}{e^{\beta\epsilon}-1} \, n^{d-1} \, d\epsilon$$

$$n^{d-1} = B_0 \, \epsilon^{(d-1)/4}$$

$$N = C_0 B_0 \int_0^\infty \frac{\epsilon^{\left(-3/4 + (d-1)/4\right)}}{e^{\beta\epsilon}-1} \, d\epsilon$$

$$= C_0 B_0 \int_0^\infty e^{-\beta\epsilon} \left(\frac{\epsilon^{\left(-3/4 + (d-1)/4\right)}}{1-e^{-\beta\epsilon}} \right) d\epsilon$$

$$= C_0 B_0 \sum_{l=1}^\infty \int_0^\infty e^{-\beta\epsilon l} \, \epsilon^{\left(-3/4 + (d-1)/4\right)} \, d\epsilon$$

$$= C_0 B_0 \underbrace{\sum_{l=1}^\infty \frac{\left(-3/4 + \frac{d-1}{4}\right)!}{(\beta l)^{d/4}}}_{\zeta\left(\frac{d}{4}\right)}$$

Therefore there are only physical solutions for $\frac{d}{4} > 1$. Therefore $d = 5$ is the lowest dimension for which a BE condensate will form.

b) $$\int f(|\vec{p}|)\, d^d p = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int f(p) p^{d-1} dp$$

$$N = \int \bar{n}(\epsilon) D(\epsilon)\, d\epsilon$$

$$= \int_0^\infty \frac{1}{e^{\beta\epsilon}-1} \frac{dN}{dn} \frac{dn}{d\epsilon}\, d\epsilon$$

$$k = \frac{n\pi}{L} \qquad p = \hbar k = \frac{\hbar n \pi}{L}$$

$$\epsilon = A\left(\frac{\hbar n \pi}{L}\right)^q \qquad \boxed{n = \frac{L}{\hbar\pi A^{1/q}} \epsilon^{1/q}}$$

$$N = \int \bar{n}(\vec{p})\, d^d p = \int \bar{n}_{(\epsilon)} D(\epsilon)\, d\epsilon$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int \frac{n^{d-1}}{e^{\beta\epsilon}-1}\, dn$$

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \left(\frac{L}{\hbar\pi A^{1/q}}\right)^{d-1} \int \frac{\epsilon^{\frac{d-1}{q}}}{e^{\beta\epsilon}-1} \left(\frac{L}{q\hbar\pi A^{1/q}}\right) \epsilon^{\frac{1-q}{q}} d\epsilon$$

$$\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^d \frac{1}{q}\int_0^\infty \frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$N = \frac{1}{2^d}\,\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^d \frac{1}{q}\int_0^\infty \frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$= C\int_0^\infty \frac{\epsilon^{\frac{d-q}{q}}}{e^{\beta\epsilon}-1}\,d\epsilon$$

$$= C\int_0^\infty \frac{e^{-\beta\epsilon}\,\epsilon^{\frac{d-q}{q}}}{1-e^{-\beta\epsilon}}\,d\epsilon$$

$$= C\int_0^\infty e^{-\beta\epsilon}\,\epsilon^{\frac{d-q}{q}}\sum_{l=0}^{\infty} e^{-\beta\epsilon l}\,d\epsilon$$

$$= C\sum_{l=1}\int_0^\infty e^{-\beta\epsilon l}\,\epsilon^{\frac{d-q}{q}}\,d\epsilon$$

$$= C\sum_{l=1}\frac{\left(\frac{d-q}{q}\right)!}{(\beta l)^{d/q}}$$

$$= C\,\Gamma(d/q)\,\frac{1}{\beta^{d/q}}\sum_{l=1}^{\infty}\frac{1}{l^{d/q}}$$

$$N = \frac{C\,\Gamma(d/q)}{\beta^{d/q}}\,\zeta(d/q)$$

for $d > q$

$$\boxed{T_c = \left(\frac{N}{C\,\Gamma(d/q)\,\zeta(d/q)}\right)^{q/d}\frac{1}{K_B}}$$

similar to Fall 2001 #6 ... more general

Calculation of conditions for BEC in an arbitrary number spatial dimensions

The conditions for the Bose-Einstein Condensate (BEC) to occur depend on the number of spatial dimensions under consideration, and the relationship of the energy to the momentum. Consider a general expression in which the energy ϵ is proportional to the momentum p, raised to some power q

$$\epsilon = Ap^q \tag{1}$$

This relation can be realized by comparison with a typical massive, non-interacting, non-relativistic boson for which $\epsilon = \frac{1}{2m}p^2$ or for a massless boson, ϵ = cp. In general, the evaluation of the condition for the BEC is done by calculation of the total number of particles through the relation

$$N = \int_0^\infty \bar{n}(\epsilon)dN = \int_0^\infty \frac{D(\epsilon)}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \tag{2}$$

where $D(\epsilon)$ is the density of states, and $\beta=(k_bT)^{-1}$. For a particle in a box of side length L, the contained modes are quantized by the condition that the wave function vanish at the walls, $\Psi(x,y,z=0)=\Psi(x,y,z=L)=0$. Thus for each spatial dimension i, we have the condition

$$k_i = \frac{n_i\pi}{L} \tag{3}$$

From the deBroglie momentum relation we have $p_i=\hbar k_i$, so in terms of n_i we get $p_i=\hbar\frac{n_i\pi}{L}$. Inserting this into eq(1) we get

$$\epsilon = A(\hbar\frac{n\pi}{L})^q \qquad \Rightarrow \qquad n = \frac{L}{\hbar\pi A^{1/q}}\epsilon^{1/q} \tag{4}$$

The density of states is evaluated by solving for the different pieces (I and II) of the derivative expansion

$$D(\epsilon) = \frac{dN}{d\epsilon} = \frac{dN}{dn}\frac{dn}{d\epsilon} = \underbrace{S_d(n)}_{(I)}\underbrace{\frac{dn}{d\epsilon}}_{(II)} \tag{5}$$

where (I) is the surface area of a d-dimentional sphere in n-space (n is the radius). This is given generally by

$$S_d(n) = \frac{2\pi^{d/2}n^{d-1}}{\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^{d-1}\epsilon^{\frac{d-1}{q}} \tag{6}$$

where we have inserted the expression for n from above. In lieu of memorizing this expression, it is sufficient to realize that the surface area term goes like n^{d-1}. From evaluation of term (II) using eq(4), we see that this term is independent of d. This is true for all such problems like this. Now solving completely for $D(\epsilon)$,

$$\begin{aligned} D(\epsilon) &= \left\{\frac{2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{L}{\hbar\pi A^{1/q}}\right)^{d-1}\epsilon^{\frac{d-1}{q}}\right\}\left\{\left(\frac{L}{q\hbar\pi A^{1/q}}\right)\epsilon^{\frac{1-q}{q}}\right\} \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{q}\left(\frac{L}{\hbar\pi}\right)^d\frac{1}{A^{d/q}}\epsilon^{(d-q)/q} \end{aligned} \tag{7}$$

Note that we have an L^d term that gives us our volume element. Inserting $D(\epsilon)$ into eq(2) we have to remember to include a factor of $1/2^d$ since we are considering only positive values of n.

$$N = \underbrace{\frac{\pi^{d/2}}{2^{d-1}\Gamma(d/2)}\frac{1}{q}\left(\frac{L}{\hbar\pi}\right)^d\frac{1}{A^{d/q}}}_{C}\int_0^\infty \frac{\epsilon^{(d-q)/q}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \tag{8}$$

Now for a couple tricks. Expand the denominator of the integrand in a power series of the form $\frac{1}{1-x} = \sum_{l=0}^\infty x^l$ valid when $|x|<1$.

$$\begin{aligned} N &= C\int_0^\infty \frac{\epsilon^{(d-q)/q}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon \\ &= C\int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}}\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\epsilon^{(d-q)/q}d\epsilon \\ &= C\int_0^\infty e^{-\beta(\epsilon-\mu)}\left(\sum_{l=0}^\infty e^{-l\beta(\epsilon-\mu)}\right)\epsilon^{(d-q)/q}d\epsilon \\ &= C\int_0^\infty \left(\sum_{l=1}^\infty e^{-l\beta(\epsilon-\mu)}\right)\epsilon^{(d-q)/q}d\epsilon \\ &= C\sum_{l=1}^\infty\left(\int_0^\infty e^{-l\beta(\epsilon-\mu)}\epsilon^{(d-q)/q}d\epsilon\right) \\ &= C\sum_{l=1}^\infty e^{l\beta\mu}\left(\int_0^\infty e^{-l\beta\epsilon}\epsilon^{(d-q)/q}d\epsilon\right) \end{aligned} \tag{9}$$

The expansion has assumed that $e^{-l\beta(\epsilon-\mu)} < 1$ and thus $\epsilon > \mu$. The validity of this condition will become apparent shortly. We can now recognize the integral as of the form

$$\int_0^\infty e^{-ax}x^n = \frac{n!}{a^{n+1}} \tag{10}$$

Therefore we have

$$\begin{aligned} N &= C\sum_{l=1}^\infty e^{l\beta\mu}\left(\frac{(\frac{d-q}{q})!}{(l\beta)^{\frac{d-q}{q}+1}}\right) \\ &= C\frac{\Gamma(d/q)}{\beta^{d/q}}\underbrace{\sum_{l=1}^\infty\frac{e^{l\beta\mu}}{l^{d/q}}}_{L_{d/q}(e^{\beta\mu})} \qquad \left\{\Gamma(d/q) = (\frac{d-q}{q})!\right\} \end{aligned} \tag{11}$$

The sum term in the underbrace is the Polylogarithmic function $L_{d/q}(e^{\beta\mu})$ also called a weighted Zeta function. For the BEC to occur $L_{d/q}(e^{\beta\mu})$ must approach it's maximum, finite value. This is given by $\mu = 0$, validating our expansion condition $\epsilon > \mu=0$ in that the energy of a single particle cannot be equal to zero. The sum then

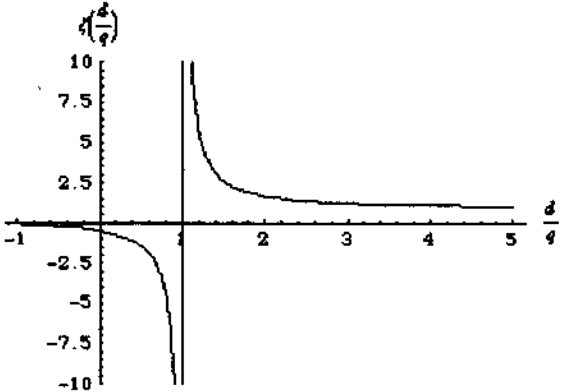


FIG. 1: The Riemann zeta function blows up at d/q $\rightarrow 1$ and is negative for d/q < 1.

becomes the standard Riemann Zeta function ζ(x). The total number of particles is therefore

$$N = C\frac{\Gamma(d/q)}{\beta^{d/q}}\zeta(d/q) \tag{12}$$

A plot of ζ(d/q) is shown in FIG(1). When d/q ≤ 1, the zeta function gives non-physical results in the form of infinite values or negative total particle numbers. Therefore the only physically tenable solutions for the BEC are obtained when the following relation is satisfied:

$$\boxed{\mathrm{d} > \mathrm{q}} \tag{13}$$

The expression for the BEC phase transition temperature can finally be obtained by rearranging eq(12).

$$T_c = \frac{1}{k_b}\left(\frac{N}{C\,\Gamma(d/q)\zeta(d/q)}\right)^{q/d} \tag{14}$$

In summary, the relationship between the momentum exponent q and the dimensionality d of the space determines whether the BEC will occur for the particle-in-a-box model. Evaluation of the integral yields the Riemann seta function for the BEC condition μ=0, which gives physical solutions *only if* $d > q$.

Problem #7 Fall 2001

$$E = \frac{p^2}{2m} + E_{\text{internal}} \qquad E_{\text{internal}} = 0, \epsilon, 2\epsilon, 3\epsilon, \ldots$$

a)

$$\sum_r e^{-\beta E\,\text{int}_r} \int_0^{\infty} \exp\left\{ \frac{-\beta}{2m}\left[(\vec{p}_1^{\,2} + \ldots \vec{p}_N^{\,2}) \right] \right\} \frac{d^3\vec{r}_1 \ldots d^3\vec{r}_N,\, d^3\vec{p}_1 \ldots d^3\vec{p}_N}{h_0^{3N}}$$

$$\sum_r e^{-\beta E\,\text{int}_r} \underbrace{e^{-\beta \frac{p^2}{2m}}}_{\text{Classical}}$$

first for one particle

$$\xi = \sum_r e^{-\beta r \epsilon} \underbrace{\int_0^{\infty} \exp\left\{ \frac{-\beta \vec{p}^{\,2}}{2m} \right\} \frac{d^3\vec{r}\, d^3\vec{p}}{(2\pi\hbar)^3}}$$

$$= \sum_r e^{-\beta r \epsilon} \, \xi'$$

$$\xi' = V\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}$$

$$\boxed{\xi = V\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\sum_r e^{-\beta r\epsilon}}$$

$$Z = \frac{(\xi)^N}{N!}$$

$$\sum_r e^{-\beta r\epsilon} = 1 + e^{-\beta\epsilon} + e^{-\beta 2\epsilon} + e^{-\beta 3\epsilon} + \ldots$$

$$= \sum_{n=0} e^{-n\beta\epsilon}$$

$$= \frac{1}{1-e^{-\beta\epsilon}} \quad \text{for } |e^{-\beta\epsilon}| < 1$$

$$\boxed{Z = \frac{1}{N!}\left(\frac{1}{1-e^{-\beta\epsilon}}\right)^N V^N\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3N/2}}$$

$$F = -KT\ln Z \qquad \frac{\partial}{\partial\beta}\ln\left(\frac{1}{\beta}\right)$$

$$(\beta)\left(-\frac{1}{\beta^2}\right)$$

$$F = -\frac{1}{\beta}\left[N\ln\left(\frac{1}{1-e^{-\beta\epsilon}}\right) + N\ln V + \frac{3N}{2}\ln\left(\frac{m}{2\pi\hbar^2\beta}\right) - N\ln N + N\right]$$

$$F = -\frac{1}{\beta}\left[N\ln V + \frac{3N}{2}\ln\left(\frac{m}{2\pi\hbar^2\beta}\right) - \underbrace{N\ln(1-e^{-\beta\epsilon})}_{\simeq N\ln(\beta\epsilon)} - N\ln N + N\right]$$

for large N

$$\boxed{F = -\frac{N}{\beta}\left[\ln\left(\frac{V}{N}\right) + \frac{3N}{2}\ln\left(\frac{m}{2\pi\hbar^2\beta}\right) - \ln(1-e^{-\beta\epsilon}) + 1\right]}$$

b) $C_p = \left(\frac{dE}{dT}\right)_P$

$$\bar{E} = \frac{\sum_r \left(e^{-\beta\epsilon r}\int_0^\infty \left(\frac{p^2}{2m} + \epsilon r\right) e^{-\frac{\beta p^2}{2m}}\, d\vec{p}\, d\vec{r}\right)}{\sum_r e^{-\beta\epsilon r}\int_0^\infty e^{-\frac{\beta p^2}{2m}}\, d\vec{p}\, d\vec{r}}$$

$$\bar{E} = -\frac{\partial}{\partial \beta} \ln Z$$

$$C_p = \left(\frac{\partial \bar{E}}{\partial T}\right)_p = \frac{\partial}{\partial T}\left(-\frac{\partial}{\partial \beta} \ln Z\right)$$

$$\ln Z = \left[\ln\left(\frac{V}{N}\right) + \frac{3N}{2}\ln\left(\frac{m}{2\pi\hbar^2\beta}\right) - \ln(1-e^{-\beta\epsilon}) + 1\right]$$

$$\frac{\partial}{\partial \beta} \ln Z = \frac{3N}{2}\left(\frac{\cancel{2\pi\hbar^2}\beta}{\cancel{m}}\right)\left(\frac{-\cancel{m}}{\cancel{2\pi\hbar^2}\beta^2}\right) - \frac{1}{1-e^{-\beta\epsilon}}\,\epsilon e^{-\beta\epsilon}$$

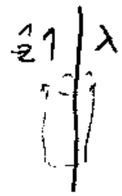
$$\frac{\partial}{\partial \beta} \ln Z = \frac{3N}{2}\left(-\frac{1}{\beta}\right) - \frac{\epsilon}{e^{\beta\epsilon}-1}$$

$$C_p = \frac{\partial}{\partial T}\left(\frac{3}{2}NKT + \frac{\epsilon}{e^{\epsilon/KT}-1}\right)_p$$

$$C_p = \frac{3}{2}NK + \frac{\epsilon^2 e^{\epsilon/KT}}{KT^2\left(e^{\epsilon/KT}-1\right)^2}$$

Fall 2001 # 8 (Like 11,13)

Change coordinates so line charge is on z-axis

a) 

$$\oint_S E\cdot da = 4\pi q \qquad \lambda = \frac{q}{L}$$

$$\underbrace{2\pi r L}_{A}\, E = 4\pi\lambda L \qquad E = \frac{2\lambda}{r}$$

$$\boxed{\vec{E} = \frac{2\lambda}{\rho}\hat{\rho}}$$ ← polar coordinates

b) Now line charge is moving with $\vec{v} = v\hat{z}$

Relativisticly

$$\beta = \frac{\vec{v}}{c}$$

Eq 559 Jackson

$$E = \gamma(\vec{E}' - \beta\times B') - \frac{\gamma^2}{\gamma+1}\beta(\beta\cdot \overset{\hat z \cdot \hat r}{E'})$$

$$B = \gamma(B' + \beta\times E') - \frac{\gamma^2}{\gamma+1}\beta(\beta\cdot B')$$

To change equation to find E', $E \to E'$, ect. $\beta \to -\beta$

$'$ means in objects rest frame

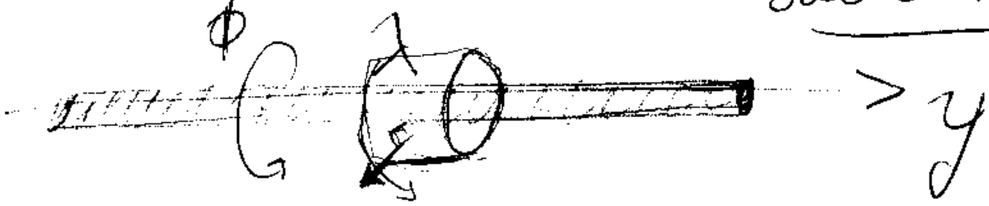
$B' = 0$

$$\vec{E} = \gamma\vec{E}' = \frac{2\gamma\lambda}{\rho}\hat{\rho}$$

$$\vec{B} = \gamma\beta\times E' = \frac{2\gamma\beta\lambda}{\rho}\hat{\Phi}$$

Problem #8 Fall 2001

Jackson (11.149)



a) $\int \vec{E} \cdot d\vec{a} = 4\pi Q_{enc}$

$E\, 2\pi r \ell = 4\pi \lambda \ell$

$\boxed{\vec{E} = \frac{4\pi\lambda}{2\pi r} = \frac{2\lambda}{r}\hat{r}}$

b) $\vec{E}_\perp = \gamma_0 \vec{E}_{0\perp}'$ $\qquad \gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}$

$$\boxed{\vec{E} = \frac{2\lambda}{r\sqrt{1 - v^2/c^2}}\hat{r}} = \boxed{\frac{2\gamma\lambda}{r}\hat{r}}$$

$I = \lambda v \hat{y}$

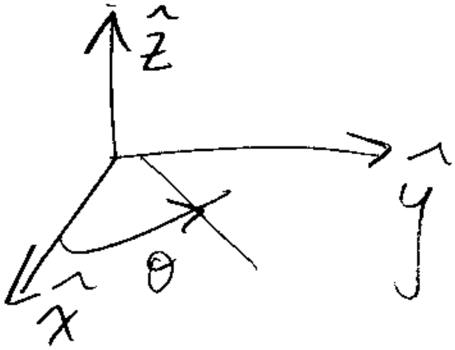
$$\vec{B}' = \gamma\left(\frac{\vec{v}}{c} \times \vec{E}\right) = \frac{v\, 2\lambda}{cr\sqrt{1 - v^2/c^2}}\hat{\phi}$$

$$\boxed{\vec{B}' = \frac{2\gamma v \lambda}{cr}\hat{\phi}}$$

$\longrightarrow$ next page better

$$\vec{E}' = \gamma\left(\vec{E} + \vec{\beta}\times\vec{B}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{E}\right)$$

$$\vec{\beta} = \frac{\vec{v}}{c}$$



$$\vec{B}' = \gamma\left(\vec{B} - \vec{\beta}\times\vec{E}\right) - \frac{\gamma^2}{\gamma+1}\,\vec{\beta}\left(\vec{\beta}\cdot\vec{B}\right)$$

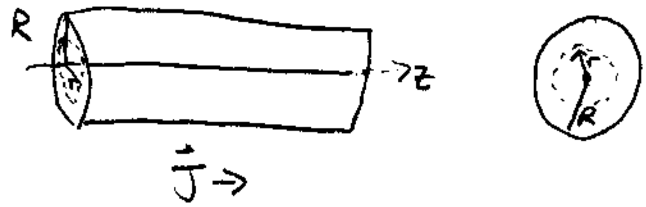
let $\vec{\beta} = -\frac{v}{c}\,\hat{y}$, $\vec{B} = 0$

$$\boxed{\vec{E}' = \gamma\vec{E}}$$ ✓

$$\vec{B}' = -\gamma\,\vec{\beta}\times\vec{E} = \frac{\gamma v}{c}\,E\,(\hat{y}\times\hat{r})$$

$$\boxed{\vec{B}' = \frac{\gamma v E}{c}\,\hat{\phi}}$$ ✓

An infinite column of a non-viscous conducting fluid carries a constant current density $\vec{J}$ and is confined to a radius R by the magnetic field that it induces. (Ignore gravity and electrostatic charge build-up)



(a) what is the electro-magnetic force density experienced by the fluid?

The force experienced by the current density is given by

$$F = \frac{1}{c}\int \vec{J}\times\vec{B}\, d^3r$$

where $\vec{J} = J\hat{z}$ and $\vec{B}$ is given by

$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enc} \quad , \quad I_{enc} = J\,\pi r^2$$

From symmetry we get

$$B\, 2\pi r = \frac{4\pi}{c}(J\,\pi r^2)$$

$$\Rightarrow \boxed{\vec{B} = \frac{2\pi}{c} J r\,\hat{\phi}} \qquad r<R$$

note: we don't care about the field outside of the cylinder since we only want the force density experienced by the fluid.

Thus, the force density is

$$f \equiv \frac{\vec{F}}{V} = \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{c}(J\hat{z})\times\left(\frac{2\pi}{c}Jr\,\hat{\phi}\right)$$

$$\therefore \boxed{\vec{f} = \frac{-2\pi}{c^2}J^2 r\,\hat{r}}$$

force in $-\hat{r}$ direction keeps current density confined

Fall 2001 #11 (p 2 of 2)

(b) In a steady state situation what is the required pressure profile of the fluid as a function of the radial distance r? That is, explicitly compute p(r).

the force due to the pressure is given by

$$\vec{p}(r) = -\nabla\left[\frac{|\vec{B}|^2}{2}\right] = -\nabla\left[\frac{1}{2}\left|\frac{2\pi}{c}Jr\right|^2\right]$$

$$= \frac{-4\pi^2}{2c^2}J^2 \quad \nabla r^2$$

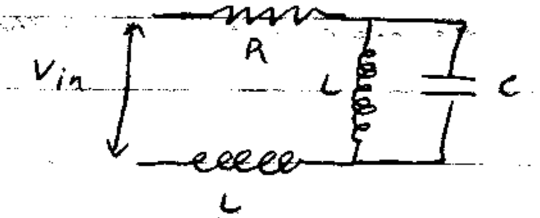
$$= \frac{-2\pi^2}{c^2}J^2 \quad \frac{\partial}{\partial r} r^2 \hat{r}$$

$$\boxed{\vec{p}(r) = -\frac{4\pi^2}{c^2}J^2 r\hat{r}} \leftarrow$$

same units as force density ????

but ok since we are considering pressure profile.

In the ac circuit shown below, the input voltage V_{in} and the circuit elements R, L, C are known quantities.



(a) Find the frequency $\omega = \omega_{res}$ at which the input impedance $Z_{in} = V_{in}/I_{in}$ is real.

$Z_R = R$; $Z_L = i\omega L$; $Z_C = \frac{-i}{\omega C}$

Need equivalent Z for $Z_L \parallel Z_C$ part

$$Z'_{eq} = \frac{Z_L Z_C}{Z_L + Z_C} = \frac{i\omega L\,(-i/\omega C)}{i\omega L - \frac{i}{\omega C}} = \frac{L/C}{i(\omega L - \frac{1}{\omega C})} \Rightarrow \frac{-iL}{\omega LC - 1/\omega}$$

Then the total equivalent impedance is just

$$Z_{eq} = Z_R + Z'_{eq} + Z_L = R + \frac{-iL}{\omega LC - 1/\omega} + iL\omega = R + i\left[\omega L - \frac{L}{\omega LC - 1/\omega}\right]$$

We are told that at $\omega = \omega_{res}$ Z_{eq} is just real (i.e. $= R$), so

$$\omega_r L - \frac{L}{\omega_r LC - 1/\omega_r} = 0 \Rightarrow \omega_r \cancel{L} = \frac{\cancel{L}}{\omega_r LC - 1/\omega_r} \Rightarrow \omega_r^2 LC - 1 = 1 \Rightarrow \omega_r^2 = \frac{2}{LC}$$

so $\omega_{res} = \omega_r = \boxed{\sqrt{\frac{2}{LC}}}$

(b) Assuming L and C are ideal (i.e. non-dissipative), then the only loss is due to the resistor:

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \cos\phi$$

where V_0 & I_0 are the maximum values of voltage and current. ϕ is the angle between the voltage and current

now

$$\phi = \arctan\left(\frac{X}{R}\right)$$

X is the reactance and R is the resistance in $Z = R + iX$

at $\omega = \omega_{res}$ $X = 0$ so $\phi = 0$ as well, hence

$$\langle P \rangle = \frac{1}{2} V_0 I_0 \quad \text{or} \quad I_0 = \frac{V_0}{R} \quad \text{or} \quad V_0 = I_0 R$$

in terms of known quantities $\boxed{\langle P \rangle = \frac{1}{2} \frac{V_0^2}{R}}$

but for (d) we will actually use $\langle P \rangle = \frac{1}{2} I_0^2 R$

(c) At $\omega = \omega_{res}$, what is the stored energy, U_{stored}?

Solving part (d) first and reverse engineering $\boxed{U_{stored} = \frac{1}{2} L I^2}$
as for why that is as opposed to $\frac{1}{2} C V^2$ I don't know.

(d) Find the quality factor $Q = \omega_{res} \dfrac{U_{stored}}{P_{dissip}}$?

From the above two answers:

$$Q = \omega_{res} \frac{\cancel{\frac{1}{2}} L \cancel{I^2}}{\cancel{\frac{1}{2}} \cancel{I^2} R} = \frac{\omega_{res} L}{R}$$

As for the reverse engineering part, first solve the simpler series R, L, C circuit (see "Basic Electronics for Scientists", Brophy, p. 82)

$$Z_{eq} = Z_R + Z_L + Z_C = R + i\left[\omega L - 1/\omega C\right]$$

at ω_{res} (where $\omega_{res} = \frac{1}{\sqrt{LC}}$) $\quad Z_{eq} = R + i\left[\sqrt{\frac{L}{C}} - \sqrt{\frac{L}{C}}\right] = R$

(which is exactly what we have).

So we can write in general

$$I = \frac{V}{|Z|} = \frac{V}{|R + i(\omega L - 1/\omega C)|} = \frac{V}{R}\frac{1}{|1 + \frac{i}{R}(\omega L - 1/\omega C)|} = \frac{V}{R}\frac{1}{\left(1 + \frac{1}{R^2}(\omega L - \frac{1}{\omega C})^2\right)^{1/2}}$$

as $|\underset{\text{resistive}}{R} + i\underset{\text{reactive}}{X}| = \sqrt{R^2 + X^2}$

which can also be writen as:

$$I = \frac{V}{R}\frac{1}{\left(1 + \underbrace{\left(\frac{\omega L}{R}\right)^2}_{\equiv Q^2 \text{ value}}\left[1 - \frac{1}{\omega^2 LC}\right]^2\right)^{1/2}}$$

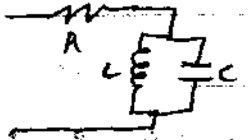
We can do a similar thing in our case:

$$I = \frac{V}{|Z|} = \frac{V}{|R + i(\omega L - \frac{L}{\omega LC - 1/\omega})|} = \frac{V}{R}\frac{1}{|1 + \frac{i}{R}(\omega L - \frac{L}{\omega LC - 1/\omega})|}$$

$$= \frac{V}{R}\frac{1}{\left(1 + \underbrace{\left(\frac{\omega L}{R}\right)^2}_{\equiv Q^2}\left[1 - \frac{1}{\omega^2 LC - 1}\right]^2\right)^{1/2}}$$

Check $\omega = \omega_{res}$: $\frac{V}{R}\frac{1}{\left(1 + Q^2\left[1 - \frac{1}{\frac{2\alpha}{\alpha} - 1}\right]^2\right)^{1/2}} = \frac{V}{R}\frac{1}{\left(1 + Q^2\left[1 - \frac{1}{2-1}\right]^2\right)^{1/2}} = \frac{V}{R}$ ✓

Note: If the series inductance were removed



then this circuit would have an infinite impedance at ω_{res} and no current would flow! Amazing what a simple inductor in the right place can do.

Selected Answers

Fall 2001

2) (a) $3^3 = 27$ states (b) $E_0 = \frac{-9\hbar^2 A}{8}$, $E_1 = \frac{\hbar^2 A}{4}$, $E_2 = \frac{3\hbar^2 A}{8}$, $E_3 = \frac{39\hbar^2 A}{8}$

(c) $7 - j = 3$ all with $s_{12} = 2$

$10 - j = 2$ 5 each w/ $s_{12} = 2, 1$

$9 - j = 1$ 3 each w/ $s_{12} = 2, 1, 0$

$1 - s = 0$ $s_{21} = 1$