# Q12 -- Permanent Magnetization

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fedbarajas@live.com

12. *Electricity and Magnetism* (Fall 2005)

An infinitely long cylinder of radius $a$ exhibits a permanent magnetization with its magnetization vector given by

$$\mathbf{M}(\mathbf{r}) = \alpha r^2 \hat{\mathbf{z}} \quad r \le a$$
$$\mathbf{M}(\mathbf{r}) = 0 \quad r > a$$

where $\alpha$ is a constant, $r$ is the (cylindrical) radial coordinate and $\hat{\mathbf{z}}$ is a unit vector along the axis of the cylinder.

(a) Find the magnetic vector field $\mathbf{B}$ for $r < a$ and for $r > a$.

*Hint*: In cylindrical coordinates

$$(\nabla \times \mathbf{F})_\phi = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}$$

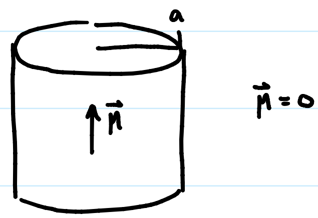
(b) Determine the value of

$$\oint \mathbf{A} \cdot \mathbf{dl}$$

along a circular path of radius $b > a$, encircling (in the $\hat{\phi}$ direction) and concentric with the magnetized cylinder. $\mathbf{A}$ is the magnetic vector potential.

(c) Find the force per unit volume experienced by the material at a location $r < a$.

(d) What will happen to the cylinder if $\alpha$ is suddenly increased to a very large value?




(a) Find $\vec{B}$:

- $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \cancel{\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}}^{\,0}$

  $\Rightarrow \vec{J} = \vec{J}_M + \cancel{\vec{J}_f}$

- no free current and no time-varying electric field

  $\vec{\nabla} \times \vec{H} = \vec{J}_f = 0 \Rightarrow \vec{H} = -\vec{\nabla}\Psi_M$ w/ $\vec{B} = \mu_0 [\vec{M} + \vec{H}]$

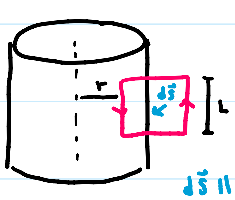
- only contribution to $\vec{B}$ is from permanent $\vec{M}$

## Method 1:* $\vec{B}$ from currents

$$\vec{J}_M = \vec{\nabla} \times \vec{M}$$
$$= \vec{\nabla} \times (\alpha r^2 \hat{z})$$
$$= -\frac{\partial}{\partial r}(\alpha r^2)\hat{\phi} \quad \leftarrow \text{from hint}$$
$$= -2\alpha r \hat{\phi}$$

$$\vec{K}_M = \vec{M} \times \hat{n}\Big|_{\vec{r}_s} \quad @\, r = a$$
$$= \left[(\alpha r^2 \hat{z}) \times \hat{r}\right]_{r=a}$$
$$= \alpha a^2 (\hat{z} \times \hat{r})$$
$$= \alpha a^2 \hat{\phi}$$

Notice that we have a solenoidal field by RHR, thus $\vec{B}_{out} = 0$




$d\vec{S} \parallel -\hat{\phi}$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$B_z(-L) = \mu_0 I_{enc} = \mu_0 \left\{ \int \vec{J} \cdot d\vec{S} \right\}$$

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\vec{B} \parallel \hat{z}$

$$= \mu_0 \left\{ \int \vec{K}\, \delta(\phi) \cdot d\vec{S} + \int \vec{J} \cdot d\vec{S} \right.$$
$$= \mu_0 \left\{ \int \vec{K} \cdot d\vec{\ell}_\perp + \int \vec{J} \cdot d\vec{S} \right.$$
$$= \mu_0 K L (\hat{\phi} \cdot -\hat{\phi}) + \mu_0 (2\alpha) \int^a r \, dr \, dz \, (-\hat{\phi} \cdot -\hat{\phi})$$

$B \parallel \hat{z}$ $\quad = \mu_0 \{ \int K \cdot dx_\perp + \int J \cdot da$

$$= \mu_0 K L(\hat{\phi} \cdot -\hat{\phi}) + \mu_0 (2\alpha) \int_r^a r\, dr\, dz\, (-\hat{\phi} \cdot -\hat{\phi})$$

$$= -\mu_0 \alpha \cancel{a^2} L + \mu_0 \alpha (\cancel{a^2} - r^2) L$$

$$= -\mu_0 \alpha r^2 L$$

$$\boxed{\vec{B} = \begin{cases} \mu_0 \alpha r^2 \hat{z} & r \le a \\ 0 & r > a \end{cases}}$$

**Method 2:** Since $\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = 0$, Ampere's Law $\vec{\nabla} \times \vec{H} = J_f = 0$ uniquely defines the field. $\vec{H}$ could be constant, but $\oint \vec{H} \cdot d\vec{l} = -HL = 0$

$$\{\vec{H} = 0\}$$

thus $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} = 0$

$$\vec{B} = \mu_0 \vec{M} \text{ everywhere}$$

$$\boxed{\vec{B} = \begin{cases} \mu_0 \alpha r^2 \hat{z} & r \le a \\ 0 & r > a \end{cases}}$$

**Method 3:** Using Fictitious Magnetic Charge (see Zangwill 415 for review)

(1) $\vec{\nabla} \cdot \vec{B} = \mu_0 \vec{\nabla} [\vec{M} + \vec{H}] = 0$

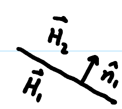
$\therefore \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = \rho^*$ (by analogy $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$)

$\Rightarrow \vec{\nabla}^2 \psi_M = -\rho^*$

However $\vec{\nabla} \cdot \vec{M} = \frac{\partial}{\partial z}(\alpha r^2) = 0$ s.t. $\rho^* = 0$ & $\vec{\nabla} \cdot \vec{H} = 0$

(2) The M.C. $\hat{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0 \Rightarrow \hat{n} \cdot \mu_0 [\vec{H}_2 + \vec{M}_2 - (\vec{H}_1 + \vec{M}_1)] = 0$

$$\hat{n} \cdot [\vec{H}_2 - \vec{H}_1]_s = -\hat{n} \cdot [\vec{M}_2 - \vec{M}_1]_s$$



$$\therefore \hat{n}_1 \cdot [\vec{H}_2 - \vec{H}_1]_s = -\hat{n}_1 \cdot [\vec{M}_2 - \vec{M}_1]_s$$

$$= \sigma^* = 0$$ (by analogy $\hat{n} \cdot [\vec{E}_2 - \vec{E}_1]_s = \sigma/\epsilon_0$)

$H_2^\perp = H_1^\perp \big|_s$ (perp to surface)

(3) $\hat{n}_1 \times [\vec{H}_2 - \vec{H}_1] = J_f = 0$

$$H_2^\parallel = H_1^\parallel \big|_s$$

Thus, $H_2^\parallel = H_1^\parallel \big|_s$, $H_2^\perp = H_1^\perp \big|_s$, $\vec{\nabla} \cdot \vec{H} = \rho^* = 0$, and $\vec{\nabla} \times \vec{H} = \vec{J}_f = 0$

Thus, $H_2^{\parallel} = H_1^{\parallel}\big|_s$ , $H_2^{\perp} = H_1^{\perp}\big|_s$ , $\vec{\nabla}\cdot\vec{H} = \rho^* = 0$ , and $\vec{\nabla}\times\vec{H} = \vec{J}_f = 0$

imply that $\vec{H} = 0$ everywhere

$$\therefore \boxed{\vec{B} = \mu_0 \vec{M}}$$

(b) Determine $\int_c \vec{A}\cdot d\vec{l}$ for $b > a$, c follows $\hat{\phi}$, concentric

Notice that $\int (\vec{\nabla}\times\vec{B})\cdot d\vec{s} = \oint \vec{B}\cdot d\vec{l}$ (form we want)

Then $\int (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \oint \vec{A}\cdot d\vec{l} = \int \vec{B}\cdot d\vec{s} = \Phi_B$

$$\oint \vec{A}\cdot d\vec{l} = \int_0^b \mu_0 \vec{M}\cdot d\vec{s} = \mu_0 \int_0^a \alpha r^2 \hat{z}\cdot\hat{z}\, r\,dr\,d\varphi + 0$$

$$= \boxed{\frac{\mu_0 \pi \alpha a^4}{2}}$$

(c) find $f$ (force/vol) @ $r < a$ on material

$$\vec{F} = \int d^3r\ \vec{J}\times\vec{B}$$

$$\vec{f} = \vec{J}\times\vec{B} = \mu_0 \vec{J}\times\vec{M} = (\vec{\nabla}\times\vec{B})\times\vec{M}$$

$$= -\vec{M}\times(\vec{\nabla}\times\vec{B}) = -\left[\vec{\nabla}(\vec{M}\cdot\vec{B}) - \vec{B}(\cancelto{0}{\vec{\nabla}\cdot\vec{M}})\right]$$

$$= -\vec{\nabla}(\vec{M}\cdot\vec{B}) = -\mu_0 \vec{\nabla}(\vec{M}^2)$$

$$\boxed{\vec{f} = -\mu_0 \vec{\nabla}(\alpha^2 r^4) = -4\mu_0 \alpha^2 r^3 \hat{r} \quad (r < a)}$$

cyl $\hat{r}$

(d) limit $\alpha \to \infty$, $\vec{f} \to -\infty$ therefore

the cylinder will collapse into a thin wire

as it compresses on itself