1. Classical Mechanics

A planet of mass m is moving in a gravitational central potential around a Sun of mass M. Assume $M >> m$.

a) Write down the Lagrangian and the Euler-Lagrange equations for the polar variables r, θ in the plane of motion.
b) Use the substitution $u = \frac{1}{r}$ to write down a differential equation for the trajectory $u(\theta)$.
c) What is the equilibrium solution of this equation? What does it represent?
d) If the planet is not initially on the equilibrium orbit, there will be small oscillations around the equilibrium point. What is the period of these oscillations?
e) Assume there is a perturbing potential $V = -B/r^2$, calculate the effect of this perturbation on the orbit.

Problem1

solutions $\quad m \sim \mu$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{GMm}{r}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \rightarrow \quad m r\dot{\vartheta}^2 - \frac{GMm}{r^2} - m\ddot{r} = 0$$

$$\frac{\partial L}{\partial \vartheta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = 0 \rightarrow \quad m r^2\dot{\vartheta} = \text{constant} = \ell$$

Using first equation and substitution

$$\ddot{r} = r\dot{\vartheta}^2 - \frac{GM}{r^2} = \frac{\ell^2}{m^2 r^3} - \frac{GM}{r^2}$$

$$\Rightarrow \frac{d^2 u}{d\vartheta^2} = -\frac{\mu^2}{\ell^2} r^2 \ddot{r} = -\frac{1}{r} + \frac{GMm^2}{\ell^2}$$

$$= -u + \frac{GMm^2}{\ell^2}$$

$$\frac{d^2 u}{d\vartheta^2} + u = \frac{GMm^2}{\ell^2}$$

equilibrium solution $u = u_0 = \frac{GMm^2}{\ell^2}$ circular orbit

general solution $u = u_0 + A\cos\vartheta + B\sin\vartheta$

small oscillation period $= 2\pi$

$\Rightarrow$ elliptical orbit

Adding perturbing potential

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{\ell^2} + \frac{2\mu^2 B}{\ell^2} u$$

$$\Rightarrow \text{period} = \frac{2\pi}{1 + \frac{2\mu^2 B}{\ell^2}}$$

equinox precession

Question 2: Classical Mechanics

Consider the classical field theory in one space dimension, parametrized by the coordinate x, with a single real scalar field $\phi(t,x)$, governed by the following action,

$$S[\phi] = S_0 \int dt\, dx \left(\frac{1}{2}(\partial_t \phi)^2 - \frac{c^2}{2}(\partial_x \phi)^2 - \omega^2 (1 - \cos\phi) \right)$$

Here, ω and c are a real constants, respectively with dimensions of frequency and velocity. The overall constant S_0 has dimensions of angular momentum divided by velocity.

(a) Use the variational principle to obtain the Euler-Lagrange equation for $\phi(t,x)$, and give the expression for the total energy E of a general field configuration.

(b) Consider solutions to the Euler-Lagrange equation of (a) of the form,

$$\phi(t,x) = f(y) \qquad\qquad y = \gamma(v)(x - vt)$$

for arbitrary constant velocity v. Show that it is possible to choose $\gamma(v)$ such that f (as a function) is governed by an equation which independent of v; determine this $\gamma(v)$, and the corresponding solution(s) f such that $\cos(f(\pm\infty)) = 1$, and $f(+\infty) \neq f(-\infty)$.

(c) Derive the relation between the total energy E of the solution and its velocity v, and show that this relation is the relativistic one. Derive the mass of the soliton.

Solution to Question 2

(a) Under an infinitesimal variation $\delta\phi$ of ϕ, the variation of the action is given by,

$$\delta S[\phi] = S_0 \int dt\, dx \Big(-\partial_t^2 \phi + c^2 \partial_x^2 \phi - \omega^2 \sin\phi \Big) \delta\phi$$

where we have freely integrated by parts on $\delta\phi$. Thus the Euler-Lagrange equation is,

$$\partial_t^2 \phi - c^2 \partial_x^2 \phi + \omega^2 \sin\phi = 0$$

The momentum canonically conjugate to ϕ is $\partial_t \phi$, so that the total energy is given as follows,

$$E = S_0 \int dx \left(\frac{1}{2}(\partial_t \phi)^2 + \frac{c^2}{2}(\partial_x \phi)^2 + \omega^2 (1 - \cos\phi) \right)$$

(b) For a field configuration of the form $\phi(t,x) = f(\gamma(x - vt))$, we have

$$\begin{aligned} \partial_t^2 \phi &= \gamma^2 v^2 f'' \\ \partial_x^2 \phi &= \gamma^2 f'' \end{aligned}$$

where the prime denotes the derivative with respect to the argument of f. The Euler-Lagrange equation on these configurations reduces to,

$$-\gamma^2 (c^2 - v^2) f'' + \omega^2 \sin f = 0 \tag{0.1}$$

The equation becomes independent of v when we choose the v-dependence to be,

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{0.2}$$

The dependence on c is not uniquely determined by (0.1), but was chosen here so that γ is dimensionless. Choosing a different c-dependence of γ amounts to redefining f. With the choice made in (0.2) for γ, the variable y becomes the spatial coordinate in the rest frame of the solution, under a relativistic change of frame by a Lorentz transformation. With the above choice of γ in (0.2) equation (0.1) becomes,

$$-c^2 f'' + \omega^2 \sin f = 0 \tag{0.3}$$

which is indeed independent of v. To integrate the equation, we multiply it by f' and integrate, which gives the following first integral of motion,

$$-\frac{1}{2}c^2(f')^2 + \omega^2(1 - \cos\phi) = \mu^2 \tag{0.4}$$

for an integration constant μ. (You can think of $-\mu^2$ as total energy of a mechanical system where cf' stands for time derivation.) Using now the boundary conditions as $y \to \pm\infty$, we conclude that $\mu = 0$. Taking the square root of the equation gives the equations,

$$\frac{f'}{2} = \pm\frac{\omega}{c}\sin\frac{f}{2} \tag{0.5}$$

Integrating this trigonometric solution, we find,

$$\tan\frac{f}{4} = e^{\pm\omega y/c} \qquad\qquad f(y) = 4\arctan(e^{\pm\omega y/c})$$

(c) Using equation (0.5) with $\mu = 0$, we find the following simplified formula for the energy,

$$E = S_0\int_{-\infty}^{+\infty} dx\, c^2(\partial_x\phi)^2 = S_0\gamma^2 c^2\int_{-\infty}^{+\infty} dx f'(\gamma(x - vt))^2 = S_0\gamma c^2\int_{-\infty}^{+\infty} dy f'(y)^2$$

By construction, the last integral is independent of the velocity v, so that the relativistic rest mass M of the solution is given by,

$$M = S_0\int_{-\infty}^{+\infty} dy f'(y)^2 = \pm\frac{2\omega S_0}{c}\int_{f(-\infty)}^{f(+\infty)} df \sin\frac{f}{2} = \frac{8\omega S_0}{c}$$

so that we have the relativistic kinetic formula, $E = \gamma Mc^2$. The second equality above was obtained by using equation (0.5) for one factor of f'. Note that we only make use of the boundary conditions to evaluate the mass, and do not need the full analytical solution for f.

QM

Problem3

A heavy quark (charm or bottom) meson can be treated as a quark and anti-quark bound state with non-relativistic quantum mechanics in a confining potential, where the potential is often assumed the form of U=A/r+Br with A<0 and B>0, and the confinement is mostly provided by the Br term. Considering the Schrodinger equation for the potential V(x)=g|x|, take $\Psi \sim e^{-ax^2}$ as a trial wavefunction and use the variational method (minimizing the expectation value of the H) to estimate the ground state energy for a particle of mass m in the confining potential.

$$\psi = c\, e^{-ax^2}$$

Normalization $|c|^2 \cdot \int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$

$$|c| = \left(\frac{2a}{\pi}\right)^{1/4}$$

The expectation value for H

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi^* \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + g|x| \right] \psi(x)\, dx$$

$$= \frac{\hbar^2 a}{2m} + \frac{g}{\sqrt{2\pi a}}$$

The condition for minimum $\langle H \rangle$

$$\frac{\partial \langle H \rangle}{d a} = 0 = \frac{\hbar^2}{2m} - \frac{g}{2\sqrt{2\pi}} \cdot a^{-3/2} = 0$$

$$a = \left(\frac{g m}{\sqrt{2\pi}\,\hbar^2}\right)^{2/3}$$

the minimum Energy

$$\langle H \rangle = \frac{\hbar^2}{2m}\left(\frac{g m}{\sqrt{2\pi}\,\hbar^2}\right)^{2/3} + \frac{g}{\sqrt{2\pi}} \cdot \left(\frac{\sqrt{2\pi}\,\hbar^2}{g m}\right)^{1/3}$$

$$= \frac{1}{\pi^{1/3}}\left[\frac{1}{2 \times 2^{1/3}} + \frac{1}{2^{1/3}}\right]\left[\frac{g^2 \hbar^2}{m}\right]^{1/3}$$

$$= 0.813 \left(\frac{g^2 \hbar^2}{m}\right)^{1/3}$$

Problem4

Consider the scattering of a particle of mass m from two identical potential centers that are some distance $\vec{a}$ apart. i.e. consider the potential

$$U(\vec{r}) = U_0(|\vec{r}|) + U_0(|\vec{r} - \vec{a}|) \tag{0.12}$$

Where $U_0(r)$ is a (rotationally symmetric) potential for scattering from one center. For this problem work in the (first) Born approximation.

a) Express the scattering amplitude $f(q)$ for the scattering from two centers in terms of the scattering amplitude from one center alone.

b) The potential (0.12) can be used as an approximation of the scattering of electrons of a diatomic molecule. Find an expression for the differential cross section for the scattering of a diatomic molecule in terms of the differential cross section of the mono-atomic gas.
Hint: To do this you have to average over all possible directions of the separation vector $\vec{a}$ since all orientations of the diatomic molecule occur with the same probability.

c) Find the relation of the total cross section for the diatomic molecule and the monoatomic gas in the limit of low energy scattering.

Solution

a) The scattering amplitude in the first Born approximation is given by the following integral

$$\begin{aligned} f(\vec{q}) &= -\frac{m}{2\pi\hbar^2}\int d^3x\Big(U_0(|\vec{x}|) + U_0(|\vec{x}-\vec{a}|)\Big)e^{-i\vec{q}\cdot\vec{x}} \\ &= -\frac{m}{2\pi\hbar^2}\int d^3x U_0(|\vec{x}|)e^{-i\vec{q}\cdot\vec{x}}\Big(1 + e^{-i\vec{q}\cdot\vec{a}}\Big) && (0.13) \\ &= f_0(q)\Big(1 + e^{-i\vec{q}\cdot\vec{a}}\Big) && (0.14) \end{aligned}$$

Note that since U_0 is rotationally symmetric the single center scattering amplitude only depends on the magnitude of $q = |\vec{q}|$.

b) The differential cross section is given by square of the scattering amplitude

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |f(q)|^2 \\ &= 2(1 + \cos\vec{q}\cdot\vec{a})|f_0(q)|^2 && (0.15) \end{aligned}$$

For the scattering of electrons from a diatomic molecule we can approximate by the (screened) potential of the two cores. Since the orientation of the separation of the cores is random in a gas we to get the differential cross section one has to average over all directions $\vec{a}$ since f_0 only depends on q the f_0^2 term is not affected and one uses

$$\begin{aligned} \overline{\cos(\vec{q} \cdot \vec{a})} &= \frac{1}{4\pi} \int \cos(\vec{q} \cdot \vec{a}) d\Omega \\ &= \frac{\sin |\vec{q}||\vec{a}|}{|\vec{q}||\vec{a}|} \end{aligned} \tag{0.16}$$

Hence the differential cross sections of the diatomic and monoatomic gas are related as follows

$$\frac{d\sigma}{d\Omega} = 2\Big(1 + \frac{\sin |\vec{q}||\vec{a}|}{|\vec{q}||\vec{a}|}\Big) \frac{d\sigma_0}{d\Omega} \tag{0.17}$$

c) For low energy scattering one has that $|\vec{q}||\vec{a}| << 1$ and hence

$$\lim_{|\vec{q}||\vec{a}| \to 0} \frac{\sin |\vec{q}||\vec{a}|}{|\vec{q}||\vec{a}|} = 1 \tag{0.18}$$

And hence

$$\frac{d\sigma}{d\Omega} \sim 4 \frac{d\sigma_0}{d\Omega} \tag{0.19}$$

After angular integration this implies

$$\sigma_{tot} = 4\sigma_{0,tot} \tag{0.20}$$

The total cross section for the diatomic gas is four times the cross section for the monoatomic gas in the low energy approximation.

Consider the Hamiltonian for a rigid rotator

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \tag{0.21}$$

Here L_i are the angular momentum operators and $I_i, i = 1, 2, 3$ are constants denoting the moments of inertia around the three axis.

a) For the case of the symmetric top, $I = I_1 = I_2$ and $I_3 \neq I_1$ and one has

$$H_0 = \frac{L_1^2}{2I} + \frac{L_2^2}{2I} + \frac{L_3^2}{2I_3} \tag{0.22}$$

derive the energy levels and their degeneracies.

b) For a slightly asymmetric top the Hamiltonian can be approximated by

$$H = H_0 + \frac{\Delta}{I}\left(\frac{L_2^2}{2I} - \frac{L_1^2}{2I}\right) \tag{0.23}$$

where $\Delta << I$ and $\Delta << I_3$. Calculate the corrections to the energy for the states with $l = 1$ to first order in Δ.

Note: Please calculate all matrix elements you need from the basic properties of angular momentum.

Solution

a) We can rewrite the Hamiltonian H_0 in terms of $\vec{L}^2$ and L_3^2 as follows

$$H_0 = \frac{1}{2I}(L_1^2 + L_2^2 + L_3^2) + \Big(\frac{1}{2I_3} - \frac{1}{2I}\Big)L_3^2 \tag{0.24}$$

For eigenstates $\mid l, m\rangle$ of $\vec{L}^2$ and L_3 one finds

$$H_0 \mid l, m\rangle = \left(\frac{\hbar^2}{2I}l(l+1) + \frac{I - I_3}{2I_3 I}\hbar^2 m^2\right) \mid l, m\rangle \tag{0.25}$$

For generic values of I, I_3 the states with $m \neq 0$ are two-fold degenerate as $\mid l, m\rangle$ and

$\mid l, -m\rangle$ have the same energy eigenvalue. The state with $m = 0$ is non degenerate.

b) It is useful to express the perturbation in terms of raising and lowering operators using the following identities

$$\begin{aligned} L_+^2 &= (L_1 + iL_2)^2 = L_1^2 + i(L_1L_2 + L_2L_1) - L_2^2 \\ L_-^2 &= (L_1 + iL_2)^2 = L_1^2 - i(L_1L_2 + L_2L_1) - L_2^2 \end{aligned} \tag{0.26}$$

And hence

$$H_1 = \frac{\Delta}{I}\left(\frac{L_2^2}{2I} - \frac{L_1^2}{2I}\right) = -\frac{\Delta}{4I^2}(L_+^2 + L_-^2) \tag{0.27}$$

- Since the state $\mid l = 1, m = 0\rangle$ is non degenerate we can apply non degenerate perturbation theory.

$$E^{(1)}_{l=1,m=0} = -\frac{\Delta}{4I^2}\langle l = 1, m = 0 \mid (L_+^2 + L_-^2) \mid l = 1, m = 0\rangle \tag{0.28}$$

Which vanishes since $L_\pm^2 \mid l = 1, m = 0\rangle = 0$, hence the first order correction is zero.

- Since the states $\mid l = 1, m = \pm 1\rangle$ are degenerate we have to apply degenerate perturbation theory. We have to evaluate the matrix element

$$\begin{pmatrix} \langle m = 1 \mid H_1 \mid l = 1, m = 1\rangle & \langle m = 1 \mid H_1 \mid m = -1\rangle \\ \langle m = -1 \mid H_1 \mid l = 1, m = 1\rangle & \langle m = -1 \mid H_1 \mid l = 1, m = -1\rangle \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\Delta\hbar^2}{2I^2} \\ -\frac{\Delta\hbar^2}{2I^2} & 0 \end{pmatrix} \tag{0.29}$$

Where we have used the fact that

$$\langle l = 1, m = -1 \mid L_-^2 \mid l = 1, m = +1\rangle = 2\hbar^2, \quad \langle l = 1, m = 1 \mid L_+^2 \mid l = 1, m = -1\rangle = 2\hbar^2, \tag{0.30}$$

The first order correction to the energies are the given by the eigenvalues of the above matrix and hence

$$E^{(1)}_{l=1,m=\pm 1} = \pm\frac{\Delta\hbar^2}{2I^2} \tag{0.31}$$

A particle with spin $\mathbf{s}$ and magnetic moment $\boldsymbol{\mu} = \gamma\mathbf{s}$ (where γ is the gyromagnetic ratio) is subjected to magnetic field $\mathbf{B}$, so that its Hamiltonian is given by

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}\,.$$

Cartesian spin projections s_i obey the usual commutation relations:

$$[s_i, s_j] = i\hbar\epsilon^{ijk}s_k\,,$$

where ϵ^{ijk} is the antisymmetric Levi-Civita tensor, and we implied summation over the repeated index k.

If the spin is measured to point along the y direction at $t = 0$, such that $\langle\mathbf{s}\rangle = s\mathbf{y}$ (where s is the magnitude of the spin and $\mathbf{y}$ is the unit vector pointing along y), find its expectation value along the x axis, i.e., $\langle s_x\rangle$, for $t > 0$. Let us orient the frame of reference such that $\mathbf{B} = B\mathbf{z}$ (where B is the magnitude of the magnetic field and $\mathbf{z}$ is the unit vector pointing along z).

Solution: Using the equation of motion in the Heisenberg picture,

$$\frac{d\mathbf{s}}{dt} = \frac{i}{\hbar}[H, \mathbf{s}] \quad \Rightarrow \quad \frac{ds_i}{dt} = \frac{i}{\hbar}\gamma B[s_i, s_z]\,,$$

we see that

$$\frac{ds_x}{dt} = \omega s_y \quad \text{and} \quad \frac{ds_y}{dt} = -\omega s_x$$

where $\omega = \gamma B$. Defining $s_+ = s_x + is_y$, we have

$$\frac{ds_+}{dt} = -i\omega s_+\,,$$

which is solved by $\langle s_+\rangle = ise^{-i\omega t}$, for the expectation value, according to the stated initial condition. We finally find

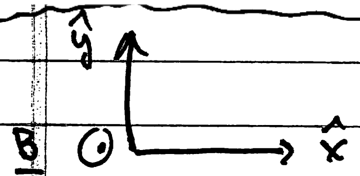
$$\langle s_x\rangle = \mathrm{Re}\langle s_+\rangle = s\sin(\omega t)\,.$$

Consider a zero-spin, nonrelativistic particle of charge q and mass m constrained to move in 2-dimensions (x, y) with a uniform magnetic field of strength B pointing along the z-direction perpendicular to the (x, y) plane.

a) Find the Hamiltonian for this system. Use a convenient gauge (Landau gauge) that simplifies the consideration of periodic boundary conditions along the x-direction and an unbounded domain along y.
b) What are the conserved quantities of the system?
c) Use your knowledge of the one-dimensional quantum harmonic oscillator to deduce the quantized energy levels for the magnetized particle.
d) Find what is the characteristic scale-length of the wave function along the y-direction.
e) Where is the center of the wave function, along the y-direction? What is the classical meaning of this center position?

1.

a)



$$\underline{B} = B\hat{z} = \nabla\times\underline{A}$$

$$H = \frac{(\underline{P}-\frac{q}{c}\underline{A})^2}{2m}$$; choose $\underline{A} = A_x\hat{x}$

$$\nabla\times\underline{A} = -\frac{\partial}{\partial y}A_x\,\hat{z} \Rightarrow A_x = -By$$

$$\boxed{H = \frac{P_y^2}{2m} + \frac{(P_x+\frac{qB}{c}y)^2}{2m} = H(P_y,P_x,y)}$$

b) $\boxed{\frac{\partial H}{\partial x} = 0 \Rightarrow P_x \text{ is conserved } + \frac{\partial H}{\partial t} = 0 \Rightarrow \underset{\underset{\text{energy}}{\uparrow}}{E} \text{ is conserved}}$

c) For 1D HO along y-direction

$$H_{HO} = \frac{P_y^2}{2m} + \frac{1}{2}\underset{\uparrow\text{effective spring constant}}{k}(y-y_0)^2$$

and oscillator frequency is $\omega^2 = \frac{k}{m}$

The quantized energy levels are: $E_n = (n+\frac{1}{2})\hbar\omega$

For the magnetized system

$$H = \frac{P_y^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2(y-y_0)^2$$

with $y_0 \equiv -\frac{P_x c}{qB}$

by analogy $\omega \to \frac{qB}{mc} = \Omega$ the cyclotron frequency

Morales QM #1 (cont..)

For magnetized system $\boxed{E_n = \left(n+\frac{1}{2}\right)\hbar\Omega = \left(n+\frac{1}{2}\right)\frac{\hbar|q|B}{mc}}$

d) The magnetized Schrödinger Eq: $H\psi = E\psi$ becomes

$$-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}\psi + \frac{1}{2}m\Omega^2(y-y_0)^2\psi = E\psi$$

Scale the y-variable to $\xi \equiv \frac{y}{y_s}$

$$\Rightarrow \quad -\frac{\hbar^2}{2m y_s^2}\frac{d^2}{d\xi^2}\psi + \frac{1}{2}m\Omega^2 y_s^2(\xi-\xi_0)^2\psi = E\psi$$

choose y_s to make a scaled Eq: $-\frac{d^2}{d\xi^2}\psi + (\xi-\xi_0)^2\psi = E'\psi$

$\Rightarrow$ Requires $\quad y_s^4 = \frac{\hbar^2}{m^2\Omega^2}$

or $\quad \boxed{y_s = \left(\frac{\hbar}{m\Omega}\right)^{1/2}}$

e) Center of wave function along y is the constant y_0

$$y_0 = -\frac{P_x c}{qB} = -\frac{P_x/m}{\Omega}$$

in Quantum case periodic B.C. along x of length L_x

$\Rightarrow \quad P_x = \frac{2\pi}{L_x} l\hbar \quad$ with $l = 0, 1, \ldots$

in classical limit $\quad P_x \to mv$

$\Rightarrow \quad \boxed{|y_0| = \frac{|v|}{\Omega} = \text{"}\rho\text{"} \text{ The Larmor Radius.}}$

3.

Morales QM #1 (ans...)

f) $$\Psi_0(x,y) = a\, e^{i\hbar P_x x}\, e^{-(y-y_0)^2/2y_s^2} \qquad \rightarrow P_x = \frac{2\pi}{L_x} l\hbar \,,\ l = 0, 1, \dots$$

$\uparrow$ normalization constant $\rightarrow = \sqrt{\int_0^{L_x} dx \int_{-\infty}^{\infty} dy\, |\Psi_0(x,y)|^2}$

because in normalized units the Schrödinger Eq is:

$$-\frac{d^2}{d\xi^2}\psi + (\xi-\xi_0)^2\psi = \left(\frac{E}{E_0}\right)\psi$$ with E_0 the ground-state energy

which is satisfied by $e^{-(\xi-\xi_0)^2/2}$ for $E = E_0$

g) $$J_x(x,y) = \frac{q}{2}\left[\psi^* \hat{v}_x \psi + \psi \hat{v}_x \psi^*\right]$$

is the current density along the x-direction

where the velocity operator is: $\hat{v}_x = \dfrac{\hat{P}_x - \frac{q}{c}A_x}{m}$ ← minus sign

and for this case $A_x = -By \Rightarrow \hat{v}_x = \dfrac{\hat{P}_x + \frac{qB}{c}y}{m}$

$$\Rightarrow \quad J_x(x,y) = \frac{q}{2m}\left[\psi^*(\hat{P}_x + \tfrac{qB}{c}y)\psi + \psi(\hat{P}_x + \tfrac{qB}{c}y)\psi^*\right]$$

$\uparrow$ given in step f)

$$J_x(x,y) = \frac{q a^2}{2m}\left(P_x + \frac{qB}{c}y\right) e^{-(y-y_0)^2/y_s^2}$$

$$\boxed{J_x(x,y) = \frac{q a^2}{2}\Omega(-y_0 + y)\, e^{-(y-y_0)^2/y_s^2}}$$

Question 8: Statistical Mechanics

Consider equilibrium between a solid and a vapor made up of the same monoatomic molecules. It is assumed that an energy φ is required per atom for transforming the solid into vapor atoms. For simplicity, use the Einstein model for the vibrations of atoms, i.e. assume that each atom is represented by a three-dimensional harmonic oscillator performing vibrations with angular frequency ω about its equilibrium position, independently of the other atoms. Evaluate the vapor pressure at equilibrium at temperature T.

Solution to Question 8

We denote by N_s and N_g respectively the number of atoms in the solid and in the gas. We use the canonical ensemble where the independent variables are T, V, N. The partition function $Z_g(T, N_g)$ for the gas is that of an ideal gas in volume V with N_g atoms at temperature T, namely,

$$Z_g(T, V, N_g) = \frac{V^{N_g}}{N_g!} \left(\frac{mkT}{\pi \hbar^2} \right)^{\frac{3}{2} N_g}$$

The partition function $Z_s(T, N_s)$ for the solid is that of N_s three-dimensional harmonic oscillators of frequency ω and binding energy $-\varphi$, and is give by,

$$\begin{aligned} Z_s(T, N_s) &= Z_1(T)^{N_s} \\ Z_1(T) &= e^{\varphi/kT} \times \left(\frac{e^{-\hbar\omega/2kT}}{1 - e^{-\hbar\omega/kT}} \right)^3 = e^{\varphi/kT} \left(2 \sinh \frac{\hbar\omega}{2kT} \right)^{-3} \end{aligned}$$

Note that the Einstein model treats the oscillators as distinguishable, as the underlying atoms are. The total partition function is given by $Z = Z_g Z_s$. Equilibrium is attained by minimizing the total free energy $-kT \ln Z$ as a function of N_g (or equivalently maximizing Z) while keeping $N_g + N_s$ fixed. In the limit of large N_g, we use Sterling's formula, and find

$$\begin{aligned} \frac{\partial \ln Z_g}{\partial N_g} &= \ln V - \ln N_g + \frac{3}{2} \ln \left(\frac{mkT}{\pi \hbar^2} \right) \\ \frac{\partial \ln Z_s}{\partial N_g} &= -\frac{\varphi}{kT} + 3 \ln \left(2 \sinh \frac{\hbar\omega}{2kT} \right) \end{aligned}$$

Hence the equilibrium equation $\frac{\partial \ln Z_g}{\partial N_g} + \frac{\partial \ln Z_s}{\partial N_g} = 0$ gives,

$$\begin{aligned} N_g &= V \left(\frac{mkT}{\pi \hbar^2} \right)^{\frac{3}{2}} e^{-\varphi/kT} \left(2 \sinh \frac{\hbar\omega}{2kT} \right)^3 \approx V \left(\frac{m\omega^2}{\pi kT} \right)^{\frac{3}{2}} e^{-\varphi/kT} \\ p &= kT \left(\frac{mkT}{\pi \hbar^2} \right)^{\frac{3}{2}} e^{-\varphi/kT} \left(2 \sinh \frac{\hbar\omega}{2kT} \right)^3 \approx kT \left(\frac{m\omega^2}{\pi kT} \right)^{\frac{3}{2}} e^{-\varphi/kT} \end{aligned}$$

where we have used the ideal gas law $pV = N_g kT$ for the gas pressure p. The approximation listed on the right side corresponds to the classical value, in the limit where $\hbar\omega \ll kT$.

Problem9

Consider an ensemble of diatomic molecules, such that each atom has three internal energy states $\epsilon = 1, 0, -1$ (in some units), independently of the other atom. The total energy of the molecule is $U = \epsilon_1 + \epsilon_2$. Calculate the ensemble averages $\langle U \rangle$ and $\langle U^2 \rangle$ at a given temperature T.

Solution: For a single atom, we have

$$\langle \epsilon \rangle = \frac{e^{-\beta} - e^{\beta}}{Z} \quad \text{and} \quad \langle \epsilon^2 \rangle = \frac{e^{-\beta} + e^{\beta}}{Z} \, ,$$

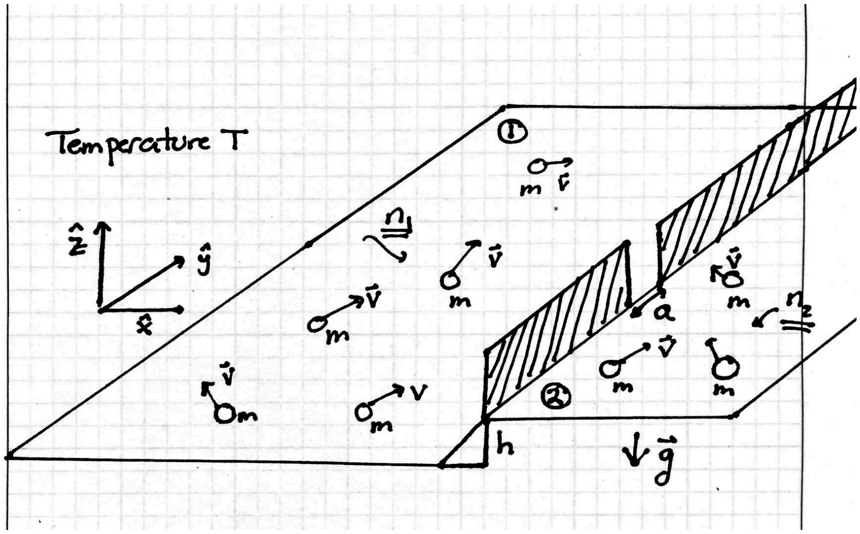
where $Z = 1 + e^{\beta} + e^{-\beta}$ is the partition function ($\beta^{-1} = k_B T$). For the full molecule,

$$\langle U \rangle = \langle \epsilon_1 \rangle + \langle \epsilon_2 \rangle = 2 \langle \epsilon \rangle = 2 \frac{e^{-\beta} - e^{\beta}}{Z}$$

and

$$\begin{aligned} \langle U^2 \rangle &= \langle (\epsilon_1 + \epsilon_2)^2 \rangle = \langle \epsilon_1^2 \rangle + \langle \epsilon_2^2 \rangle + 2 \langle \epsilon_1 \rangle \langle \epsilon_2 \rangle = 2(\langle \epsilon^2 \rangle + \langle \epsilon \rangle^2) \\ &= 2 \left[\frac{e^{-\beta} + e^{\beta}}{Z} + \left(\frac{e^{-\beta} - e^{\beta}}{Z} \right)^2 \right] = \frac{2}{Z^2} \left[e^{-\beta} + e^{\beta} + 2(e^{-2\beta} + e^{2\beta}) \right] \, . \end{aligned}$$

Problem10



1. Consider a two-dimensional gas of particles of mass m sliding (without rolling) on a table as shown in the figure above. There is a constant gravitational field $-g\hat{z}$ directed normal to the surface of the table as shown in the figure. The table is flat expect for a step of height h that runs parallel to the y-axis. There is an impenetrable wall along the top of the step except for a small slot of width a through which the particles may pass. In thermal equilibrium at temperature T and when the slot in the ramp is open so that particles can be exchanges between the two levels, the area density (number of particles) on the lower and upper parts of the table are n_1 and n_2 respectively. In the following you may assume that the particles are much small than the slot's width a and each may be treated as an ideal gas.

a) (5 points) Calculate the equilibrium value of the ratio of $\left.\frac{n_2}{n_1}\right|$ eq in terms of the parameters given.

b) (5 points) Now assume that the density of the particles on the right and left sides are given by $\tilde{n}_2$ and $\tilde{n}_1$ respectively. Calculate the rate $R_{1\to2}$ at which particles pass through the slot from side 1 to side 2 in thermal equilibrium in terms of these densities and the parameters given above. Calculate the analogous rate from particles going from side 2 to side 1 $R_{2\to1}$.

c) (5 points) From your answer in part b, determine the ratio $\tilde{n}_2/\tilde{n}_1$ required to make the net flux of particles through the slot vanish. Compare this to your answer in part a. Explain in one sentence what this means.

Solutions

①

1.

a) $\frac{n_2}{n_1} = e^{-\frac{mgh}{k_B T}}$ Just a Boltzmann factor since each state in the upper plane has energy mgh more than the corresponding state in the lower plane.

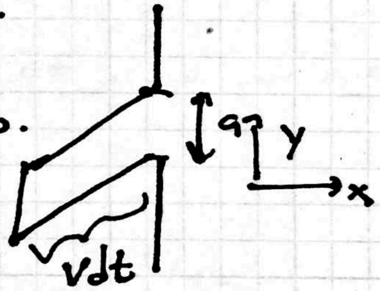
b) We have a maxwell Boltzmann velocity distribution

$$P(\vec{v}) = \tilde{n}_1 \left(\frac{m}{2\pi k_B T}\right) \exp\left\{\frac{-m|\vec{v}|^2}{2k_B T}\right\}$$

in the lower level.

Calculate $R_{1\to2}$ using this.

flux of particles up the ramp.



$\Phi(\vec{v})\, d^2\vec{v}$ = # particles hitting the slot w/ velocity between $\vec{v}$ and $\vec{v}+d\vec{v}$ per unit time.

$$\Phi(\vec{v})\, d^2\vec{v} = d^2\vec{v}\, P(\vec{v})\, v_x \quad ; \quad \tfrac{1}{2} m v_x^2 = mgh$$

Integrate over all v_y and $v_x > \sqrt{2gh}$ — minimum speed to get up the ramp.

$$R_{1\to2} = a \int_{-\infty}^{\infty} dv_y \int_{\sqrt{2gh}}^{\infty} dv_x\, P(\vec{v})\, v_x \quad \text{or}$$

Flux of Particles

$$R_{1\to 2} = \tilde{n}_1 a \frac{m^{1/2}}{(2\pi k_B T)^{1/2}} \int_{\sqrt{2gh}}^{\infty} v_x \, e^{-\frac{m v_x^2}{2k_B T}} \, dv_x$$

$$z = \frac{m v_x^2}{2 k_B T} \Rightarrow \frac{dz \, k_B T}{m} = v_x \, dv_x \, ; \quad z_{min} = \frac{m 2gh}{2 k_B T}$$

$$R_{1\to 2} = \tilde{n}_1 a \sqrt{\frac{k_B T}{2\pi m}} \int_{mgh/k_BT}^{\infty} dz \, e^{-z} = \tilde{n}_1 a \sqrt{\frac{k_B T}{2\pi m}} \, e^{-mgh/k_BT}$$

Going the other way is the same except we integrate over $-\infty < v_x < 0$ same as above w/ $h \to 0$.

$$R_{2\to 1} = \tilde{n}_2 a \sqrt{\frac{k_B T}{2\pi m}}$$

c) $\frac{R_{1\to 2}}{R_{2\to 1}} = 1 \Rightarrow \frac{\tilde{n}_1 e^{-mgh/k_BT}}{\tilde{n}_2} = 1$ or

$\frac{\tilde{n}_2}{\tilde{n}_1} = e^{-mgh/k_BT}$. same as equilibrium result in part a.

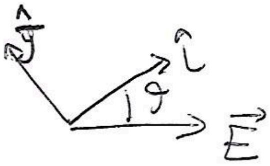
$\Rightarrow$ In equilibrium rate of going up = rate of going down. Example of detailed balance

Problem11

A horizontally polarized plane wave of wavenumber k is incident normally on a thin birefringent crystal of thickness $d = \frac{\pi}{2k\Delta n}$ where Δn is the difference in the indexes of refraction for the fast and slow axes of the crystal. The fast axis of the crystal is oriented at an angle θ with respect to the laser polarization.

a) Assuming perfect transmission, calculate the horizontal and vertical component of the electric field of the output plane wave as a function of θ.
b) For what θ angles is the output wave linearly polarized?
c) For what θ angles is the output wave circularly polarized?
d) What happens if I double the length of the crystal?

solutions



$$\vec{E} = E_0 \left(\cos\vartheta\, \hat{i} - \sin\vartheta\, \hat{j}\right)$$ at crystal entrance

$$\vec{E} = E_0 \left(\cos\vartheta\, \hat{i} - \sin\vartheta\, e^{i\Delta n k d}\, \hat{j}\right)$$ at the output

Going back to $\hat{x}$ and $\hat{y}$

$$\vec{E} = E_0 \left[\cos\vartheta \left(\cos\vartheta\, \hat{x} + \sin\vartheta\, \hat{y}\right) - \sin\vartheta\, e^{i\Delta n k d} \left(-\sin\vartheta\, \hat{x} + \cos\vartheta\, \hat{y}\right)\right]$$

$$= E_0 \left[\left(\cos^2\vartheta + \sin^2\vartheta\, e^{i\Delta n k d}\right)\hat{x} + \cos\vartheta \sin\vartheta \left(1 - e^{i\Delta n k d}\right)\hat{y}\right]$$

for $d = \frac{\pi}{2k\Delta n}$

$$\vec{E} = E_0 \left(\cos^2\vartheta + i\sin^2\vartheta\right)\hat{x} + \cos\vartheta \sin\vartheta \left(1 - i\right)\hat{y}$$

$\vartheta = 0, 90°$ linearly polarized wave

$\vartheta = -45°, 45°$ circularly polarized wave

for $d = \frac{\pi}{k\Delta n}$

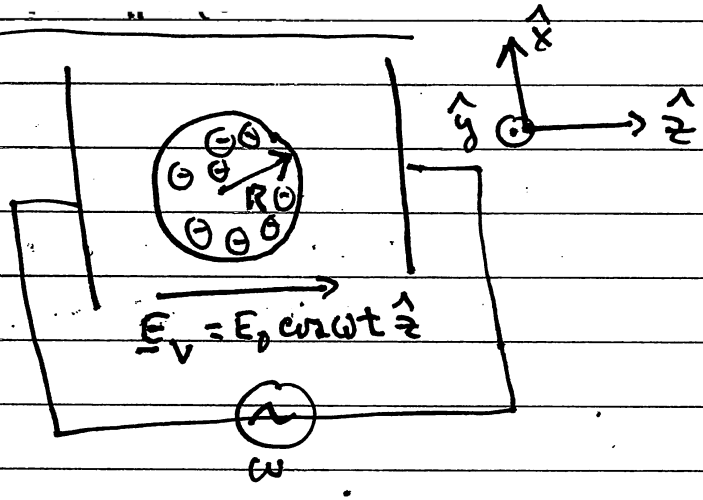
$$\vec{E} = E_0 \cos 2\vartheta\, \hat{x} + \sin 2\vartheta\, \hat{y}$$ rotated linearly polarized wave

Problem12

Consider a small, cold plasma ball of radius R. It consists of free electrons, of mass m and density n, which are neutralized by immobile (very heavy) background ions. The ball is placed between a pair of very large capacitor plates driven by a voltage generator oscillating at frequency ω . In the absence of the ball the vacuum electric field within the capacitor points along the z-direction, i.e., $\vec{E}_V = E_0 \cos(\omega t)\hat{z}$, where E_0 is a constant. If the value of frequency is chosen equal to the plasma frequency, i.e., $\omega = \omega_p \equiv \sqrt{4\pi e^2 n/m}$

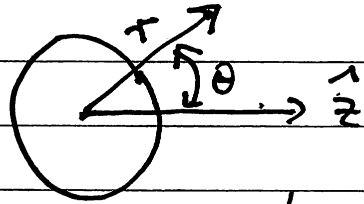
a) Find the electric field vector within the ball--Ignore resistive effects.

b) Obtain the total time-averaged electric energy stored within the ball.



a) Spherical symmetry of ball implies that potential is given by $\phi(r,\theta,t) = \frac{\tilde{\phi}(r,\theta)e^{-i\omega t} + c.c.}{2}$

with $\underline{E}(\underline{r},t) = -\nabla\phi$ and with coordinates



with $r\cos\theta = z$

because ball is symmetric about $\hat{z}$-axis.

For $r > R$: $\tilde{\phi}(r,\theta) = \sum_{l=1}^{\infty}\left[a_l r^l + \frac{b_l}{r^{l+1}}\right]P_l(\cos\theta)$

and For $r < R$: $\tilde{\phi}(r,\theta) = \sum_{l=1}^{\infty} c_l r^l P_l(\cos\theta)$

& asymptotically $z \gg R \Rightarrow \tilde{\phi}(r,\theta) \to -E_0 z = -E_0 r\cos\theta$

& since $P_1(\cos\theta) = \cos\theta \Rightarrow a_1 = -E_0$

& $a_l = 0$ for $l \neq 1$

Morales E+M #1 (ans..)

suitable solutions are: $r > R \Rightarrow \quad \tilde{\phi} = \left(-E_0 r + \frac{b_1}{r^2}\right)\cos\theta$

$$r < R \Rightarrow \quad \tilde{\phi} = c_1 r \cos\theta$$

Apply B.C.. $\quad c_1 R\cos\theta = \left(-E_0 R + \frac{b_1}{R^2}\right)\cos\theta \quad \|$ continuity of $\tilde{\phi}$

+ also continuity of radial component of $\underline{D}$, i.e.,

$$\left(\epsilon \frac{\partial}{\partial r}\tilde{\phi}\right)_{r=R_-} = \left(\frac{\partial}{\partial r}\tilde{\phi}\right)_{r=R_+}$$

but for the free-electron, cold plasma ball: $\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$

$$\epsilon c_1 \cos\theta = \left(-E_0 - \frac{2b_1}{R^3}\right)\cos\theta$$

$$\Rightarrow \quad \epsilon c_1 = -E_0 - \frac{2b_1}{R^3} = -E_0 - 2(c_1 + E_0)$$

$$\Rightarrow \quad \epsilon c_1 = -3E_0 - 2c_1 \quad \text{or} \quad c_1 = -\frac{3E_0}{\epsilon + 2}$$

+ for $r < R$: $\quad \tilde{\phi}(r,\theta) = -\frac{3E_0}{\epsilon+2} r\cos\theta = -\frac{3E_0}{\epsilon+2} z$

The electric field inside is $\underline{E} = -\nabla\phi = -\nabla\left[\frac{-\frac{3E_0}{\epsilon+2} z e^{-i\omega t} + c.c.}{2}\right]$

or $$\boxed{\underline{E} = \frac{3E_0}{1 - \frac{\omega_p^2}{\omega^2} + 2} \cos\omega t \, \hat{z}}$$

for the choice $\omega = \omega_p$ $\quad \boxed{\underline{E} \rightarrow \frac{3}{2} E_0 \cos\omega t \, \hat{z}}$ independent of position!

Morales E+M #1 (ans..)

b) The total time-averaged energy density for an electric field at frequency ω is: $\langle U_E \rangle = \frac{\partial}{\partial \omega}(\epsilon \omega) \frac{|E|^2}{16\pi}$

In this case, $\frac{\partial}{\partial \omega}(\epsilon \omega) = \epsilon + \omega \frac{\partial \epsilon}{\partial \omega}$

but $\epsilon = 0$ at $\omega = \omega_p$ & $\frac{\partial \epsilon}{\partial \omega} = \frac{\partial}{\partial \omega}\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \frac{2\omega_p^2}{\omega^3}$

$\Rightarrow$ at $\omega = \omega_p$ $\frac{\partial}{\partial \omega}(\epsilon \omega) = 2$

Since $\underline{E}$ is independent of position for $r < R$

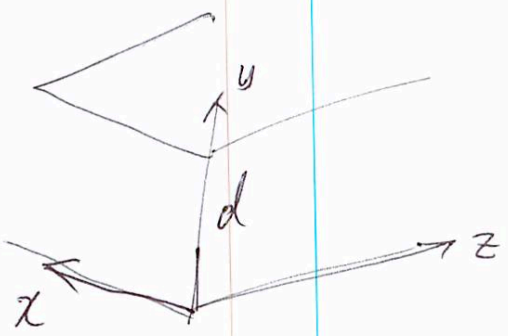
$$\Rightarrow \quad \langle U_E \rangle = \frac{\partial}{\partial \omega}(\epsilon \omega) \frac{|\underline{E}|^3}{16\pi} \underbrace{\left(\frac{4\pi R^3}{3}\right)}_{\text{Volume}}$$

$$\langle U_E \rangle = (2) \frac{\left(\frac{3}{2} E_0\right)^2}{16\pi} \left(\frac{4\pi}{3} R^3\right)$$

$$\boxed{\langle U_E \rangle = \left(\frac{3}{8}\right) R^3 E_0^2}$$

Problem13

Two large parallel conductor plates are separated by a distance d. Electromagnetic waves are propagating along the z direction parallel to the plates. Assuming the wave is uniform along the x direction, find the possible waveform and cut-off frequency of the electromagnetic waves propagating in the z direction.



No wave in x direction $k_x=0$

$$E(y,z\,t)= E(y)\cdot e^{i(k_z z-wt)}$$

$$\frac{d^2E(y)}{dy^2}+k_y^2E(y)=0$$

general solution

$$E(y)=A\cos k_y y+B\sin k_y y$$

boundary condition

at $y=0$, $E_x=E_z=0$, $\frac{\partial E_y}{\partial x}=0$

$$\begin{cases} E_x=A_1\sin k_y y\, e^{i(k_z z-wt)} \\ E_y=A_2\cos k_y y\, e^{i(k_z z-wt)} \\ E_z=A_3\sin k_y y\cdot e^{i(k_z z-wt)} \end{cases}$$

at $y=d$, $E_x=E_z=0$

$$k_y=\frac{m\pi}{d} \quad (m=0,1,2,\cdots)$$

$$k_z^2=k^2-k_y^2=\left(\frac{w}{c}\right)^2-\left(\frac{m\pi}{d}\right)^2$$

The cut-off frequency $w_0=c\left(\frac{m\pi}{d}\right)$

Also $\nabla\cdot E=0 \Rightarrow -A_2k_y+iA_3k_z=0$

$A_2=i\frac{k_z}{k_y}A_3$ for Amplitudes

14. Electromagnetism

An infinitely long perfectly conducting straight wire of radius *r* carries a constant current *i* and charge density zero as seen by a fixed observer A. The current is due to an electron stream of uniform density moving with relativistic velocity *u*. A second observer B travels parallel to the wire with relativistic velocity *v*. As seen by the observer B:

a) What is the electromagnetic field?
b) What is the charge density in the wire implied by this field?
c) With what velocities do the electron and ion streams move?
d) How do you account for the presence of a charge density seen by B but not by A?

Solution

(a) Let Σ and Σ' be the rest frames of the observers A and B respectively, the common x-axis being along the axis of the conducting wire, which is fixed in Σ, as shown in Fig. 5.6. In Σ, $\rho = 0$, $\mathbf{j} = \frac{i}{\pi r_0^2}\mathbf{e}_x$, so the electric and magnetic fields in Σ are respectively

$$\mathbf{E} = 0\,,$$

$$\mathbf{B}(r) = \begin{cases} \frac{\mu_0 i r}{2\pi r_0^2}\mathbf{e}_\varphi\,, & (r < r_0) \\ \frac{\mu_0 i}{2\pi r}\mathbf{e}_\varphi\,, & (r > r_0) \end{cases}$$

where $\mathbf{e}_x$, $\mathbf{e}_r$, and $\mathbf{e}_\varphi$ form an orthogonal system. Lorentz transformation gives the electromagnetic field as seen in Σ' as

$$E'_{\parallel} = E_{\parallel} = 0, \quad B'_{\parallel} = B_{\parallel} = 0\,,$$

$$\mathbf{E}' = \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}_\perp) = -\gamma v B\mathbf{e}_r = \begin{cases} -\frac{\mu_0\gamma i v r}{2\pi r_0^2}\mathbf{e}_r\,, & (r < r_0) \\ -\frac{\mu_0\gamma i v}{2\pi r}\mathbf{e}_r\,, & (r > r_0) \end{cases}$$

$$\mathbf{B}' = \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}_\perp}{c^2}\right) = \gamma B\mathbf{e}_\varphi = \begin{cases} \frac{\mu_0 i\gamma r}{2\pi r_0^2}\mathbf{e}_\varphi, & (r < r_0) \\ \frac{\mu_0 i\gamma}{2\pi r}\mathbf{e}_\varphi, & (r > r_0) \end{cases}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, and the lengths r and r_0 are not changed by the transformation.

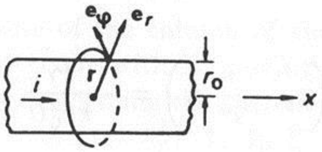


Fig. 5.6

(b) Let the charge density of the wire in Σ' be ρ', then the electric field produced by ρ' for $r < r_0$ is given by Gauss' law

$$2\pi r E'_r = \rho'\pi r^2/\varepsilon_0$$

to be

$$\mathbf{E}' = \frac{\rho' r}{2\varepsilon_0}\mathbf{e}_r\,. \quad (r < r_0)$$

Comparing this with the expression for $\mathbf{E}'$ above we have

$$\rho' = -\frac{vi\gamma}{\pi r_0^2 c^2}\,,$$

where we have used $\mu_0\varepsilon_0 = \frac{1}{c^2}$.

(c) In Σ the velocity of the electron stream is $\mathbf{v}_\mathrm{e} = -U\mathbf{e}_x$, while the ions are stationary, i.e. $v_\mathrm{i} = 0$. Using the Lorentz transformation of velocity we have in Σ'

$$\mathbf{v}'_\mathrm{e} = -\frac{v\ +U}{1+\frac{vU}{c^2}}\,\mathbf{e}_x\,,\quad \mathbf{v}'_\mathrm{i} = -v\mathbf{e}_x\,. \tag{6}$$

(d) The charge density is zero in Σ. That is, the positive charges of the positive ions are neutralized by the negative charges of the electrons. Thus $\rho_\mathrm{e} + \rho_\mathrm{i} = 0$, where ρ_e and ρ_i are the charge densities of the electrons and ions. As

$$\rho_\mathrm{e} = \frac{j}{-U} = -\frac{i}{\pi r_0^2 U}$$

we have

$$\rho_\mathrm{i} = \frac{i}{\pi r_0^2 U}\,.$$

However, the positive ions are at rest in Σ and do not give rise to a current. Hence

$$\mathbf{j}_\mathrm{e} = \mathbf{j} = \frac{i}{\pi r_0^2}\,\mathbf{e}_x\,,\quad \mathbf{j}_\mathrm{i} = 0\,.$$

$(\frac{\mathbf{j}}{c}, \rho)$ form a four-vector, so the charge densities of the electrons and ions in Σ' are respectively

$$\rho'_\mathrm{e} = \gamma\left(\rho_\mathrm{e} - \frac{v}{c^2}\,j_\mathrm{e}\right) = -\frac{i\gamma}{\pi r_0^2 U} - \frac{vi\gamma}{\pi r_0^2 c^2}\,,$$

$$\rho'_\mathrm{i} = \gamma\rho_\mathrm{i} = \frac{i\gamma}{\pi r_0^2 U}\,.$$

Obviously, $\rho'_\mathrm{e} + \rho'_\mathrm{i} \neq 0$, but the sum of ρ'_e and ρ'_i is just the charge density ρ' detected by B.