CLASSICAL MECHANICS 220

Final Exam – Fall 2016

Tuesday 6 December 2016 from 3 pm to 6 am in room PAB-2-434

- Please write clearly and print your name on every page used, including this one;
- Make clear which question you are answering on each page;
- Present your arguments and calculations clearly;
- No books, notes, computers, or calculators are allowed during the exam;
- Please turn off and store away cell-phones, iPhones, iPods, iPads, Kindles, and other electronics.

Grades

Q1.	/ 16
Q2.	/ 17
Q3.	/ 16
Q4.	/ 16
Total	/ 65

Angular velocities $\omega_1, \omega_2, \omega_3$ in terms of Euler angles (θ, ϕ, ψ)

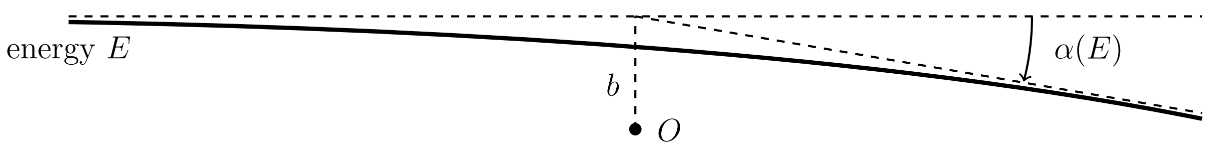
$$\begin{aligned}
\omega_1 &= \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi \\
\omega_2 &= \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \\
\omega_3 &= \dot\phi \cos\theta + \dot\psi
\end{aligned}$$

Navier-Stokes Equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \eta \Delta \mathbf{v} + \left(\zeta + \frac{1}{3}\eta \right) \nabla(\nabla \cdot \mathbf{v})$$

PROBLEM 1 [16 points]

A beam of non-relativistic particles with mass m is sent into a spherically symmetrical potential $V(r)$ centered at the point O. It is assumed that $V(r) \to 0$ as $r \to \infty$ and that, as $r \to 0$, the product $r^2 V(r)$ tends to zero or to a finite positive limit. The impact parameter b of the beam is fixed, the energy of the beam is denoted E, and the angle by which the beam is scattered is denoted by $\alpha(E)$, as indicated in the figure below (shown here for an attractive potential V).



1. Show that the motion of the beam is in the plane containing the initial beam velocity and the point O, and obtain two conserved quantities for the motion of the beam.
2. Derive a general formula for $\alpha(E)$ as a function of $V(r)$.
   [Hint: it is convenient to use the variable $u = b/r$.]
3. Evaluate $\alpha(E)$ for the potential $V(r) = B/r^2$ with $B > 0$.
4. Does your result, including the sign of $\alpha(E)$, make physical sense for small B/E ? Explain.

PROBLEM 2 [17 points]

A symmetrical rigid body has mass m and moments of inertia $I_x = I_y \neq I_z$ with respect to its center of mass. The rigid body interacts with a uniform electric field $\mathcal{E}$ through its electric dipole moment μ which is parallel to its principal z-axis. No other forces act on the rigid body which is free to move throughout space. You can think of this system as a molecule with a dipole moment in an electric field.

1. Derive the Lagrangian for the rigid body in terms of its center of mass position $\mathbf{x}$ and Euler angles θ, ϕ, ψ. The angular velocities in terms of Euler angles are given on the front page.
2. Obtain all the continuous symmetries of the system, and explain their physical meaning.
3. Obtain the momenta canonically conjugate to $\mathbf{x}, \theta, \phi, \psi$ and the conserved Noether charges associated with the continuous symmetries obtained in the previous item.
4. Derive the Hamiltonian, and show that all equations of motion can be solved by quadrature, but do not attempt to carry out the integrals or solve the differential equations.
5. Show that the differential equation for θ, expressed in the variable $z = \cos\theta$, is of the form,

$$\dot{z}^2 = az^3 + bz^2 + cz + d$$

   (Its solution is known to be given by the Weierstrass elliptic function.) Determine a, b, c, d in terms of the parameters of the problem.

PROBLEM 3 [16 points]

We consider a relativistic field theory in four space-time dimensions with a single scalar field $\phi(x^\mu)$, governed by the following action,

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2(\phi^2 - \phi_0^2)^2 \right) \tag{0.1}$$

where m and ϕ_0 are real positive non-zero constants, and ∂_μ represents the derivative with respect to x^μ which are the usual space-time coordinates with $\mu = 0, 1, 2, 3$.

1. Use the variational principle to obtain the Euler-Lagrange equation for ϕ, and give the expression for the total energy E of a general field configuration.
2. A *domain wall solution* in the x^1, x^2 plane is independent of x^1, x^2 and depends on the remaining coordinates $x^0 = ct$ and x^3 as follows $\phi(x^\mu) = f(y)$, with $y = \gamma(v)(x^3 - vt)$, for arbitrary constant velocity v, such that $f(\pm\infty) = \pm\phi_0$. Show that the Euler-Lagrange equation of (a) admits domain-wall solutions, such that f (as a function) is independent of v and γ. Determine this $\gamma(v)$ and the corresponding function f.
3. Derive the relation between the total energy per unit area ε in the x^1, x^2 plane of the domain wall and its velocity v, and show that this relation is the relativistic one.
4. Derive the mass per unit area μ of the domain wall. Verify that your result has the correct dimensions, given that the action $S[\phi]$ has dimensions of energy $\times$ time.

QUESTION 4 [16 points]

Consider the steady-state motion of a viscous incompressible fluid between two infinite coaxial cylinders with radii $R_2 > R_1 > 0$, rotating about their axis with angular frequencies Ω_2 and Ω_1.

1. Give the general form of the velocity field $\mathbf{v}$ and pressure p under these conditions;
2. Write down the equations satisfied by the velocity and the pressure;
3. Obtain the general solution to these equations given the above conditions;
4. Solve for the velocity field and pressure by imposing the appropriate boundary conditions.

[Hint: in cylindrical coordinates (r, ϕ, z), the Laplacian is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

while the unit vectors $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$ are such that

$$\frac{\partial\, \mathbf{e}_r}{\partial \phi} = -\mathbf{e}_\phi \qquad\qquad \frac{\partial\, \mathbf{e}_\phi}{\partial \phi} = +\mathbf{e}_r$$

while the derivatives with respect to r and z of these vectors vanish.]

Solutions

Problem 1

1. [4 points] Angular momentum around the point O is conserved. From the initial conditions of the beam, its direction is perpendicular to the plane containing the initial velocity and the point O, so angular momentum conservation will keep motion in that plane. The other conserved quantity is energy. In polar coordinates (r, θ) in the plane, with θ being zero to the infinite left and π to the infinite right, angular momentum and energy are given by,

$$\begin{aligned} mbv &= mr^2\dot{\theta} \\ E = \frac{1}{2}mv^2 &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V(r) \end{aligned} \tag{0.2}$$

2. [6 points] Eliminating $\dot{\theta}$ from the energy equation using angular momentum gives,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mv^2\frac{b^2}{r^2} + V(r) \tag{0.3}$$

Eliminating the time derivative of r in favor of its θ-derivative using the angular momentum conservation equation, and changing variables to $u = b/r$ gives,

$$E = E\left(\frac{du}{d\theta}\right)^2 + Eu^2 + V(b/u) \tag{0.4}$$

As $u \to 0$, we have $V(b/u) \to 0$. The beam reaches a maximum value of u, which is a minimum of u, at the turning point where $dr/d\theta = 0$, which is the lowest value u_0 such that,

$$E(1 - u_0^2) = V(b/u_0) \tag{0.5}$$

The angle $\alpha(E)$ is then obtained by quadrature,

$$\alpha(E) = \alpha_0 + 2\int_0^{u_0} \frac{du}{\sqrt{1 - u^2 - V(b/u)/E}} \tag{0.6}$$

When $V = 0$, we must have $\alpha(E) = 0$, so that $\alpha_0 = -\pi$

3. [4 points] For $V = B/r^2$, we have $V = Bu^2/b^2$, and therefore u_0 is given by $1 = (1 + B/(b^2E))u_0^2$, and α is given by,

$$\alpha(E) = -\pi + 2\int_0^{u_0} \frac{du}{\sqrt{1 - u^2 - (B/b^2E)u^2}} = -\pi + \frac{\pi b}{\sqrt{b^2 + B/E}} \tag{0.7}$$

4. [2 points] For $B/E \to 0$, we find $\alpha(E) \to 0$ which makes sense. For small positive B/E, $\alpha(E) < 0$, which makes sense with the conventions we have used.

Problem 2

1. [4 points] The potential energy of the dipole coupling is given by $V = -\mu\mathcal{E}\cos\theta$. In terms of the variables $\mathbf{x}, \theta, \phi, \psi$, the Lagrangian is given by,

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{1}{2}I_x\dot{\theta}^2 + \frac{1}{2}I_x\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_z(\dot{\phi}\cos\theta + \dot{\psi})^2 + \mu\mathcal{E}\cos\theta \tag{0.8}$$

2. [3 points] The continuous symmetries are as follows,

- time translations;
- space translations of $\mathbf{x}$;
- shift in ϕ gives rotations in the inertial frame (note that we do not have full rotation invariance in the inertial frame since the electric field breaks full rotation symmetry down to just rotations about the electric field);
- shift in ψ gives rotations in the body fixed frame.

3. [3 points] The canonical momenta are,

$$\begin{aligned}
\mathbf{p} &= m\dot{\mathbf{x}} \\
p_\theta &= I_x\dot{\theta} \\
p_\phi &= I_x\dot{\phi}\sin^2\theta + I_z\cos\theta(\dot{\phi}\cos\theta + \dot{\psi}) \\
p_\psi &= I_z(\dot{\phi}\cos\theta + \dot{\psi})
\end{aligned} \tag{0.9}$$

The momenta $\mathbf{p}, p_\phi$ and p_ψ are the conserved charges for respectively space-translations, and rotations in ϕ and ψ.
4. [4 points] The Hamiltonian is given by,

$$H = \dot{\mathbf{x}}\cdot\mathbf{p} + \dot{\theta}p_\theta + \dot{\phi}p_\phi + \dot{\psi}p_\psi - L \tag{0.10}$$

Eliminating the velocities in terms of momenta we get,

$$H = \frac{\mathbf{p}^2}{2m} + \frac{p_\theta^2}{2I_x} + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_x\sin^2\theta} + \frac{p_\psi^2}{2I_z} - \mu\mathcal{E}\cos\theta \tag{0.11}$$

which conserved in view of time translation invariance. Since $\mathbf{p}, p_\phi$ and p_ψ are conserved, we first consider the equation for θ alone. Energy conservation, and recasting p_θ in terms of $\dot{\theta}$ gives,

$$E = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}I_x\dot{\theta}^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_x\sin^2\theta} + \frac{p_\psi^2}{2I_z} - \mu\mathcal{E}\cos\theta \tag{0.12}$$

This equation can be solved by quadrature for θ in terms of elliptic integrals as we shall see below. Having θ, we solve for ϕ using the constancy of p_ϕ and p_ψ as well as the solution for θ

obtained earlier. Then one does the same thing for ψ. Finally, the solution for the center of mass is obviously linear in time.
5. [3 points] Multiply the above equation by $2I_x \sin^2\theta$, and express it in terms of $z = \cos\theta$. One finds,

$$I_x^2 \dot{z}^2 = \varepsilon(1-z^2) + 2I_x \mu\varepsilon z(1-z^2) - (p_\phi - p_\psi \cos\theta)^2 \tag{0.13}$$

where

$$\varepsilon = 2I_x \left(E - \frac{p_\psi^2}{2I_z} - \frac{\mathbf{p}^2}{2m}\right) \tag{0.14}$$

Hence we may read off,

$$\begin{aligned} a &= -\frac{2\mu\varepsilon}{I_x} & b &= -\frac{\varepsilon}{I_x^2} - \frac{p_\psi^2}{2I_x^2} \\ c &= -a + 2\frac{p_\phi p_\psi}{I_x^2} & d &= \frac{\varepsilon}{I_x^2} - \frac{p_\phi^2}{I_x^2} \end{aligned} \tag{0.15}$$

Problem 3

1. [4 points] Under an arbitrary variation $\delta\phi$ of the field ϕ, the action changes as follows,

$$\delta S[\phi] = \int d^4x \left(-\partial^\mu \delta\phi \partial_\mu \phi - 2m^2(\phi^2 - \phi_0^2)\phi\delta\phi\right) \tag{0.16}$$

Integrating the first term by parts gives the field equations,

$$\partial^\mu \partial_\mu \phi - 2m^2 \phi(\phi^2 - \phi_0^2) = 0 \tag{0.17}$$

2. [4 points] Given the dependence of ϕ in terms of the function f, the derivative four vector $\partial_\mu\phi$ and $\partial^\mu\partial_\mu\phi$ are given by,

$$\begin{aligned} \partial_\mu \phi &= \left(-\frac{v}{c}\gamma(v) f', 0, 0, \gamma(v) f'\right) \\ \partial^\mu \partial_\mu \phi &= \gamma(v)^2 \left(1 - \frac{v^2}{c^2}\right) f'' \end{aligned} \tag{0.18}$$

where f' denotes differentiation with respect to the argument y of f. Clearly, the choice,

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{0.19}$$

namely the standard γ-factor of special relativity, makes the equation for f independent of v and $\gamma(v)$, and the corresponding equation is given by,

$$f'' - 2m^2 f(f^2 - \phi_0^2) \tag{0.20}$$

The equation may be integrated by multiplying by f', and we obtain,

$$(f')^2 - m^2(f^2 - \phi_0^2)^2 = a \tag{0.21}$$

The constant a may be fixed by using the boundary conditions, and we find $a = 0$. This equation may be integrated as well, using the arrangement,

$$\frac{df}{\phi_0^2 - f^2} = \pm m dy \tag{0.22}$$

or

$$f(y) = \pm \phi_0 \tanh\Big(m\phi_0(y - y_0)\Big) \tag{0.23}$$

where y_0 is an arbitrary integration constant which represents the possible translations of the domain wall in the x^3 direction.
3. [4 points] The energy density per unit area in the x^1, x^2 direction is given by,

$$\varepsilon = c \int_{-\infty}^{\infty} dx^3 \left(\frac{1}{2}\frac{v^2}{c^2}(f')^2 + \frac{1}{2}\gamma^2 (f')^2 + \frac{1}{2} m^2 (f^2 - \phi_0^2)^2 \right) \tag{0.24}$$

Using the expression $(f')^2 = m^2(f^2 - \phi_0^2)^2$ derived earlier, and the relation $dy = \gamma dx^3$, we find the following expression for the energy density ε

$$\varepsilon = \gamma c \int_{-\infty}^{\infty} dy (f')^2 \tag{0.25}$$

which is the relativistic dependence of energy on velocity.
4. [4 points] The mass density μ is related to the energy density at zero velocity, and we may evaluate the integral,

$$\mu = \frac{1}{c} \int_{-\infty}^{\infty} dy (f')^2 = 2m\phi_0^3 / c \tag{0.26}$$

The dimensions that follow from the dimension of the action, are obtained by,

$$[\phi_0^2] = M/T \qquad [m^2] = T/M/L^2 \qquad [m\phi_0] = 1/L \qquad [\mu] = M/L^2 \tag{0.27}$$

The latter is the correct dimension for a mass density. We recognize $y = \gamma(x^3 - vt)$ as the Lorentz transform $y = n_\mu \Lambda^\mu{}_\nu x^\nu$ where n_μ is any fixed unit space-like vector, in this case $m^\mu = (0, 0, 0, 1)$, and Λ is the Lorentz boost in the x^3-direction. Clearly, we may take the Lorentz transformation to be in any direction, including rotations, and we may translate the solution arbitrarily in space and in time by a vector x_0^μ,

$$\phi(x) = f\Big(n_\mu \Lambda^\mu{}_\nu (x^\mu - x_0^\mu)\Big) \tag{0.28}$$

The existence of this general solution is guaranteed by the Poincar'e invariance of the action and field equations obeyed by ϕ.

Problem 4

1. [4 points] In Cartesian coordinates (x, y, z), the pressure and velocity field are independent of z and of time, since the flow is in steady-state. The pressure depends only on the radius $r = \sqrt{x^2 + y^2}$, while the velocity field is given by,

$$v_x = -\omega(r) y \qquad\qquad v_y = +\omega(r) x \qquad\qquad v_z = 0 \tag{0.29}$$

for some unction $\omega(r)$ which depends only on r.
2. [4 points] Since the fluid is incompressible we have $\nabla \cdot \mathbf{v} = 0$, which is automatic on the velocity field given above. The remaining equation is the Navier-Stokes equation under these conditions, which reduces to,

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \eta \Delta \mathbf{v} \tag{0.30}$$

3. [4 points] Since pressure and velocity field are independent of z, and $v_z = 0$ the z-component of the Navier-Stokes equation is trivially satisfied. To compute the remaining equations, we use,

$$\mathbf{v} \cdot \nabla = \omega(r)\Big(x\partial_y - y\partial_x\Big) \tag{0.31}$$

Using this operator in the Navier-Stokes equations gives,

$$\begin{aligned} \rho\omega(r)^2 x &= \partial_x p + \eta\Delta\Big(\omega(r)y\Big) \\ \rho\omega(r)^2 y &= \partial_y p - \eta\Delta\Big(\omega(r)x\Big) \end{aligned} \tag{0.32}$$

Expressing the Laplacian in polar coordinates, these equations reduce to,

$$\begin{aligned} \rho\omega(r)^2 x &= \partial_x p + \eta y\Big(\omega''(r) + 3\omega'(r)/r\Big) \\ \rho\omega(r)^2 y &= \partial_y p - \eta x\Big(\omega''(r) + 3\omega'(r)/r\Big) \end{aligned} \tag{0.33}$$

Projection onto the radial direction gives,

$$\rho r \omega(r)^2 = \partial_r p \tag{0.34}$$

while projection onto the angular direction gives,

$$\omega''(r) + \frac{3}{r}\omega' = 0 \tag{0.35}$$

whose general solution is given by,

$$\omega(r) = \omega_0 + \frac{\omega_1}{r^2} \tag{0.36}$$

while the pressure is given by,

$$p(r) = p_0 + \frac{1}{2}\rho\omega_0^2 r^2 + 2\rho\omega_0\omega_1 \ln(r) - \frac{1}{2}\rho\frac{\omega_1^2}{r^2} \tag{0.37}$$

4. [4 points] The boundary conditions require $\omega(R_i) = \Omega_i$ for $i = 1, 2$. This gives,

$$\omega_0 = \frac{\Omega_1 R_1^2 - \Omega_2 R^2}{R_1^2 - R_2^2} \qquad \omega_1 = \frac{R_1^2 R_2^2(\Omega_1 - \Omega_2)}{R_2^2 - R_1^2} \tag{0.38}$$

so that the velocity field is given by,

$$\omega(r) = \Omega_1 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\frac{1}{r^2} - \frac{1}{R_2^2}\right) + \Omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\frac{1}{R_1^2} - \frac{1}{r^2}\right) \tag{0.39}$$