18. For sufficiently small displacements, motion along the direction of the rods is decoupled from motion normal to the rods. Denoting displacements along the rod by (x_1, x_2, x_3) and those normal to the rod by (y_1, y_2, y_3), we find that the Lagrangian is

$$\mathscr{L} = \frac{m}{2}(\dot{x}_1^2 + 2\dot{x}_2^2 + \dot{x}_3^2 + \dot{y}_1^2 + 2\dot{y}_2^2 + \dot{y}_3^2) - \frac{k}{2}[(x_1 - x_2)^2 + (x_3 - x_2)^2]$$
$$- \frac{k'}{2}[(y_1 - y_2) - (y_2 - y_3)]^2.$$

The last term is constructed so as to vanish when all the particles lie on a straight line.

Motion in the x-direction is governed by the equations

$$m\ddot{x}_1 + k(x_1 - x_2) = 0, \qquad m\ddot{x}_3 + k(x_3 - x_2) = 0,$$
$$2m\ddot{x}_2 - k(2x_2 - x_1 - x_3) = 0.$$

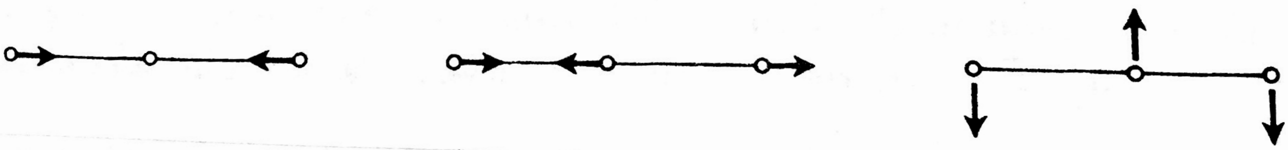
Conservation of momentum requires that $\ddot{x}_1 + 2\ddot{x}_2 + \ddot{x}_3 = 0$. This condition implies that there are only two normal modes for vibration in the x-direction. The normal modes are

$$x_1 - x_3 \quad \text{with frequency} \quad (k/m)^{1/2}$$

and

$$x_1 - 2x_2 + x_3 \quad \text{with frequency} \quad (2\,k/m)^{1/2}.$$

Motion in the y-direction is constrained by conservation of angular momentum, as well as linear momentum. Consequently there is only one mode of vibration, with frequency $(4k'/m)^{1/2}$. The modes may be sketched as in the accompanying figure.



40. (a) Any collection of particles has a center-of-mass velocity

$$\mathbf{v} = \frac{\mathbf{p}\ (\text{total})}{E\ (\text{total})},$$

as can be seen from the Lorentz transformation of momentum $p'_\parallel = \gamma(p_\parallel - Ev)$; $p'_\perp = p_\perp$. Choosing $\mathbf{v}$ parallel to $\mathbf{p}$, we see $\mathbf{p}' = \mathbf{0}$ when $\mathbf{v} = \mathbf{p}/E$. In our case the result is $\mathbf{v} = (\mathbf{p}_+ + \mathbf{p}_-)/(E_+ + E_-)$.

(b) The total energy and momentum constitute a Lorentz four-vector; hence the quantity $[(E_+ + E_-)^2 - (\mathbf{p}_+ + \mathbf{p}_-)^2]$ is an invariant. The barycentric frame is the one in which $\mathbf{p}'_+ + \mathbf{p}'_- = \mathbf{0}$; as $m_+ = m_-$, one also has $E'_+ = E'_-$ in this frame. Thus

$$4(E'_+)^2 = (E_+ + E_-)^2 - (\mathbf{p}_+ + \mathbf{p}_-)^2$$

or finally

$$E'_+ = E'_- = \frac{\sqrt{(E_+ + E_-)^2 - (\mathbf{p}_+ + \mathbf{p}_-)^2}}{2}.$$

(c) Consider the invariant $I = (\mathbf{p}_+ - \mathbf{p}_-)^2 - (E_+ - E_-)^2$. In the rest frame of the electron, $\mathbf{p}_- = 0$, $E_- = m$, we find that $E_+ = m/\sqrt{1 - v_{\text{rel}}^2}$ where

v_{rel} is the relative velocity. Thus

$$I = \frac{2m^2}{\sqrt{1 - v_{\text{rel}}^2}} - 2m^2, \text{ and we have } v_{\text{rel}} = \left[1 - \frac{1}{\left(1 + \frac{I}{2m^2}\right)^2}\right]^{1/2}.$$

3. *Quantum Mechanics* (Fall 2006)

Consider two flavours of massive neutrinos, denote $|\nu_e\rangle$ the electron neutrino flavour eigenstateand $|\nu_\mu\rangle$ the muon neutrino flavour eigenstate. These are related to the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ by

$$\begin{aligned} |\nu_e\rangle &= \cos(\theta)\ |\nu_1\rangle - \sin(\theta)\ |\nu_2\rangle \\ |\nu_\mu\rangle &= \sin(\theta)\ |\nu_1\rangle + \cos(\theta)\ |\nu_2\rangle \end{aligned}$$

(a) Show that flavour eigenstates and energy eigenstates are related by a unitary transformation.

(b) The energy of the eigenstate $|\nu_i\rangle$ is

$$E_i = \sqrt{\mathbf{p}^2c^2 + m_i{}^2c^4}, \quad i = 1, 2$$

Assume that an electron neutrino is produced in the sun with momentum $\mathbf{p}$ such that $|\mathbf{p}| \gg m_i c$. Find the probability for the electron neutrino to oscillate into a muon neutrino after travelling a distance L.

a) $$\vec{\Psi}_f = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{\Psi}_e = U\vec{\Psi}_e$$

$$|\Psi\rangle = \vec{\Psi}_f \cdot |\vec{\beta}_f\rangle = \Psi_{f1}|\nu_e\rangle + \Psi_{f2}|\nu_\mu\rangle = \vec{\Psi}_e \cdot |\vec{\beta}_e\rangle = \Psi_{e1}|\nu_1\rangle + \Psi_{e2}|\nu_2\rangle$$

where, e.g., $\vec{\Psi}_f = \begin{pmatrix} \Psi_{f1} \\ \Psi_{f2} \end{pmatrix}$ $\quad |\vec{\beta}_f\rangle = \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$

$$U^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow U^\dagger U = \begin{pmatrix} C^2 + S^2 & -SC + CS \\ -CS + SC & -(-S^2) + C^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

$\Rightarrow U^\dagger = U^{-1} \Rightarrow U$ is unitary

b) $|\Psi_0\rangle = |\nu_e\rangle$ with $|\vec{p}| >> m_1 c, m_2 c$, traveling $L \Rightarrow \Delta t \approx \frac{L}{c} \equiv \tau$

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi_0\rangle = e^{-i\omega_1 t}\cos\theta\,|\nu_1\rangle - e^{-i\omega_2 t}\sin\theta\,|\nu_2\rangle \quad \text{where } \omega_i \equiv \frac{E_i}{\hbar}$$

$$P_{\nu_e\to\nu_\mu}(L) = |\langle\nu_\mu|\Psi(\Delta t)\rangle|^2$$

$$\langle\nu_\mu|\Psi(t)\rangle = (\sin\theta \quad \cos\theta)\begin{pmatrix} e^{-i\omega_1 t}\cos\theta \\ -e^{-i\omega_2 t}\sin\theta \end{pmatrix} = (e^{-i\omega_1 t} - e^{-i\omega_2 t})\sin\theta\cos\theta$$

$$= \tfrac{1}{2}\sin 2\theta\,(e^{-i\omega_1 t} - e^{-i\omega_2 t})$$

$$P_{\nu_e\to\nu_\mu}(L) = \tfrac{1}{4}\sin^2 2\theta\,[1 - e^{i(\omega_2-\omega_1)\tau} - e^{-i(\omega_2-\omega_1)\tau} + 1] = \tfrac{1}{4}\sin^2 2\theta\,[2 - 2\cos(\omega_2-\omega_1)\tau]$$

$$= \sin^2 2\theta\,(\tfrac{1}{2} - \tfrac{1}{2}\cos(\omega_2-\omega_1)\tau) = \sin^2 2\theta\,\sin^2(\tfrac{1}{2}(\omega_2-\omega_1)\tau)$$

$$\left(E_2 - E_1 = \sqrt{p^2c^2 + m_2^2c^4} - \sqrt{p^2c^2 + m_1^2c^4} = pc[(1 + m_2^2c^2/p^2)^{1/2} - (1 + m_1^2c^2/p^2)^{1/2}] \quad \frac{m_i c}{p} << 1 \right.$$
$$\left. \approx pc[1 + \tfrac{1}{2}m_2^2c^2/p^2 - 1 - \tfrac{1}{2}m_1^2c^2/p^2] = c^3\Delta(m^2)/2p \right)$$

$$\approx \sin^2 2\theta\,\sin^2[c^2\Delta(m^2)L/4\hbar p]$$

18. Let the amplitudes for states 1 and 2 be C_1 and C_2. Then

$$i\frac{dC_1}{dt} = H_{11}C_1 + H_{12}C_2 = E_1C_1 + V_{12}C_2,$$

$$i\frac{dC_2}{dt} = H_{21}C_1 + H_{22}C_2 = V_{12}^*C_1 + E_2C_2.$$

If $C_1 = A_1e^{-iwt}$ and $C_2 = A_2e^{-iwt}$, one has

$$A_1(W - E_1) - V_{12}A_2 = 0, \qquad A_1V_{12}^* + (E_2 - W)A_2 = 0.$$

Self-consistency requires

$$\frac{V_{12}}{W - E_1} = \frac{W - E_2}{V_{12}^*}, \quad \text{or} \quad W_\pm = \frac{(E_1 + E_2)}{2} \pm \frac{[(E_1 - E_2)^2 + 4|V_{12}|^2]^{1/2}}{2}.$$

Then, one may write

$$C_1 = A_1e^{-iw_+t} + B_1e^{-iw_-t}, \qquad C_2 = A_2e^{-iw_+t} + B_2e^{-iw_-t}.$$

The coefficients obey the constraints (not all of them independent):

$$\frac{A_1}{A_2} = \frac{2V_{12}}{(E_2 - E_1) + \sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}},$$

$$\frac{B_1}{B_2} = \frac{2V_{12}}{(E_2 - E_1) - \sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}},$$

and

$$A_1 + B_1 = 1, \qquad A_2 + B_2 = 0, \qquad A_1^2 + B_1^2 + A_2^2 + B_2^2 = 1,$$
$$A_1B_1 + A_2B_2 = 0.$$

The solutions are

$$A_1 = \frac{2|V_{12}|^2}{(E_1 - E_2)^2 + 4|V_{12}|^2 + (E_2 - E_1)\sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}},$$

$$B_1 = \frac{2|V_{12}|^2}{(E_1 - E_2)^2 + 4|V_{12}|^2 - (E_2 - E_1)\sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}},$$

$$A_2 = \frac{V_{12}^*}{\sqrt{(E_1 - E_2)^2 + 4|V_{12}|^2}} = -B_2.$$

35. The first-order Born-approximation scattering amplitude is

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r}) e^{i\mathbf{K}\cdot\mathbf{r}} d^3\mathbf{r},$$

where $\mathbf{K} = \mathbf{k}_i - \mathbf{k}_f$. Therefore

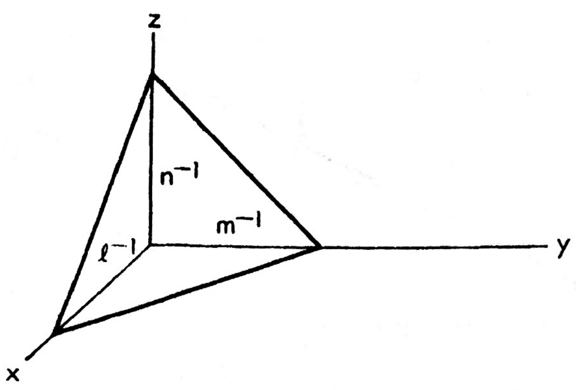
$$f(\theta, \phi) = a \sum_i \int \delta(\mathbf{r} - \mathbf{r}_i) e^{i\mathbf{K}\cdot\mathbf{r}} d^3\mathbf{r} = a \sum_i e^{i\mathbf{K}\cdot\mathbf{r}_i}.$$

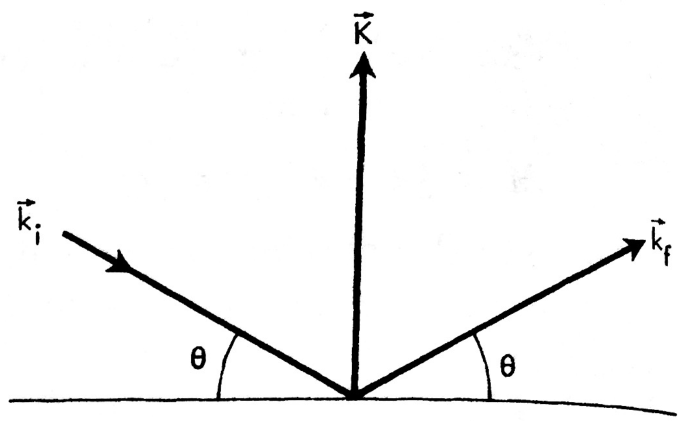
One obtains maximum scattering when the contributions from each lattice point are in phase. We choose a lattice point $\mathbf{r}_i = d(n_1\hat{\mathbf{x}} + n_2\hat{\mathbf{y}} + n_3\hat{\mathbf{z}})$ where the n_i are integers, and this condition becomes

$$d\mathbf{K}\cdot\hat{\mathbf{x}} = 2\pi l; \qquad d\mathbf{K}\cdot\hat{\mathbf{y}} = 2\pi m; \qquad \text{and} \qquad d\mathbf{K}\cdot\hat{\mathbf{z}} = 2\pi n,$$

where (l, m, n) are integers (the so-called Miller indices). Thus $\mathbf{K}$ is normal to the set of lattice planes defined by (lmn) (see the figure on p. 174). The magnitude of K then satisfies

$$Kd = 2\pi(l^2 + m^2 + n^2)^{1/2}. \tag{1}$$





In terms of the scattering angle from the reflection planes,

$$K^2 = (\mathbf{k}_i - \mathbf{k}_f)^2 = 2k^2(1 - \cos 2\theta) = 4k^2 \sin^2\theta,$$

Eq. (1) becomes $kd \sin\theta = \pi(l^2 + m^2 + n^2)^{1/2}$, which is the Bragg condition for reflection from a set of planes of spacing $d(l^2 + m^2 + n^2)^{-1/2}$.

Solution 3.3. a) The general form of the Hamiltonian in the presence of an electromagnetic field is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi, \tag{12.31}$$

where **A** and ϕ are the vector and scalar potentials. A convenient choice of gauge which minimizes cross terms in (12.31) is

$$\mathbf{A} = (Bz, 0, 0), \quad \phi = -Ez, \tag{12.32}$$

where we can check that $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi$. The Schrödinger equation in this gauge is

$$H\psi = \left[\frac{1}{2m}\left\{\left(p_x - \frac{e}{c}Bz\right)^2 + p_y^2 + p_z^2\right\} - eEz\right]\psi = \mathcal{E}\psi, \tag{12.33}$$

with $\mathcal{E}$ the energy.

b) To separate variables, we note that equation (12.33) has no terms involving either x or y, which suggests a simple solution for these two variables. It is easy to check that the solutions in the x- and y-directions are plane waves, so we write

$$\psi(x,y,z) = e^{ik_x x + ik_y y}\phi(z). \tag{12.34}$$

Substituting this into the Schrödinger equation gives

$$\left[\frac{1}{2m}\left\{\left(\hbar k_x - \frac{e}{c}Bz\right)^2 + \hbar^2 k_y^2 + p_z^2\right\} - eEz\right]\phi = \mathcal{E}\phi, \tag{12.35}$$

which is a one-dimensional problem.

c) Rearranging terms and collecting the constants into $\mathcal{E}'$ gives us the equation

$$\left\{p_z^2 + \left(\frac{eB}{c}z - \hbar k_x - \frac{mEc}{B}\right)^2\right\}\phi = \mathcal{E}'\phi, \tag{12.36}$$

which we recognize to be that of a simple harmonic oscillator (centered around a point other than the origin). The expectation value of z in this case is simply the position z where the potential is a minimum:

$$\langle z \rangle = \frac{c}{eB}\left(\hbar k_x + \frac{mEc}{B}\right). \tag{12.37}$$

We want to find the expectation value of v_x. Using the standard result that $i\hbar v \equiv i\hbar\, dx/dt = [x, H]$ and the commutation relation $[x, p_x] = i\hbar$ we find

$$\langle v_x \rangle = \frac{1}{m}\left(\langle p_x \rangle - \frac{eB}{c}\langle z \rangle\right) = -\frac{Ec}{B}, \qquad (12.38)$$

where we have used $\langle p_x \rangle = \hbar k_x$.

We recognize this as the classical result, found by requiring that the total electromagnetic force on the particle in the x-direction vanish:

$$F_x = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \hat{\mathbf{x}} = 0. \qquad (12.39)$$

Solution 3.4. We can write the Hamiltonian as $H = H_0 + V'$, where

$$\begin{aligned} H_0 &= \frac{\mathbf{p}^2}{2m} + \frac{1}{2}k(x^2 + y^2 + z^2), \text{ and} \quad (12.40) \\ V' &= qAe^{-(t/\tau)^2} z, \quad (12.41) \end{aligned}$$

and where $V'(t)$ is assumed to be small.

It is usually easier to solve problems involving a simple harmonic oscillator potential using raising and lowering operators. We can write the unperturbed hamiltonian H_0 as

$$H_0 = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}), \quad (12.42)$$

where we have defined:

$$\begin{aligned} \omega &\equiv \sqrt{\frac{k}{m}}, \quad (12.43) \\ a_x^\dagger &\equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(x - \frac{i}{m\omega}p_x\right), \quad (12.44) \\ a_x &\equiv \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(x + \frac{i}{m\omega}p_x\right), \quad (12.45) \end{aligned}$$

with analogous definitions for a_y, $a_y^\dagger$, a_z, and $a_z^\dagger$. The operators a and $a^\dagger$ are the annihilation and creation operators from which we can form the number operator,

$$a^\dagger a\,|n\rangle = n\,|n\rangle, \tag{12.46}$$

where n is some integer. The eigenstates of H_0 are therefore

$$|\mathbf{n}\rangle = |n_x, n_y, n_z\rangle, \tag{12.47}$$

where n_x, n_y and n_z are integers. The energies are given by

$$H_0\,|n_x, n_y, n_z\rangle = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})\,|n_x, n_y, n_z\rangle. \tag{12.48}$$

Using the eigenstates of H_0 as a basis we can write an arbitrary wavefunction as

$$|\psi\rangle = \sum_{\mathbf{n}} c_{\mathbf{n}}(t)\,|\mathbf{n}\rangle e^{-iE_{\mathbf{n}}t/\hbar}, \tag{12.49}$$

where the $c_{\mathbf{n}}(t)$ are complex coefficients. If the initial state at $t = -\infty$ is $|s\rangle$, then $c_s(-\infty) = 1$. According to time-dependent perturbation theory at $t = +\infty$, to first order in the perturbing potential V',

$$|c_{\mathbf{n}}(+\infty)|^2 = \frac{1}{\hbar^2}\left|\int_{-\infty}^{+\infty} V'_{\mathbf{ns}}(t')e^{i\omega_{\mathbf{ns}}t'}dt'\right|^2, \tag{12.50}$$

where $V'_{\mathbf{ns}} = \langle\mathbf{n}\,|V'|\,\mathbf{s}\rangle$, and $\omega_{\mathbf{ns}} = (E_{\mathbf{n}} - E_{\mathbf{s}})/\hbar$. This result is not hard to derive from Schrödinger's equation if we write the wavefunction in the form (12.49). In this problem, $\mathbf{s}$ labels the ground state: $\mathbf{s} = (0,0,0)$. Therefore the probability that the system is in any excited state at $t = +\infty$ is given by the sum

$$P = \sum_{\mathbf{n}\neq\mathbf{s}} |c_{\mathbf{n}}(+\infty)|^2. \tag{12.51}$$

To evaluate this sum, we need the matrix elements of $V'(t)$, which are

$$\langle n_x, n_y, n_z\,|V'(t)|\,0,0,0\rangle = qAe^{-(t/\tau)^2}\langle n_x, n_y, n_z\,|z|\,0,0,0\rangle. \tag{12.52}$$

Rewriting z in terms of the raising and lowering operators, we can see that V' only connects states whose values of n_z differ by one, so that

$$V'_{\mathbf{ns}} = qAe^{-(t/\tau)^2}\left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}\delta_{n_z,1}\,\delta_{n_x,0}\,\delta_{n_y,0}\,, \tag{12.53}$$

and only one term in the sum is nonzero. The desired probability is

$$P = \frac{1}{2\hbar m\omega} q^2 A^2 |I|^2, \tag{12.54}$$

with

$$I = \int_{-\infty}^{+\infty} e^{-(t/\tau)^2} e^{i\omega t} dt. \tag{12.55}$$

This integral can be evaluated by substituting $u = t/\tau$, completing the square in the exponent, and evaluating the resulting gaussian integral. This yields

$$P = \frac{q^2 A^2 \tau^2 \pi}{2m\omega\hbar} e^{-\omega^2\tau^2/2}. \tag{12.56}$$