1. (Quantum Mechanics)

Two distinguishable spin-1/2 particles interact via the Hamiltonian $H_0 = g\,\mathbf{S}_1 \cdot \mathbf{S}_2$.

(a) What are the energy eigenstates and eigenvalues? Express the states in terms of eigenstates of $S_{1,z}$ and $S_{1,z}$.

We now add a time-dependent perturbation: $H = H_0 + \epsilon \exp\left(-\frac{t^2}{2\alpha^2}\right) S_{1,z}$.

(b) Assume the system is an eigenstate of H_0 at $t = -\infty$. Compute the probabilities for the system to be in a given eigenstate of H_0 at $t = +\infty$, working to lowest nontrivial order in ϵ. In particular, you are being asked to consider transitions between all posssible initial and final eigenstates of H_0.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

Most times a quantum mechanics problem talks about two particles with spin, the problem is about addition of angular momentum. Here is some key information about addition of angular momentum:

Addition of angular momentum

Suppose we are adding a spin-j_1 particle to a spin-j_2 particle (this also works for adding orbital and spin angular momentum of a single particle). Let $\mathbf{J}_i$ be the ith angular momentum operator (where $i = 1, 2$ throughout), and define $\mathbf{J} \equiv \mathbf{J}_1 + \mathbf{J}_2$. We can express the state of the system in two different bases:

Original basis: $|m_1\rangle\,|m_2\rangle$

$$\text{This basis simultaneously diagonalizes} \quad \mathbf{J}_1^2,\ \mathbf{J}_2^2,\ J_{1z},\ J_{2z} \tag{1}$$

$$\text{Possible quantum numbers:} \quad m_i = -j_i, -j_i + 1, \ldots, j_i - 1, j_i \tag{2}$$

$$\mathbf{J}_i^2 \text{ eigenvalues:} \quad \mathbf{J}_i^2\,|m_1\rangle\,|m_2\rangle = \hbar^2 j_i(j_i + 1)\,|m_1\rangle\,|m_2\rangle \tag{3}$$

$$J_{i,z} \text{ eigenvalues:} \quad J_{i,z}\,|m_1\rangle\,|m_2\rangle = \hbar\, m_i\,|m_1\rangle\,|m_2\rangle \tag{4}$$

$$\text{Dimension of space:} \quad (2j_1 + 1)(2j_2 + 1) \quad \text{different basis states} \tag{5}$$

Combined basis: $|j, m\rangle$

$$\text{This basis simultaneously diagonalizes} \quad \mathbf{J}_1^2,\ \mathbf{J}_2^2,\ \mathbf{J}^2,\ J_z \tag{6}$$

$$\text{Possible quantum numbers:} \quad j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2| \tag{7}$$

$$m = -j, -j + 1, \ldots, j - 1, j \tag{8}$$

$$\mathbf{J}_i^2 \text{ eigenvalues:} \quad \mathbf{J}_i^2\,|j, m\rangle = \hbar^2 j_i(j_i + 1)\,|j, m\rangle \tag{9}$$

$$\mathbf{J}^2 \text{ eigenvalues:} \quad \mathbf{J}^2\,|j, m\rangle = \hbar^2 j(j + 1)\,|j, m\rangle \tag{10}$$

$$J_z \text{ eigenvalues:} \quad J_z\,|j, m\rangle = \hbar m\,|j, m\rangle \tag{11}$$

$$\text{Dimension of space:} \quad \sum_{j=|j_1 - j_2|}^{j_1 + j_2} (2j + 1) = (2j_1 + 1)(2j_2 + 1) \quad \text{different basis states} \tag{12}$$

$$\text{Relation between bases:} \quad m_1 + m_2 = m \tag{13}$$

We also need one weird trick for this problem (and many problems like it), which is so important that it deserves its own box:

Dot product trick:

For addition of angular momentum problems, dot products in the Hamiltonian must be simplified as follows:

$$\mathbf{S}_1 \cdot \mathbf{S}_2 = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right) \quad \text{where } \mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2 \tag{14}$$

(a) Since particles 1 and 2 are spin-1/2 particles, then the spin quantum number s_i associated with each particle is equal to 1/2. We need to add these two spin-1/2 particles.

To start this problem, apply the dot product trick to the given Hamiltonian:

$$\begin{aligned} H_0 &= g\,\mathbf{S}_1 \cdot \mathbf{S}_2 \\ &= \frac{g}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right) \end{aligned} \tag{15}$$

This Hamiltonian is now expressed in terms of the operators $\mathbf{S}^2$, $\mathbf{S}_1^2$, and $\mathbf{S}_2^2$. By (6), we know that the combined basis $|j,m\rangle$ simultaneously diagonalizes these three operators (and also the operator S_z, which does not appear in the Hamiltonian H_0). We will therefore work in the $|j,m\rangle$ basis and search for the energy eigenstates and values.

Applied to a generic state $|j,m\rangle$, the three operators in the Hamiltonian yield

$$\mathbf{S}^2\,|j,m\rangle = \hbar^2 j(j+1)\;|j,m\rangle \quad \text{by (10)} \tag{16}$$

$$\begin{aligned} \mathbf{S}_1^2\,|j,m\rangle &= \hbar^2 s_1(s_1+1)\;|j,m\rangle \quad \text{by (9)} \\ &= \frac{3}{4}\hbar^2\,|j,m\rangle \quad \text{since } s_1 = 1/2 \text{ for the spin-1/2 particle 1} \end{aligned} \tag{17}$$

$$\begin{aligned} \mathbf{S}_2^2\,|j,m\rangle &= \hbar^2 s_2(s_1+1)\;|j,m\rangle \quad \text{by (9)} \\ &= \frac{3}{4}\hbar^2\,|j,m\rangle \quad \text{since } s_2 = 1/2 \text{ for the spin-1/2 particle 2} \end{aligned} \tag{18}$$

Putting these equations together, we get the action of the Hamiltonian on an arbitrary state $|j,m\rangle$:

$$\begin{aligned} H_0\,|j,m\rangle &= \frac{g}{2}\left(\mathbf{S}^2 - \mathbf{S}_1^2 - \mathbf{S}_2^2\right)|j,m\rangle \\ &= \frac{g}{2}\left(\hbar^2 j(j+1) - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2\right)|j,m\rangle \quad \text{by (16), (17), and (18)} \\ H_0\,|j,m\rangle &= \frac{g\hbar^2}{2}\left(j(j+1) - \frac{3}{2}\right)|j,m\rangle \end{aligned}$$

This tells us that the state $|j,m\rangle$ is an energy eigenstate of H_0 with energy eigenvalue

$$E_{j,m} = \frac{g\hbar^2}{2}\left(j(j+1) - \frac{3}{2}\right) \tag{19}$$

Now, we need to determine the possible values of j and m. Since we are dealing with two spin-1/2 particles, $s_1 = s_2 = 1/2$. Therefore, by (7), the possible values of j are

$$j = s_1 + s_2, \ldots, |s_1 - s_2| = 1 \text{ and } 0 \tag{20}$$

We can use (8) to find the possible values of m for each possible value of j. For $j = 1$, the possible values of m are equal to

$$j = 1: \qquad m = 1, 0, -1 \tag{21}$$

For $j = 0$, the only possible value of m is zero:

$$j = 0: \qquad m = 0 \tag{22}$$

We can now draw a "wedding cake" diagram for the possible values of j and m for this system of two spin-1/2 particles:

$$\begin{array}{ll} |1,1\rangle & \\ |1,0\rangle & \qquad |0,0\rangle \\ |1,-1\rangle & \end{array} \tag{23}$$

Note that as we expect for two spin-1/2 particles, the total number of spin states is $2 \times 2 = 4$. For the specific case of two spin-1/2 particles, the three states $|1,1\rangle$, $|1,0\rangle$, and $|1,-1\rangle$ are referred to as the "triplet," and the state $|0,0\rangle$ is referred to as the "singlet."

The energy $E_{j,m}$ only depends on j, so by (19), we can write the possible energy eigenvalues:

$$E_{1,1} = E_{1,0} = E_{1,-1} = \frac{g\hbar^2}{2}\left(1(1+1) - \frac{3}{2}\right) = \frac{g\hbar^2}{4} \tag{24}$$

$$E_{0,0} = \frac{g\hbar^2}{2}\left(0(0+1) - \frac{3}{2}\right) = -\frac{3g\hbar^2}{4} \tag{25}$$

The problem asks us to evaluate the energy eigenstates in terms of eigenstates of $S_{1,z}$ and $S_{2,z}$. We already know that these energy eigenstates are the states $|j,m\rangle$, but the problem is asking us to write them in the original angular momentum basis $|m_1\rangle\,|m_2\rangle$, since that one diagonalizes $S_{1,z}$ and $S_{2,z}$ by (1).

By (2), m_1 and m_2 have possible values $\pm\frac{1}{2}$, corresponding to spin-up ($\uparrow$) and spin-down ($\downarrow$). Therefore, we need to write the combined basis states $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$, and $|0,0\rangle$ in terms of the original basis states $|\uparrow\rangle\,|\uparrow\rangle$, $|\uparrow\rangle\,|\downarrow\rangle$, $|\downarrow\rangle\,|\uparrow\rangle$, and $|\downarrow\rangle\,|\downarrow\rangle$. There is a general method for deriving these relations, which will be explained after the solution to this problem. However, for the sum of two spin-1/2 particles, you might already know them:

$$\begin{aligned} |1,1\rangle &= |\uparrow\rangle\,|\uparrow\rangle \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\,\big] \qquad |0,0\rangle = \tfrac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\,\big] \\ |1,-1\rangle &= |\downarrow\rangle\,|\downarrow\rangle \end{aligned} \tag{26}$$

We can summarize our result as follows:

Energy eigenvalue	Energy eigenstates
$E_1 = \frac{g\hbar^2}{4}$	Triplet: $|1,1\rangle = |\uparrow\rangle\,|\uparrow\rangle$ $|1,0\rangle = \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\,\big]$ $|1,-1\rangle = |\downarrow\rangle\,|\downarrow\rangle$
$E_0 = -\frac{3g\hbar^2}{4}$	Singlet: $|0,0\rangle = \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\,\big]$

(b) This is a first-order time-dependent perturbation theory problem. Here is the relevant derivation:

Time-dependent perturbation theory:
The key to deriving the formulas for time-dependent perturbation theory is to work in the interaction picture. For an unperturbed, time-independent Hamiltonian H_0 added to a time-dependent perturbation $V(t)$,

$$H(t) = H_0 + V(t) \tag{27}$$

we write the interaction picture by folding the time-evolution of each state under H_0 into the quantum operators. If $\mathcal{O}_S$ is an operator in the (typical) Schrödinger picture, the equivalent operator $\mathcal{O}_I$ in the interaction picture is defined by

$$\mathcal{O}_I(t) \equiv e^{iH_0t/\hbar}\,\mathcal{O}_S\,e^{-iH_0t/\hbar} \tag{28}$$

To make sure that the expectation value $\langle\psi|\mathcal{O}|\psi\rangle$ is the same in both pictures, we must change the state $|\psi\rangle$ accordingly. If $|\psi_S(t)\rangle$ is a time-evolved ket in the Schrödinger picture, the equivalent ket $|\psi_I(t)\rangle$ in the interaction picture is defined by

$$|\psi_I(t)\rangle \equiv e^{iH_0t/\hbar}\,|\psi_S(t)\rangle \tag{29}$$

Kets in the interaction picture obey the Schrödinger equation for the perturbation Hamiltonian $V_I(t)$ in the interaction picture:

$$i\hbar\frac{\partial}{\partial t}\,|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle \tag{30}$$

We can integrate this equation (applying the initial condition for the state ψ at a reference time t_0) to get

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt'\,V_I(t')\,|\psi_I(t')\rangle \tag{31}$$

To lowest order in perturbation theory, $|\psi_I(t')\rangle \approx |\psi_I(t_0)\rangle$, so this equation becomes

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt'\,V_I(t')\,|\psi_I(t_0)\rangle \quad \text{to lowest order} \tag{32}$$

Now suppose that at $t = t_0$, the system is in an eigenstate $|n\rangle$ of H_0, and we are interested in the transition amplitude to another eigenstate $|m\rangle$. We can then take the inner product of (32) with $\langle m|$:

$$\begin{aligned}\langle m|\psi_I(t)\rangle &= \langle m|n\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt'\ \langle m|V_I(t')|n\rangle \\ &= \delta_{mn} - \frac{i}{\hbar}\int_{t_0}^{t} dt'\,e^{i(E_m-E_n)t'/\hbar}\,\langle m|V_S(t')|n\rangle \quad \text{to lowest order}\end{aligned} \tag{33}$$

In the second line, we have applied the definition of an operator in the interaction picture (28). Since $\langle m|\psi_I(t)\rangle = e^{-iE_mt/\hbar}\,\langle m|\psi_S(t)\rangle$ by (29), this is what we need to calculate transition probabilities.

In this problem, the perturbation in the Schrödinger picture (which we will call $V(t)$ is

$$\begin{aligned}V(t) &= \epsilon\exp\left(-\frac{t^2}{2\alpha^2}\right)S_{1,z} \\ &= \epsilon\,\frac{\hbar}{2}\exp\left(-\frac{t^2}{2\alpha^2}\right)\sigma_{1,z} \quad \text{since } S_z = \frac{\hbar}{2}\sigma_z \text{ for a spin-1/2 particle}\end{aligned} \tag{34}$$

and we want to consider the transition probabilities between *all* initial and final eigenstates. Recall from part (a) that there are four such eigenstates, given by (26). Equation (33) tells us how to start:

For all time-dependent perturbation theory problems, start by calculating the matrix elements of the perturbation Hamiltonian between initial and final states.

The only portion of $V(t)$ that is an operator (as opposed to a prefactor) is $\sigma_{1,z}$. Since the Pauli matrix $\sigma_{1,z}$ is equal to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ when applied to the spinor $\begin{pmatrix}a\\ b\end{pmatrix} \cong a\,|\uparrow\rangle + b\,|\downarrow\rangle$, we can write the effect of σ_z in Dirac notation as

$$\sigma_z\,|\uparrow\rangle = |\uparrow\rangle \quad \text{and} \quad \sigma_z\,|\downarrow\rangle = -\,|\downarrow\rangle \tag{35}$$

Therefore, as applied to the eigenstates in (26), the operator $\sigma_{1,z}$ (which acts on the first particle's spin only) yields

$$\begin{aligned}
\sigma_{1,z}\left|1,1\right\rangle &= \sigma_{1,z}\left|\uparrow\right\rangle\left|\uparrow\right\rangle & \sigma_{1,z}\left|1,0\right\rangle &= \sigma_{1,z}\frac{1}{\sqrt{2}}\big[\left|\uparrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle\left|\uparrow\right\rangle\big] \\
&= \left|\uparrow\right\rangle\left|\uparrow\right\rangle & &= \frac{1}{\sqrt{2}}\big[\left|\uparrow\right\rangle\left|\downarrow\right\rangle - \left|\downarrow\right\rangle\left|\uparrow\right\rangle\big] \\
\sigma_{1,z}\left|1,1\right\rangle &= \left|1,1\right\rangle & \sigma_{1,z}\left|1,0\right\rangle &= \left|0,0\right\rangle
\end{aligned} \tag{36}$$

$$\begin{aligned}
\sigma_{1,z}\left|1,-1\right\rangle &= \sigma_{1,z}\left|\downarrow\right\rangle\left|\downarrow\right\rangle & \sigma_{1,z}\left|0,0\right\rangle &= \sigma_{1,z}\frac{1}{\sqrt{2}}\big[\left|\uparrow\right\rangle\left|\downarrow\right\rangle - \left|\downarrow\right\rangle\left|\uparrow\right\rangle\big] \\
&= -\left|\downarrow\right\rangle\left|\downarrow\right\rangle & &= \frac{1}{\sqrt{2}}\big[\left|\uparrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle\left|\uparrow\right\rangle\big] \\
\sigma_{1,z}\left|1,-1\right\rangle &= -\left|1,-1\right\rangle & \sigma_{1,z}\left|0,0\right\rangle &= \left|1,0\right\rangle
\end{aligned} \tag{37}$$

Equation (31) tells us that in order for a transition from a state $\left|m\right\rangle$ to a different state $\left|n\right\rangle$ to be possible, the state element $V(t)\left|n\right\rangle$ must have nonzero overlap with $\left|m\right\rangle$. Note that this is true to all orders in perturbation theory.

Therefore, from (36), we can tell that there are no transitions between the state $\left|1,1\right\rangle$ and any other states under the perturbation, since the state $\sigma_{1,z}\left|1,1\right\rangle$ has no overlap with any other states. Similarly, from (37), there are no transitions between the state $\left|1,-1\right\rangle$ and any other states.

All that is left is to calculate the transition probabilities between the state $\left|1,0\right\rangle$ and the state $\left|0,0\right\rangle$. By (33), the transition amplitude between $\left|0,0\right\rangle$ and $\left|1,0\right\rangle$ is

$$\langle 1,0|\psi_I(+\infty)\rangle = -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\,\exp\left(\frac{i(E_{1,0}-E_{0,0})t'}{\hbar}\right)\langle 1,0|V(t')|0,0\rangle \tag{38}$$

We start by calculating the matrix elements of $V(t)$ corresponding to these states:

$$\begin{aligned}
\langle 1,0|V(t)|0,0\rangle &= \epsilon\,\frac{\hbar}{2}\exp\left(-\frac{t^2}{2\alpha^2}\right)\langle 1,0|\sigma_{1,z}|0,0\rangle \quad \text{by (34)} \\
&= \epsilon\,\frac{\hbar}{2}\exp\left(-\frac{t^2}{2\alpha^2}\right)\langle 1,0|1,0\rangle \quad \text{by (37)} \\
&= \epsilon\,\frac{\hbar}{2}\exp\left(-\frac{t^2}{2\alpha^2}\right) \quad \text{since the eigenstates } |j,m\rangle \text{ are normalized}
\end{aligned} \tag{39}$$

We also need to know the energy difference between the two states, which we calculated in part (a):

$$E_{1,0}-E_{0,0} = \frac{g\hbar^2}{4} - \left(-\frac{3g\hbar^2}{4}\right) = g\hbar^2 \tag{40}$$

Plugging all this into (38), we get the integral

$$\begin{aligned}
\langle 1,0|\psi_I(+\infty)\rangle &= -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\,\exp\left(\frac{i(g\hbar^2)t'}{\hbar}\right)\epsilon\,\frac{\hbar}{2}\exp\left(-\frac{(t')^2}{2\alpha^2}\right) \\
&= -i\frac{\epsilon}{2}\int_{-\infty}^{\infty} dt'\,\exp\left(-\frac{(t')^2}{2\alpha^2} + ig\hbar t'\right)
\end{aligned} \tag{41}$$

All that remains is to take the Gaussian integral. The starting point is completing the square in the exponential:

$$\begin{aligned} -\frac{(t')^2}{2\alpha^2} + ig\hbar t' &= -\frac{1}{2\alpha^2}\left((t')^2 - 2ig\hbar\alpha^2 t'\right) \\ &= -\frac{1}{2\alpha^2}\left(\left(t' - ig\hbar\alpha^2\right)^2 + g^2\hbar^2\alpha^4\right) \\ &= -\frac{1}{2\alpha^2}\left(t' - ig\hbar\alpha^2\right)^2 - \frac{g^2\hbar^2\alpha^2}{2} \end{aligned} \tag{42}$$

Then, the integral in (41) simplifies to

$$\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{(t')^2}{2\alpha^2} + ig\hbar t'\right) = \exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{1}{2\alpha^2}\left(t' - ig\hbar\alpha^2\right)^2\right)$$

Making the change of variables $u \equiv \left(\frac{1}{2\alpha^2}\right)^{1/2}\left(t' - ig\hbar\alpha^2\right)$, we get that

$$\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{(t')^2}{2\alpha^2} + ig\hbar t'\right) = \exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\left(2\alpha^2\right)^{1/2}\int_{-\infty}^{\infty} du \, e^{-u^2} \tag{43}$$

Using the known Gaussian integral $\int_{-\infty}^{\infty} du \, e^{-u^2} = \pi^{1/2}$, this gives us

$$\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{(t')^2}{2\alpha^2} + ig\hbar t'\right) = \exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\left(2\pi\alpha^2\right)^{1/2} \tag{44}$$

Plugging this into (41), we get

$$\begin{aligned} \langle 1,0|\psi_I(+\infty)\rangle &= -i\frac{\epsilon}{2}\exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\left(2\pi\alpha^2\right)^{1/2} \\ &= -i\epsilon\left(\frac{\pi}{2}\right)^{1/2}\alpha\exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right) \end{aligned} \tag{45}$$

This is the transition *amplitude*; to get the transition *probability*, we must take the square of its absolute value:

$$\begin{aligned} P_{|0,0\rangle\to|1,0\rangle} &= \left|\langle 1,0|\psi_I(+\infty)\rangle\right|^2 \\ &= \left|-i\epsilon\left(\frac{\pi}{2}\right)^{1/2}\alpha\exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\right|^2 \\ &= \frac{\pi\epsilon^2\alpha^2}{2}\exp\left(-g^2\hbar^2\alpha^2\right) \end{aligned} \tag{46}$$

The probability for the reverse process $|1,0\rangle \to |0,0\rangle$ is the same as for the forward process $|0,0\rangle \to |1,0\rangle$. Since $E_{0,0} - E_{1,0} = -g\hbar^2$, the setup for calculating the amplitude when the case when the system starts in state $|1,0\rangle$ (the equivalent of (41)) is

$$\begin{aligned} \langle 0,0|\psi_I(+\infty)\rangle &= -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt' \, \exp\left(\frac{i(E_{0,0} - E_{1,0})t'}{\hbar}\right)\langle 0,0|V(t')|1,0\rangle \\ &= -i\frac{\epsilon}{2}\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{(t')^2}{2\alpha^2} - ig\hbar t'\right) \quad \text{since } \langle 0,0|V(t')|1,0\rangle = \langle 1,0|V(t')|0,0\rangle^* \end{aligned} \tag{47}$$

The integral here is just the complex conjugate of the integral in (41). We calculated that integral in (44). When evaluated, that integral turned out to be real, so it is equal to its complex conjugate and we have

$$\int_{-\infty}^{\infty} dt' \, \exp\left(-\frac{(t')^2}{2\alpha^2} - ig\hbar t'\right) = \exp\left(-\frac{g^2\hbar^2\alpha^2}{2}\right)\left(2\pi\alpha^2\right)^{1/2} \tag{48}$$

Following the rest of the same process as we used to get $P_{|0,0\rangle\to|1,0\rangle}$, we get the same answer:

$$\begin{aligned} P_{|1,0\rangle\to|0,0\rangle} &= \left|\langle 1,0|\psi_I(+\infty)\rangle\right|^2 \\ &= \frac{\pi\epsilon^2\alpha^2}{2}\exp\left(-g^2\hbar^2\alpha^2\right) \end{aligned} \tag{49}$$

In summary, we have the following transition probabilities:

$$\boxed{P_{|1,0\rangle\to|0,0\rangle} = P_{|0,0\rangle\to|1,0\rangle} = \frac{\pi\epsilon^2\alpha^2}{2}\exp\left(-g^2\hbar^2\alpha^2\right) \quad \text{to order } \epsilon^2} \tag{50}$$

The states $|1,1\rangle$ and $|1,-1\rangle$ cannot transition to any other states.

Deriving the spin-1/2 Clebsch-Gordan coefficients:
We now derive the relations between the combined basis and the original basis in (26). Recall the "wedding cake" diagram of the four states in the combined basis (23):

$$\begin{array}{ll} |1,1\rangle & \\ |1,0\rangle & \qquad |0,0\rangle \\ |1,-1\rangle & \end{array}$$

Here is the appropriate method:

Wedding cake method of computing Clebsch-Gordon coefficients:

$$\begin{array}{lcl} |1,1\rangle & & \\ |1,0\rangle & \longrightarrow & |0,0\rangle \\ |1,-1\rangle & & \end{array} \tag{51}$$

Start at the $|j,m\rangle$ state with largest m (top of the diagram). For each curvy arrow, use the lowering operators

$$J_-\,|j,m\rangle = \hbar\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle \tag{52}$$

$$J_{1-}\,|m_1\rangle\,|m_2\rangle = \hbar\sqrt{(j_1+m_1)(j_1-m_1+1)}\,|m_1-1\rangle\,|m_2\rangle \tag{53}$$

$$J_{2-}\,|m_1\rangle\,|m_2\rangle = \hbar\sqrt{(j_2+m_2)(j_2-m_2+1)}\,|m_1\rangle\,|m_2-1\rangle \tag{54}$$

with $J_- = J_{1-} + J_{2-}$.
For each straight arrow, use the orthogonality of different eigenstates.

We'll now explain how to apply this method in the context of this problem.

To avoid getting stuck in a quagmire of algebra, and to keep the focus on the problem-solving method, we will pre-calculate some values of the proportionality constant $f(j,m) \equiv \sqrt{(j+m)(j-m+1)}$ that appears in the formulas for the lowering operator:

(j,m)	$f(j,m) \equiv \sqrt{(j+m)(j-m+1)}$
$(1,1)$	$\sqrt{2}$
$(1,0)$	$\sqrt{2}$
$(1/2,1/2)$	1

(55)

Then, in the context of this problem, since we have two spin-1/2 particles, equations (52)-(54)

become

$$S_- \left| j, m \right\rangle = \hbar f(j,m) \left| j, m-1 \right\rangle \tag{56}$$
$$S_{1-} \left| m_1 \right\rangle \left| m_2 \right\rangle = \hbar\, f(1/2, m_1) \left| m_1 - 1 \right\rangle \left| m_2 \right\rangle \tag{57}$$
$$S_{2-} \left| m_1 \right\rangle \left| m_2 \right\rangle = \hbar\, f(1/2, m_2) \left| m_1 \right\rangle \left| m_2 - 1 \right\rangle \tag{58}$$

We are now ready to start working our way through the wedding cake diagram:

Starting point: $\left|1,1\right\rangle$
Recall that $m_1 + m_2 = m$ (by (13)). In this case, $m = 1$. Since we have two spin-1 particles, m_1 and m_2 can be at most 1/2 (by (2)). Thus, the only possible original eigenket that can contribute to the combined eigenket $\left|1,1\right\rangle$ is $\left|\uparrow\right\rangle\left|\uparrow\right\rangle$ (recall that $\left|\uparrow\right\rangle$ has $m_i = 1/2$). We can set the normalization of $\left|1,1\right\rangle$ so that the prefactor is equal to 1, getting

$$\left|1,1\right\rangle = \left|\uparrow\right\rangle\left|\uparrow\right\rangle \tag{59}$$

Lowering operator: $\left|1,1\right\rangle \curvearrowright \left|1,0\right\rangle$

$$\begin{tikzcd} \left|1,1\right\rangle \arrow[d, bend right] & \\ \left|1,0\right\rangle \arrow[d, bend right] \arrow[r] & \left|0,0\right\rangle \\ \left|1,-1\right\rangle & \end{tikzcd}$$

Lowering $\left|1,1\right\rangle$ with the S_- lowering operator for total angular momentum and applying table (55), we get

$$S_- \left|1,1\right\rangle = \hbar f(1,1) \left|1,\, 1-1\right\rangle = \sqrt{2}\hbar \left|1,0\right\rangle \tag{60}$$

But $S_- = S_{1-} + S_{2-}$, so we can also use perform this lowering operation in the original basis (recalling that $\left|\downarrow\right\rangle$ has $m_i = -1/2$):

$$\begin{aligned} S_- \left|1,1\right\rangle &= (S_{1-} + S_{2-}) \left|1,1\right\rangle \\ &= (S_{1-} + S_{2-}) \left|\uparrow\right\rangle\left|\uparrow\right\rangle \quad \text{by our earlier calculation of } \left|1,1\right\rangle \text{ in the original basis (59)} \\ &= S_{1-} \left|\uparrow\right\rangle\left|\uparrow\right\rangle + S_{2-} \left|\uparrow\right\rangle\left|\uparrow\right\rangle \\ &= \hbar\, f(1/2,1/2) \left|\downarrow\right\rangle\left|\uparrow\right\rangle + \hbar\, f(1/2,1/2) \left|\uparrow\right\rangle\left|\downarrow\right\rangle \quad \text{by (53) and (54)} \\ &= \hbar\, \left|\downarrow\right\rangle\left|\uparrow\right\rangle + \hbar\, \left|\uparrow\right\rangle\left|\downarrow\right\rangle \quad \text{by table (55)} \end{aligned} \tag{61}$$

Setting (60) and (61) equal to one another, we get

$$\sqrt{2}\hbar \left|1,0\right\rangle = S_- \left|1,1\right\rangle = \hbar\, \left|\downarrow\right\rangle\left|\uparrow\right\rangle + \hbar\, \left|\uparrow\right\rangle\left|\downarrow\right\rangle$$

Simplifying, we get an expression for $\left|1,0\right\rangle$ in the original basis:

$$\left|1,0\right\rangle = \frac{1}{\sqrt{2}} \left[\, \left|\uparrow\right\rangle\left|\downarrow\right\rangle + \left|\downarrow\right\rangle\left|\uparrow\right\rangle \,\right] \tag{62}$$

Note that this expression is correctly normalized, which is a useful check that our work is correct. (We could have skipped calculating the overall constant in (60) and used the normalization to calculate it. Calculating the overall constant is a useful algebra check, though, so we have opted to include it.)

Lowering operator: $\left|1,0\right\rangle \curvearrowright \left|1,-1\right\rangle$

$$\begin{tikzcd} \left|1,1\right\rangle \arrow[d, bend right] & \\ \left|1,0\right\rangle \arrow[d, bend right] \arrow[r] & \left|0,0\right\rangle \\ \left|1,-1\right\rangle & \end{tikzcd}$$

This is exactly the same process. Lowering $|1,0\rangle$ with the S_- lowering operator for total angular momentum and using table (55), we get

$$S_- \,|1,0\rangle = \hbar f(1,0)\,|1,0-1\rangle = \sqrt{2}\hbar\,|1,-1\rangle \tag{63}$$

But $S_- = S_{1-} + S_{2-}$, so we can also perform this lowering operation in the original basis (noting that $|\downarrow\rangle$ vanishes upon lowering):

$$\begin{aligned}
S_- \,|1,0\rangle &= (S_{1-} + S_{2-})\,|1,0\rangle \\
&= (S_{1-} + S_{2-})\left(\frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\,\big]\right) \quad \text{by (62)} \\
&= \frac{1}{\sqrt{2}}\Big(S_{1-}\,|\uparrow\rangle\,|\downarrow\rangle + S_{1-}\,|\downarrow\rangle\,|\uparrow\rangle + S_{2-}\,|\uparrow\rangle\,|\downarrow\rangle + S_{12}\,|\downarrow\rangle\,|\uparrow\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\Big(\hbar\, f(1/2,1/2)\,|\downarrow\rangle\,|\downarrow\rangle + \hbar\, f(1/2,1/2)\,|\downarrow\rangle\,|\downarrow\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\Big(\hbar\,\,|\downarrow\rangle\,|\downarrow\rangle + \hbar\,\,|\downarrow\rangle\,|\downarrow\rangle\Big) \quad \text{by table (55)} \\
&= \sqrt{2}\hbar\,|\downarrow\rangle\,|\downarrow\rangle
\end{aligned} \tag{64}$$

Setting (63) and (64) equal to one another, we get

$$\sqrt{2}\hbar\,|1,-1\rangle = S_-\,|1,0\rangle = \sqrt{2}\hbar\,|\downarrow\rangle\,|\downarrow\rangle \tag{65}$$

Simplifying, we get an expression for $|1,-1\rangle$ in the original basis:

$$|1,-1\rangle = |\downarrow\rangle\,|\downarrow\rangle \tag{66}$$

As before, this state is correctly normalized.

Orthogonality: $|1,0\rangle \to |0,0\rangle$

$$\begin{tikzcd}
{|1,1\rangle} \arrow[d, bend right] & \\
{|1,0\rangle} \arrow[d, bend right] \arrow[r, red] & {|0,0\rangle} \\
{|1,-1\rangle} &
\end{tikzcd}$$

Since $m_1 + m_2 = m$ and $m_i = 1/2, -1/2$, we know that $|0,0\rangle$ must be the sum of $|\uparrow\rangle\,|\downarrow\rangle$ and $|\uparrow\rangle\,|\downarrow\rangle$. But since $|j,m\rangle$ is an orthonormal basis, $|0,0\rangle$ must be orthogonal to $|1,0\rangle$. Recall our expression for $|1,0\rangle$ in the original basis (62)

$$|1,0\rangle = \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\,\big]$$

There is only one vector that is orthogonal to this one, and (up to an overall phase) we can set it equal to $|0,0\rangle$:

$$|0,0\rangle = \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\,\big] \tag{67}$$

This completes the derivation of the eigenstates in the original basis:

$$\begin{array}{rcl@{\qquad}rcl}
|1,1\rangle & = & |\uparrow\rangle\,|\uparrow\rangle & & & \\
|1,0\rangle & = & \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle + |\downarrow\rangle\,|\uparrow\rangle\,\big] & |0,0\rangle & = & \frac{1}{\sqrt{2}}\big[\,|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle\,\big] \\
|1,-1\rangle & = & |\downarrow\rangle\,|\downarrow\rangle & & &
\end{array}$$

The coefficients which relate the combined basis to the original basis are known as “Clebsch-Gordan coefficients.” For small values of j, you can look them up in a table (just make sure you know how to read such a table first).

Angular momentum problems are very frequent on the comp. For more practice, try 2020 Q3, 2017 Q3, and 2015 Q4. For a special challenge, try 2015 Q6 and 2011 Q4.