1. *Quantum Mechanics* (Fall 2006)

(a) For a spherically symmetric potential, show that the radial part of a wave function obeys the radial Schrödinger equation

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{2mr^2} + V(r) - E \right) u_l(r) = 0$$

Assume that the potential $V(r)$ vanishes rapidly for large r and is less singular than $1/r^2$ for small r.

(b) Derive the behavior of $u_l(r)$ for $r \to 0$.

(c) Derive the behavior of $u_l(r)$ for large r ($r \to \infty$) when it describes a bound state.

2. *Quantum Mechanics* (Fall 2006)

The spin degree of freedom of a spin 1/2 particle with mass m can be described in a basis $|\pm\rangle$, where

$$\sigma_3 \left|+\right\rangle = + \left|+\right\rangle, \quad \sigma_3 \left|-\right\rangle = - \left|-\right\rangle,$$

and where σ_3 is the third Pauli matrix. The spin operator for a single fermion is $S_3 = \frac{\hbar}{2}\sigma_3$.

(a) Two identical fermions of spin 1/2 are initially assumed to be noninteracting. For this part of the problem take only the spin degrees of freedom into account. Construct the singlet state, i.e., the state for which the total spin of the two-fermion system satisfies $S_3 = 0$ and $S^2 = 0$.

Now consider that the two spin 1/2 fermions are both moving in a one dimensional infinite square well with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ -a & 0 < x < L \\ \infty & x > L \end{cases}$$

For the rest of the problem take both the spin degrees of freedom and the spatial wavefunction into account.

(b) What does the Fermi exclusion principle imply for the wavefunction of the two-fermion system? What does this imply for the spatial wavefunctions of the singlet state?

(c) Find the normalized wavefunction of the two-fermion system which has the lowest energy and is a singlet. Find the energy eigenvalue for this state.

(d) Now assume that there is a small interaction of the form

$$V_{\text{int}}(x_1, x_2) = -\alpha\, \delta(x_1 - x_2)$$

To lowest order in perturbation theory find the change in energy of the ground state due to the interaction.

3. *Quantum Mechanics* (Fall 2006)

Consider two flavours of massive neutrinos, denote $|\nu_e\rangle$ the electron neutrino flavour eigenstateand $|\nu_\mu\rangle$ the muon neutrino flavour eigenstate. These are related to the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ by

$$\begin{aligned} |\nu_e\rangle &= \cos(\theta)\ |\nu_1\rangle - \sin(\theta)\ |\nu_2\rangle \\ |\nu_\mu\rangle &= \sin(\theta)\ |\nu_1\rangle + \cos(\theta)\ |\nu_2\rangle \end{aligned}$$

(a) Show that flavour eigenstates and energy eigenstates are related by a unitary transformation.

(b) The energy of the eigenstate $|\nu_i\rangle$ is

$$E_i = \sqrt{\mathbf{p}^2 c^2 + {m_i}^2 c^4}, \quad i = 1, 2$$

Assume that an electron neutrino is produced in the sun with momentum $\mathbf{p}$ such that $|\mathbf{p}| \gg m_i c$. Find the probability for the electron neutrino to oscillate into a muon neutrino after travelling a distance L.

4. *Quantum Mechanics* (Fall 2006)

Consider a quantum mechanical system with Hamiltonian

$$H = a^\dagger a$$

Where a and $a^\dagger$ are operators satisfying the following relations

$$a^2 = 0, \quad (a^\dagger)^2 = 0 \quad a^\dagger a + a a^\dagger = 1$$

(a) Show that the Hamiltonian satisfies

$$H^2 = H$$

(b) Find the eigenvalues of the Hamiltonian H.

(c) If $|0\rangle$ is the **unique** normalized ground state of the system (i.e., the state with the lowest energy eigenvalue) find

$$a\,|0\rangle = \ ?$$

Under the assumption above, what dimension can the complete Hilbert space of states have?

5. *Quantum Mechanics* (Fall 2006)

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r}$$

(a) Imagine you could find the solution $\psi(\mathbf{r})$ to the time-independent Schrödinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\mathbf{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

(b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

(c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

6. *Statistical Mechanics and Thermodynamics* (Fall 2006)

Consider a set of spin one-half particles in a magnetic field, $\mathbf{H}$, oriented in the z direction. There is no interaction between the particles, and the energy of interaction of particles with this magnetic field is given by

$$E_H = -\mu_{\mathrm{B}} \sum_{i=1}^{N} s_i H$$

where μ_{B} is proportional to a Bohr magneton, and the quantities s_i can take on the values ± 1. We will forget about the g factor for the moment. The Boltzmann factor governing the equilibrium statistics of these moments is $e^{-\beta E_H}$, where $\beta = 1/k_{\mathrm{B}}T$.

(a) Obtain an expression for the total magnetic moment of this system of moments in the presence of the external field.

(b) The specific heat of this system in the absence of an externally applied field is given by

$$C_H|_{H=0} = AT^2$$

Use this information and the solution to part a) of the question to obtain an expression for the magnetic Gibbs potential, $G(T, H, N)$ of this system of spins. This expression may contain undetermined coefficients.

(c) From your result above, find the condition that applies when an external magnetic field is introduced or removed *adiabatically*.

7. *Statistical Mechanics and Thermodynamics* (Fall 2006)

Consider a gas of noninteracting particles for which the kinetic energy of each has the following dependence on momentum

$$E(\mathbf{p}) = |\mathbf{p}|\, c$$

These particles obey Boltzmann statistics. There are N of them, occupying a volume V.

(a) Find the partition function of this system of particles.

(b) What is the heat capacity at constant pressure of this system of particles?

8. *Statistical Mechanics and Thermodynamics* (Fall 2006)

(a) Consider a grand canonical ensemble of particles, at fixed temperature T and in a container of volume V. Show that the mean square fluctuation in the number of particles $\overline{(\Delta N)^2}$ is:

$$\overline{(\Delta N)^2} = k_{\mathrm{B}} T \frac{\partial \bar{N}}{\partial \mu}.$$

(b) Using the relation:

$$S\mathrm{d}T - V\mathrm{d}p + N\mathrm{d}\mu = 0$$

express the solution in terms of $(\partial\rho/\partial p)_{T,V}$ where $p =$ pressure and $\rho = N/V$ is the density of the system.

(c) Since intensive quantities are independent of extensive quantities by definition, we can change external constraints to obtain:

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,V} = \left(\frac{\partial \rho}{\partial p}\right)_{T,N}.$$

Using this relation, find an expression for $\overline{(\Delta N)^2}$ in terms of the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\partial V/\partial p\right)_{T,N}$.

9. *Statistical Mechanics and Thermodynamics* (Fall 2006)

Consider an idealized "white dwarf" star made up of ionized helium only. We make several simplifying assumptions, namely:

- there is no radiation pressure
- the electrons form a completely degenerate (i.e., "$T = 0$"), ultrarelativistic ("$m_{\rm e} = 0$") Fermi gas
- the density ρ is uniform.

Set up the condition for mechanical equilibrium of the star under the opposing influences of the gravitational force and the pressure of the Fermi gas. You will find that, with these approximations, equilibrium is possble for only one particular value of the mass of the star, M (this mass ic called the "Chandrasekhar limit"). Give the value of M in terms of fundamental constants.

(Note: This calculation, with $m_{\rm e} = 0$, overestimates the pressure; in reality equilibrium is possible for masses smaller and up to the Chandrasekhar limit.)

10. *Electricity and Magnetism* (Fall 2006)

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

(a) What is the total charge of this system? Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

(b) Identify any point charges in this system and give their location and charge.

(c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

11. *Electricity and Magnetism* (Fall 2006)

Consider an electromagnetic wave incident from vacuum onto a dielectric with a dielectric constant, ϵ. The surface normal lies along the $\hat{\mathbf{z}}$ axis.

(a) Derive the reflection and transmission coefficients if the wave is incident along the $\hat{\mathbf{z}}$ direction.

(b) Derive the reflection and transmission coefficients if the wave is incident in the $\hat{\mathbf{x}}\hat{\mathbf{z}}$ plane with an angle θ_i with respect to the surface normal and the electric field is in the y direction. Is there an angle for which there is no reflected energy?

12. *Electricity and Magnetism* (Fall 2006)

Consider a single electron interacting with electric and magnetic fields obtained from the corresponding scalar and vector potentials.

(a) If the fields do not depend explicitly on time then the energy is conserved. Start from the equation for the time rate of change of energy for a single charged particle and derive the relativistically correct expression for the energy.

(b) Consider a one-dimensional problem where the fields only depend on one spatial variable. Suppose the fields are described by a scalar potential of the form $\phi = \phi_0 \cos(kz - \omega t)$. What is the constant of the motion in the laboratory frame now?

Hint: Take a linear combination of the conservation of energy equation and the conservation of momentum equation. Use this constant to determine how large ϕ_0 must be in order that an electron that starts from rest at $z = 0$ at $t = 0$ is trapped by the wave and to determine the maximum energy that the electron can obtain.

(c) Consider a fully three-dimensional case. If both the scalar and vector potential are functions of $(x, y, z - v_\phi t)$, where v_ϕ is the phase velocity, then the energy is no longer a constant. What is the new constant?

Hint: Take a linear combination of the conservation of energy equation and the component of the conservation of momentum equation in the $\hat{\mathbf{z}}$ direction.

13. *Electricity and Magnetism* (Fall 2006)

Consider a charge q moving on a circle of radius a (centered at the origin) on the xy-plane, with constant angular velocity ω.

(a) In the dipole approximation, calculate the power radiated per unit solid angle in the direction defined by the polar angle θ (i.e., θ is the angle with the z-axis).

(b) Still in the dipole approximation, what is the state of polarization of the radiation emitted in the direction $\theta = 0$? And $\theta = \pi/2$?

(c) Going beyond the dipole approximation, show that radiation is emitted also at frequencies other than ω (what frequencies?).

You may want to follow the steps below:

- show that if $\rho(\mathbf{x}, t)$ is periodic in time (but not necessarily of the form $\rho(\mathbf{x})\, e^{-i\omega t}$) with period $T = 2\pi/\omega$, then one can write

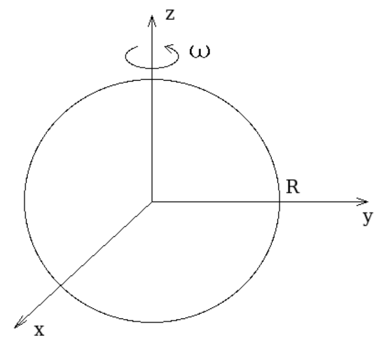
$$\rho(x,t) = \frac{1}{2}\rho_0(x) + \sum_{n=1}^{\infty} \mathrm{Re}\left[\rho_n(x)\, e^{-in\omega t}\right] \qquad \text{where} \quad \rho_n(x) = \frac{2}{T}\int_0^T \mathrm{d}t\ \rho(x,t)\, e^{in\omega t} \quad (n \geq 1)$$

- recall that the multipole moments are $q_{lm} = \int \mathrm{d}^3x\ Y_{lm}(\theta,\phi)\, r^l\, \rho(x)$

Write $\rho(\mathbf{x})$ in spherical coordinates for this problem, and using the expression above, find the frequencies at which the different multipole terms radiate.

14. *Electricity and Magnetism* (Fall 2006)

Consider a rotating sphere with radius R. A charge Q is distributed homogeneously over the sphere. The sphere rotates counter-clockwise around the z-axis with angular velocity ω. (See figure below.)



(a) Find the charge density ρ and the current density $\mathbf{j}$ in terms of delta functions. Show that $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$.

(b) Find the vector potential $\mathbf{A}(\mathbf{x})$ in the Coulomb gauge ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$).

Hint: To do the integral it is advantageous to choose $\mathbf{r} = r\mathbf{e}_z$ and choose $\boldsymbol{\omega}$ to be arbitrary.

(c) Calculate the magnetic field $\mathbf{B}$ from the vector potential $\mathbf{A}$.

1. *Quantum Mechanics* (Fall 2006)

(a) For a spherically symmetric potential, show that the radial part of a wave function obeys the radial Schrödinger equation

$$\left(-\frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{2mr^2}+V(r)-E\right)u_l(r)=0$$

Assume that the potential $V(r)$ vanishes rapidly for large r and is less singular than $1/r^2$ for small r.

(b) Derive the behavior of $u_l(r)$ for $r \to 0$.

(c) Derive the behavior of $u_l(r)$ for large r ($r \to \infty$) when it describes a bound state.

a) $\hat{H}\Psi = \hat{E}\Psi \qquad \Psi = \Psi(\vec{r}, t)$

$$\left[\frac{\hat{p}^2}{2m} + \hat{V}\right]\Psi = \hat{E}\Psi$$

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\Psi = i\hbar\,\partial_t\,\Psi$$

Assume $\Psi(\vec{r},t) = \psi(\vec{r})\,\tau(t) = R(r)\,Y(\theta,\phi)\,\tau(t)$

where a general solution is a linear combination (add, integrate) of terms of this type

$$\frac{\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\psi(\vec{r})}{\psi(\vec{r})} = \frac{i\hbar\, d_t\,\tau(t)}{\tau(t)} \equiv E$$

$$\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\vec{\nabla}^2 = \left[\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right\}\right]$$

$$\equiv \left[\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{L^2(\theta,\phi)}{r^2}\right]$$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}\left\{\partial_r(r^2\partial_r) - L^2(\theta,\phi)\right\} + V(r)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$\frac{\left[-\frac{\hbar^2}{2m}d_r(r^2 d_r) + r^2V(r) - r^2E\right]R(r)}{R(r)} = -\frac{\frac{\hbar^2}{2m}L^2(\theta,\phi)\,Y(\theta,\phi)}{Y(\theta,\phi)} = -\frac{\hbar^2\,l(l+1)}{2m},$$

using the lemma that $L^2(\theta,\phi)\,Y(\theta,\phi) = l(l+1)\,Y(\theta,\phi)$, $l \in \mathbb{N} = \{0,1,2,\dots\}$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{r^2}d_r(r^2 d_r) + \frac{\hbar^2\,l(l+1)}{2mr^2} + V(r) - E\right]R_l(r) = 0$$

multiplying by $r \Rightarrow \left[-\frac{\hbar^2}{2m}\frac{1}{r}d_r(r^2 d_r) + \left\{\frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right\}r\right]R_l(r) = 0$

and defining $u_l(r) = r\,R_l(r)$ so that

$d_r^2 u_l = d_r^2(rR_l) = d_r(R_l + r\,d_rR_l) = 2\,d_rR_l + r\,d_r^2R_l$ and

$\frac{1}{r}d_r(r^2 d_r)R_l = \frac{1}{r}(2r\,d_rR_l + r^2 d_r^2 R_l) = 2\,d_rR_l + r\,d_r^2R_l,$ we have

$$\left[-\frac{\hbar^2}{2m}d_r^2 + \frac{\hbar^2\,l(l+1)}{2mr^2} + V(r) - E\right]u_l(r) = 0 \qquad \checkmark$$

1. Quantum Mechanics (Fall 2006)

b) For $r \to 0$, the equation is dominated by the $\frac{\hbar^2 \ell(\ell+1)}{r^2}$ term (since $V(r)$ is less singular than $1/r^2$ for small r):

$$\frac{\hbar^2}{2m}\left[-d_r^{\,2} + \frac{\ell(\ell+1)}{r^2}\right] u_\ell(r) = 0$$

Assuming $u_\ell(r) = \sum_{k=1}^{\infty} a_k r^k$ (there can be no nonpositive powers since $u_\ell(0) = 0 R_\ell(0) = 0$), we have

$$\left[-\sum_{k=}^{\infty} k(k-1) a_k r^{k-2} + \ell(\ell+1) \sum_{k=}^{\infty} a_k r^{k-2}\right] = 0$$

$\Rightarrow \quad k(k-1) = \ell(\ell+1) \quad \Rightarrow \quad k = \ell+1, -\ell$

but $-\ell$ is not possible since $\ell \in \mathbb{N}$ and $k \geq 1$

$\Rightarrow \quad u_\ell(r) \propto r^{\ell+1}$ asymptotically as $r \to 0$

c) For $r \to \infty$ and $E < 0$ (bound states), the $\frac{\hbar^2 \ell(\ell+1)}{r^2}$ term and $V(r)$ vanish (since $V(r)$ vanishes rapidly for large r):

$$\left[-\frac{\hbar^2}{2m} d_r^{\,2} - E\right] u_\ell(r) = 0$$

$\Rightarrow \quad d_r^{\,2} u_\ell(r) = -\frac{2mE}{\hbar^2} u_\ell(r) \equiv k^2 u_\ell(r)$ for $k = \sqrt{-\frac{2mE}{\hbar^2}} \in \mathbb{R}$

$\Rightarrow \quad u_\ell(r) \propto e^{kr}$ asymptotically as $r \to \infty$

2. *Quantum Mechanics* (Fall 2006)

The spin degree of freedom of a spin 1/2 particle with mass m can be described in a basis $|\pm\rangle$, where

$$\sigma_3 |+\rangle = +|+\rangle, \quad \sigma_3 |-\rangle = -|-\rangle,$$

and where σ_3 is the third Pauli matrix. The spin operator for a single fermion is $S_3 = \frac{\hbar}{2}\sigma_3$.

(a) Two identical fermions of spin 1/2 are initially assumed to be noninteracting. For this part of the problem take only the spin degrees of freedom into account. Construct the singlet state, i.e., the state for which the total spin of the two fermion system satisfies $S_3 = 0$ and $S^2 = 0$.

Now consider that the two spin 1/2 fermions are both moving in a one dimensional infinite square well with potential

$$V(x) = \begin{cases} \infty & x < 0 \\ -a & 0 < x < L \\ \infty & x > L \end{cases}$$

For the rest of the problem take both the spin degrees of freedom and the spatial wavefunction into account.

(b) What does the Fermi exclusion principle imply for the wavefunction of the two fermion system? What does this imply for the spatial wavefunctions of the singlet state?

(c) Find the normalized wavefunction of the two fermion system which has the lowest energy and is a singlet. Find the energy eigenvalue for this state.

(d) Now assume that there is a small interaction of the form

$$V_{\text{int}}(x_1, x_2) = -\alpha\, \delta(x_1 - x_2)$$

To lowest order in perturbation theory find the change in energy of the ground state due to the interaction.

a) $|m_1, m_2\rangle \rightarrow |j, m\rangle'$ $\quad m = m_1 + m_2 \quad -j \le m \le j$

$|0,0\rangle' = ?$

$|1,1\rangle' = |++\rangle$ by necessity

$|1,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$ using $J_- = J_{1-} + J_{2-}$ $\ast$

$\boxed{|0,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)}$ by orthogonality to $|1,0\rangle'$ $\circledast$

The same can be found using $|1,-1\rangle' = |--\rangle$ and $J_+ = J_{1+} + J_{2+}$

$\ast$ $\quad J_-|1,1\rangle' = \sqrt{1(1+1) - 1(1-1)}\,\hbar\,|1,0\rangle' = \sqrt{2}\,\hbar\,|1,0\rangle'$

$= (J_{1-} + J_{2-})|++\rangle = \sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\,\hbar\,(|-+\rangle + |+-\rangle)$

$= \hbar(|+-\rangle + |-+\rangle)$

$\Rightarrow |1,0\rangle' = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$

2. Quantum Mechanics (Fall 2006)

a) (continued)

⊗ and noting that $|1,0\rangle'$ must incorporate both indistinguishable states $|+-\rangle$ and $|-+\rangle$

b) 2 identical spin-1/2 particles

$\Rightarrow$ fermionic odd (antisymmetric) exchange parity

$$P_x \Upsilon(x_1, x_2, m_1, m_2, t) = \boxed{\Upsilon(x_2, x_1, m_2, m_1, t) = -\Upsilon(x_1, x_2, m_1, m_2, t)}$$

$\Rightarrow$ Pauli exclusion principle ("Fermi exclusion principle"?)

$$H \neq H(\text{spin}) \Rightarrow \Upsilon(x_1, x_2, m_1, m_2, t) = \Psi(x_1, x_2, t)\,\chi(m_1, m_2) \quad \text{(separable)}$$

Singlet state $\chi(+,-) = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ (more properly $\chi(\{m_1, m_2\})$) is odd under P_x and Υ is odd under P_x,

so Ψ must be even under P_x: $\boxed{\Psi(x_2, x_1, t) = +\Psi(x_1, x_2, t)}$

c) $H \neq H(\text{interaction}, t) \Rightarrow \Psi(x_1, x_2, t) = \psi_1(x_1)\,\psi_2(x_2)\,\tau(t)$

$$H = H_1 + H_2 \qquad H_i = \vec{p}_i^{\,2}/2m + V(x_i) \qquad \tau = e^{-iEt/\hbar}$$

$$H_i\psi_i = \left[-\frac{\hbar^2}{2m} d_{x_i}^{\,2} - a\right]\psi_i = E_i\psi_i \quad \text{for } 0 \le x_i \le L, \quad E = E_1 + E_2$$

$$\Rightarrow \left[d_{x_i}^{\,2} + \frac{2m}{\hbar^2}(E_i + a)\right]\psi_i = 0 \quad \Rightarrow \quad \psi_i(x_i) = A_i \sin(k_i x_i) \text{ due to B.C.s}$$

where $k_i^{\,2} = \frac{2m}{\hbar^2}(E_i + a) = \left(\frac{n_i\pi}{L}\right)^2$

and $n_i \in \mathbb{Z}^+$

$$\int_0^L\int_0^L |\Psi(x_1, x_2, t)|^2\,dx_1\,dx_2 = 1 \quad \Rightarrow \quad \int_0^L |\psi_i(x_i)|^2\,dx_i = 1 \Rightarrow A_i = \sqrt{\frac{2}{L}}$$

$$\Rightarrow \Upsilon_{n_1 n_2}(x_1, x_2, m_1, m_2, t) = \left(\frac{2}{L}\right)\sin\left(\frac{n_1\pi}{L}x_1\right)\sin\left(\frac{n_2\pi}{L}x_2\right)e^{-iEt/\hbar}\,\chi(m_1, m_2)$$

$$E = E_{n_1 n_2} = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2(n_1^{\,2} + n_2^{\,2}) - 2a$$

(in full detail, you'll find $E_1 = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2 n_1^{\,2} - a - U$ and $E_2 = \frac{1}{2m}\left(\frac{\hbar\pi}{L}\right)^2 n_2^{\,2} - a + U$, where U is arbitrary)

2. Quantum Mechanics (Fall 2006)

c) (continued)

Lowest singlet energy eigenstate

$\Rightarrow$ spatially P_x-even state with lowest $E_{n_1 n_2}$

$\Rightarrow$ $n_1 = n_2$ and $n_1 = n_2 = 1$

$$\Rightarrow \Upsilon_{11}(x_1, x_2, +, -, t) = \left(\frac{2}{L}\right) \sin\left(\frac{\pi}{L}x_1\right) \sin\left(\frac{\pi}{L}x_2\right) e^{-iE_{11}t/\hbar} \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

with $E_{11} = \frac{1}{m}\left(\frac{\hbar\pi}{L}\right)^2 - 2a$

d) $$\Delta E^{(1)} = \langle \Upsilon | V_{int} | \Upsilon \rangle = -\alpha \int_0^L \int_0^L \left(\frac{2}{L}\right)^2 \sin^2\left(\frac{\pi}{L}x_1\right) \sin^2\left(\frac{\pi}{L}x_2\right) \delta(x_1 - x_2)\, dx_1\, dx_2$$

$$= -\frac{4\alpha}{L^2} \int_0^L \sin^4\left(\frac{\pi}{L}x\right) dx \quad \text{where} \quad \sin^4\theta = \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)^2 = \frac{1}{4} - \cos 2\theta + \frac{1}{4}\cos^2 2\theta$$

$$= -\frac{4\alpha}{L^2} \int_0^L \left[\frac{1}{4} - \cos\left(\frac{2\pi}{L}x\right) + \frac{1}{4}\cos^2\left(\frac{2\pi}{L}x\right)\right] dx$$

$$= -\frac{4\alpha}{L^2}\left[\frac{L}{4} - \left[\cancelto{0}{\frac{L}{2\pi}\sin\left(\frac{2\pi}{L}x\right)}\right]_0^L + \frac{1}{4}\left(\frac{L}{2}\right)\right] = -\frac{4\alpha}{L^2}\frac{L}{8} = -\frac{\alpha}{2L}$$

3. *Quantum Mechanics* (Fall 2006)

Consider two flavours of massive neutrinos, denote $|\nu_e\rangle$ the electron neutrino flavour eigenstateand $|\nu_\mu\rangle$ the muon neutrino flavour eigenstate. These are related to the energy eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$ by

$$\begin{aligned} |\nu_e\rangle &= \cos(\theta)\,|\nu_1\rangle - \sin(\theta)\,|\nu_2\rangle \\ |\nu_\mu\rangle &= \sin(\theta)\,|\nu_1\rangle + \cos(\theta)\,|\nu_2\rangle \end{aligned}$$

(a) Show that flavour eigenstates and energy eigenstates are related by a unitary transformation.

(b) The energy of the eigenstate $|\nu_i\rangle$ is

$$E_i = \sqrt{\mathbf{p}^2c^2 + m_i{}^2c^4}, \quad i = 1, 2$$

Assume that an electron neutrino is produced in the sun with momentum $\mathbf{p}$ such that $|\mathbf{p}| \gg m_i c$. Find the probability for the electron neutrino to oscillate into a muon neutrino after travelling a distance L.

a) $\vec{\Psi}_f = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \vec{\Psi}_e$ $\qquad |\Psi\rangle = \vec{\Psi}_f \cdot |\vec{\beta}_f\rangle = \Psi_{f1}|\nu_e\rangle + \Psi_{f2}|\nu_\mu\rangle$

$= U\vec{\Psi}_e$ $\qquad = \vec{\Psi}_e \cdot |\vec{\beta}_e\rangle = \Psi_{e1}|\nu_1\rangle + \Psi_{e2}|\nu_2\rangle$

where, e.g., $\vec{\Psi}_f = \begin{pmatrix} \Psi_{f1} \\ \Psi_{f2} \end{pmatrix}$ $\quad |\vec{\beta}_f\rangle = \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}$

$U^\dagger = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow U^\dagger U = \begin{pmatrix} C^2+S^2 & -SC+CS \\ -CS+SC & -(-S^2)+C^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\Rightarrow U^\dagger = U^{-1} \Rightarrow$ U is unitary

b) $|\Psi_0\rangle = |\nu_e\rangle$ with $|\vec{p}| \gg m_1c, m_2c$, traveling $L \Rightarrow \Delta t \approx \frac{L}{c} \equiv \tau$

$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi_0\rangle = e^{-i\omega_1 t}\cos\theta|\nu_1\rangle - e^{-i\omega_2 t}\sin\theta|\nu_2\rangle$ where $\omega_i \equiv \frac{E_i}{\hbar}$

$P_{\nu_e\to\nu_\mu}(L) = |\langle\nu_\mu|\Psi(\Delta t)\rangle|^2$

$\langle\nu_\mu|\Psi(t)\rangle = (\sin\theta \;\; \cos\theta)\begin{pmatrix} e^{-i\omega_1 t}\cos\theta \\ -e^{-i\omega_2 t}\sin\theta \end{pmatrix} = (e^{-i\omega_1 t} - e^{-i\omega_2 t})\sin\theta\cos\theta$

$= \frac{1}{2}\sin 2\theta\,(e^{-i\omega_1 t} - e^{-i\omega_2 t})$

$P_{\nu_e\to\nu_\mu}(L) = \frac{1}{4}\sin^2 2\theta\left[1 - e^{i(\omega_2-\omega_1)\tau} - e^{-i(\omega_2-\omega_1)\tau} + 1\right] = \frac{1}{4}\sin^2 2\theta\left[2 - 2\cos(\omega_2-\omega_1)\tau\right]$

$= \sin^2 2\theta\left(\frac{1}{2} - \frac{1}{2}\cos(\omega_2-\omega_1)\tau\right) = \sin^2 2\theta\,\sin^2\left(\frac{1}{2}(\omega_2-\omega_1)\tau\right)$

$\Big(E_2 - E_1 = \sqrt{p^2c^2+m_2^2c^4} - \sqrt{p^2c^2+m_1^2c^4} = pc\left[(1+m_2^2c^2/p^2)^{1/2} - (1+m_1^2c^2/p^2)^{1/2}\right] \quad \frac{m_i c}{p} \ll 1$

$\approx pc\left[1 + \frac{1}{2}m_2^2c^2/p^2 - 1 - \frac{1}{2}m_1^2c^2/p^2\right] = c^3\Delta(m^2)/2p\Big)$

$\approx \sin^2 2\theta\,\sin^2\left[c^2\Delta(m^2)L/4\hbar p\right]$

4. *Quantum Mechanics* (Fall 2006)

Consider a quantum mechanical system with Hamiltonian

$$H = a^\dagger a$$

Where a and $a^\dagger$ are operators satisfying the following relations

$$a^2 = 0, \quad (a^\dagger)^2 = 0 \quad a^\dagger a + a a^\dagger = 1$$

(a) Show that the Hamiltonian satisfies

$$H^2 = H$$

(b) Find the eigenvalues of the Hamiltonian H.

(c) If $|0\rangle$ is the **unique** normalized ground state of the system (i.e., the state with the lowest energy eigenvalue) find

$$a|0\rangle = \ ?$$

Under the assumption above, what dimension can the complete Hilbert space of states have?

a) $H^2 = (a^+a)^2 = a^+a\,a^+a = a^+(1-a^+a)a = a^+a - \cancel{a^{+2}}\cancel{a^2} = a^+a = H$

b) $H|E\rangle = E|E\rangle \Rightarrow H^2|E\rangle = E^2|E\rangle$

$= H|E\rangle = E|E\rangle$

$\Rightarrow E^2 = E \Rightarrow E \in \{0, 1\}$

c) $H|0\rangle = E_0|0\rangle = 0|0\rangle = 0 \qquad H|1\rangle = E_1|1\rangle = |1\rangle$

$H(a|0\rangle) = a^+a\,a|0\rangle = a^+\cancel{a^2}|0\rangle = 0 \Rightarrow$ either $a|0\rangle = 0$ or $a|0\rangle \propto |0\rangle$

$Ha = a^+a\,a = a^+\cancel{a^2} = 0 = q(0) = qHa$ with q arbitrary

$[H, a] = Ha - aH = \cancel{a^+a^2} - a a^+ a = -a(1 - a a^+) = -a + \cancel{a^2}a^+ = -a$

$H(a|0\rangle) = (Ha)|0\rangle = q(Ha)|0\rangle = q([H,a] + aH)|0\rangle = q(-a + a\cancel{\emptyset})|0\rangle$

$= -q(a|0\rangle) = 0$

with q arbitrary

pick $q \neq 0 \Rightarrow a|0\rangle = 0$

The complete Hilbert space of states could be 2D (if the E=1 state is nondegenerate) or (n+1)D (if the E=1 state is degenerate with multiplicity n).

5. *Quantum Mechanics* (Fall 2006)

A neutron (mass M) scatters off a very heavy nucleus, and the force between them is given by a Yukawa potential:

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r}$$

(a) Imagine you could find the solution $\psi(\mathbf{r})$ to the time-independent Schrödinger equation (with an incident wave in the $+z$ direction) with this potential for positive energy E. Write a formula for the scattering amplitude in terms of this wave function. Don't try to calculate $\psi(\mathbf{r})$. Define any symbols you introduce, other than those in $V(r)$ above and natural constants.

(b) What is the first Born approximation to the scattering amplitude $f(\theta, \phi)$?

(c) What is the total cross section in the limit that the scattering neutron has zero kinetic energy?

a) Scattering off heavy nucleus $\Rightarrow$ elastic scattering off fixed scatterer/potential $(k' = k)$

singular potential (at origin) $\Rightarrow$ strong potential

$\Rightarrow$ (I would think) partial wave analysis rather than the Born approximation, but perhaps the singularity is "small enough" to use the Born approximation

$$f(\theta,\phi) = -\frac{(2\pi)^{3/2}}{4\pi}\frac{2M}{\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}'}\, V(\vec{r}')\, \psi(\vec{r}')\, d^3r'$$

where $\vec{k}' = k\hat{r}$ is the scattering wave vector and $k = |\vec{k}| = |k\hat{z}|$ is the incident wave number

b) Born approx: $\psi(\vec{r}) \approx \psi_0(\vec{r}) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}} = (2\pi)^{-3/2} e^{ikz}$

$$f^{(1)}(\theta,\phi) = -\frac{1}{4\pi}\frac{2M}{\hbar^2}\int e^{-i\vec{k}'\cdot\vec{r}'}\, V(r')\, e^{i\vec{k}\cdot\vec{r}'}\, d^3r'$$

$$= -\frac{2M}{\hbar^2}\frac{1}{q}\int_0^\infty r' V(r') \sin(qr')\, dr'$$

(by spherical symmetry of potential... skipped steps)

$\vec{q} \equiv \vec{k} - \vec{k}'$ $\quad q = |\vec{k} - \vec{k}'| = 2k\sin\frac{\theta}{2}$

$$= -\frac{2M}{\hbar^2 q}\int_0^\infty \frac{V_0}{\mu} e^{-\mu r'} \sin(qr')\, dr'$$

$$= -\frac{2MV_0}{\hbar^2 \mu q}\int_0^\infty \left[qr' - \frac{1}{3!}q^3 r'^3 + \frac{1}{5!}q^5 r'^5 - \dots\right] e^{-\mu r'}\, dr'$$

5. Quantum Mechanics (Fall 2006)

b) (continued)

$$= -\frac{2MV_0}{\hbar^2 \mu q}\left[q\left(\frac{1}{\mu}\right)^2 - \frac{1}{3!} q^3 \left(\frac{1}{\mu}\right)^4 3! + \frac{1}{5!} q^5 \left(\frac{1}{\mu}\right)^6 5! - \cdots \right]$$

since $\int_0^\infty x^n e^{-x/a} dx = a^{n+1}\Gamma(n+1) = a^{n+1} n!$

$$= -\frac{2MV_0}{\hbar^2 \mu q}\frac{1}{q}\left[\left(\frac{q}{\mu}\right)^2 - \left(\frac{q}{\mu}\right)^4 + \left(\frac{q}{\mu}\right)^6 - \cdots \right] = +\frac{2MV_0}{\hbar^2 \mu q^2}\sum_{n=1}^{\infty}\left[-\left(\frac{q}{\mu}\right)^2\right]^n$$

$$= \frac{2MV_0}{\hbar^2 \mu q^2}\frac{-\left(\frac{q}{\mu}\right)^2}{1-\left[-\left(\frac{q}{\mu}\right)^2\right]} \qquad \text{assuming } \frac{q}{\mu} < 1 \Rightarrow 2k\sin\frac{\theta}{2} < \mu \Rightarrow k < \frac{\mu}{2}$$

$$= -\frac{2MV_0}{\hbar^2 \mu q^2}\frac{\left(\frac{q}{\mu}\right)^2}{1+\left(\frac{q}{\mu}\right)^2} = -\frac{2MV_0}{\hbar^2 \mu q^2}\frac{q^2}{\mu^2+q^2} = -\frac{2MV_0}{\hbar^2 \mu}\frac{1}{\mu^2+q^2}$$

$$\Rightarrow f^{(1)}(\theta,\phi) = f^{(1)}(\theta) = -\frac{2MV_0}{\hbar^2 \mu}\frac{1}{\mu^2 + 4k^2\sin^2\frac{\theta}{2}}$$

c) $E \to 0 \qquad E = \frac{p^2}{2M} = \frac{\hbar^2 k^2}{2M} \Rightarrow k \to 0 \quad (\lambda \to \infty)$

$$\Rightarrow f^{(1)}(\theta) \to -\frac{2MV_0}{\hbar^2 \mu^3} = -f_0$$

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta,\phi)|^2 d\Omega \to \int |f_0|^2 d\Omega = 4\pi |f_0|^2$$

$$\sigma_{tot} \to 4\pi\, 4\left(\frac{MV_0}{\hbar^2\mu^3}\right)^2 = \pi\left(\frac{4MV_0}{\hbar^2\mu^3}\right)^2$$

8. *Statistical Mechanics and Thermodynamics* (Fall 2006)

(a) Consider a grand canonical ensemble of particles, at fixed temperature T and in a container of volume V. Show that the mean square fluctuation in the number of particles $\overline{(\Delta N)^2}$ is:

$$\overline{(\Delta N)^2} = k_B T \frac{\partial \bar{N}}{\partial \mu}.$$

(b) Using the relation:

$$S\mathrm{d}T - V\mathrm{d}p + N\mathrm{d}\mu = 0 \qquad (1)$$

express the solution in terms of $\left(\frac{\partial \rho}{\partial p}\right)_{T,V}$ where $p =$ pressure and $\rho = \frac{N}{V}$ is the density of the system.

(c) Since intensive quantities are independent of extensive quantities by definitino, we can change external constraints to obtain:

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,V} = \left(\frac{\partial \rho}{\partial p}\right)_{T,N}.$$

Using this relation, find an expression for $\overline{(\Delta N)^2}$ in terms of the isothermal compressibility $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N}$.

a) $\overline{(\Delta N)^2} = \overline{N^2} - \bar{N}^2$

Grand partition function: $\mathcal{Z} = \sum_N Z(N) e^{\beta \mu N}$ where $Z(N) = \sum_{R(N)} e^{-\beta E_{R(N)}}$

$\Rightarrow \mathcal{Z} = \sum_N \sum_{R(N)} e^{-\beta(E_{R(N)} - \mu N)} = \sum_i e^{-\beta(E_i - \mu N_i)}$

$$\bar{N} = \frac{\sum_i N_i e^{-\beta(E_i - \mu N_i)}}{\mathcal{Z}} = \frac{1}{\beta}\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \mu}; \quad \overline{N^2} = \frac{\sum_i N_i^2 e^{-\beta(E_i - \mu N_i)}}{\mathcal{Z}} = \frac{1}{\beta^2}\frac{1}{\mathcal{Z}}\frac{\partial^2 \mathcal{Z}}{\partial \mu^2}$$

$$kT\frac{\partial \bar{N}}{\partial \mu} = \frac{1}{\beta}\frac{\partial}{\partial \mu}\left(\frac{1}{\beta}\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \mu}\right) = \frac{1}{\beta^2}\left(-\frac{1}{\mathcal{Z}^2}\frac{\partial \mathcal{Z}}{\partial \mu}\right)\frac{\partial \mathcal{Z}}{\partial \mu} + \frac{1}{\beta^2}\frac{1}{\mathcal{Z}}\frac{\partial^2 \mathcal{Z}}{\partial \mu^2}$$

$$= -\bar{N}^2 + \overline{N^2} = \overline{(\Delta N)^2}$$ ✓

b) $\left(\frac{\partial \rho}{\partial P}\right)_{T,V} \Rightarrow \rho = \rho(P,T,V) = \frac{N'(P,T,V)}{V}$ $\qquad N'(P,T,V) = N(\mu(P,T),T,V)$

$N = N(\mu, T, V)$

$$\left(\frac{\partial \bar{\rho}}{\partial P}\right)_{T,V} = \frac{1}{V}\left(\frac{\partial \bar{N}'}{\partial P}\right)_{T,V} \overset{\star}{=} \frac{1}{V}\left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V}\left(\frac{\partial \mu}{\partial P}\right)_T \Rightarrow \left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V} = V\left(\frac{\partial \bar{\rho}}{\partial P}\right)_{T,V} \Big/ \left(\frac{\partial \mu}{\partial P}\right)_T$$

$$(1) \Rightarrow d\mu = \frac{VdP - SdT}{N} = \frac{V}{N}dP - \frac{S}{N}dT \Rightarrow \mu = \mu(P,T) \text{ and } \left(\frac{\partial \mu}{\partial P}\right)_T = \frac{V}{N}$$

$$\Rightarrow kT\left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V} = kTN\left(\frac{\partial \bar{\rho}}{\partial P}\right)_{T,V}$$

8. Statistical Mechanics and Thermodynamics (Fall 2006)

c) $\left(\frac{\partial \rho}{\partial P}\right)_{T,N} \Rightarrow \rho = \rho(P,T,N) = \frac{N}{V(P,T,N)}$

$$\Rightarrow \left(\frac{\partial \rho}{\partial P}\right)_{T,N} = \frac{\partial}{\partial P}\left(\frac{N}{V}\right)_{T,N} = -\frac{N}{V^2}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{N}{V}\kappa_T$$

$$\Rightarrow \overline{(\Delta N)^2} = kTN\left(\frac{\partial \rho}{\partial P}\right)_{T,V} = kTN\left(\frac{\partial \rho}{\partial P}\right)_{T,N} = \boxed{kT\frac{N^2}{V}\kappa_T}$$

★ Proof:

$\bar{N}' = \bar{N}'(P,T,V) \qquad d\bar{N}' = \left(\frac{\partial \bar{N}'}{\partial P}\right)_{T,V} dP + \left(\frac{\partial \bar{N}'}{\partial T}\right)_{P,V} dT + \left(\frac{\partial \bar{N}'}{\partial V}\right)_{P,T} dV$

$\bar{N} = \bar{N}(\mu,T,V) \qquad d\bar{N} = \left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V} d\mu + \left(\frac{\partial \bar{N}}{\partial T}\right)_{\mu,V} dT + \left(\frac{\partial \bar{N}}{\partial V}\right)_{\mu,T} dV$

$\mu = \mu(P,T) \qquad = \left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V}\left[\left(\frac{\partial \mu}{\partial P}\right)_T dP + \left(\frac{\partial \mu}{\partial T}\right)_P dT\right] + \left(\frac{\partial \bar{N}}{\partial T}\right)_{\mu,V} dT + \left(\frac{\partial \bar{N}}{\partial V}\right)_{\mu,T} dV$

$= d\bar{N}'$

equating the coefficients of $dP \Rightarrow \left(\frac{\partial \bar{N}'}{\partial P}\right)_{T,V} = \left(\frac{\partial \bar{N}}{\partial \mu}\right)_{T,V}\left(\frac{\partial \mu}{\partial P}\right)_T$ ✓

10. *Electricity and Magnetism* (Fall 2006)

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

(a) What is the total charge of this system? Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

(b) Identify any point charges in this system and give their location and charge.

(c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

a) Letting $r \to \infty$, $\quad \Phi \to \frac{B}{r} \quad \Rightarrow \quad \boxed{Q_{tot} = \frac{B}{K}}$ where $K = \frac{1}{4\pi\varepsilon_0}$

b) Since the potential is singular at $r=0$, the origin is suspect:

Letting $r \to 0$, $\quad \Phi \to \frac{A}{r} \quad \Rightarrow \quad \boxed{q_{origin} = \frac{A}{K}}$

There are no other singularities in Φ, so there are no other point charges.

c) $\rho(\vec{r}) = \rho_d(r) + \frac{A}{K}\delta(\vec{r})$, where ρ_d is the non-singular distribution

$$\rho(\vec{r}) = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = -\varepsilon_0\vec{\nabla}\cdot(\vec{\nabla}\Phi) = -\varepsilon_0\vec{\nabla}\cdot\left(\partial_r \frac{f(r)}{r}\hat{r}\right)$$

$$= -\varepsilon_0\vec{\nabla}\cdot\left(f'(r)\frac{\hat{r}}{r} - f(r)\frac{\hat{r}}{r^2}\right)$$

$$= -\varepsilon_0\left(\vec{\nabla}f'(r)\cdot\frac{\hat{r}}{r} + f'(r)\vec{\nabla}\cdot\frac{\hat{r}}{r} - \vec{\nabla}f(r)\cdot\frac{\hat{r}}{r^2} - f(r)\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)$$

$$= -\varepsilon_0\left(f''(r)\hat{r}\cdot\frac{\hat{r}}{r} + f'(r)\left[\frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\frac{1}{r}\right)\right] - f'(r)\hat{r}\cdot\frac{\hat{r}}{r^2} - f(r)\,4\pi\delta(\vec{r})\right)$$

$$= -\varepsilon_0\left(\frac{f''(r)}{r} + \cancel{\frac{f'(r)}{r^2}} - \cancel{\frac{f'(r)}{r^2}} - 4\pi f(0)\delta(\vec{r})\right)$$

$$= -\varepsilon_0\frac{f''(r)}{r} + 4\pi\varepsilon_0 A\,\delta(\vec{r})$$

$$= -\varepsilon_0\frac{f''(r)}{r} + \frac{A}{K}\delta(\vec{r})$$

Note: Care was needed in dealing with the divergence to retain the Dirac delta.

10. *Electricity and Magnetism* (Fall 2006)

A spherically symmetric potential $\Phi(r)$ is given by

$$\Phi(r) = \frac{f(r)}{r}$$

where $f(r) \to A$ as $r \to 0$ and $f(r) \to B$ as $r \to \infty$. $f(r)$ is a non-singular function.

(a) What is the total charge of this system? Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

(b) Identify any point charges in this system and give their location and charge.

(c) Find the charge density $\rho(r)$ for this system. Give the answer in terms of A, B, $f(r)$, and (possibly) derivatives of $f(r)$.

a) Examine a sphere of radius R and the total charge enclosed Q_R:

$$Q_R = \int_{V_R} \rho \, dv = \varepsilon_0 \int_{V_R} (\vec\nabla \cdot \vec E)\, dv = \varepsilon_0 \int_{S_R} \vec E \cdot d\vec a = -\varepsilon_0 \int_{S_R} \vec\nabla \Phi \cdot d\vec a$$

$$\vec\nabla \Phi = \partial_r \Phi(r)\, \hat r = \left[\frac{f'(r)}{r} - \frac{f(r)}{r^2} \right] \hat r$$

$$Q_R = -\varepsilon_0 \left[\frac{f'(R)}{R} - \frac{f(R)}{R^2} \right] 4\pi R^2 = -4\pi \varepsilon_0 \left[f'(R) R - f(R) \right]$$

Let $R \to \infty$ to encompass the whole system (since $f(R) \to B$, $f'(R) \to 0$):

$$Q_{tot} = \lim_{R \to \infty} Q_R = 4\pi \varepsilon_0 B \quad \text{if } f'(R) \to 0 \text{ faster than } \tfrac{1}{R} \text{ as } R \to \infty$$

$$Q_{tot} = 4\pi \varepsilon_0 B$$

b) The potential is singular at the origin, so let's use Q_R again but take $R \to 0$:

$$q_{origin} = \lim_{R \to 0} Q_R = 4\pi \varepsilon_0 A \quad \text{if } f'(R) \text{ does not diverge or diverges slower than } \pm \tfrac{1}{R} \text{ as } R \to 0$$

$$q_{origin} = 4\pi \varepsilon_0 A$$

c) $\rho(\vec r) = \rho_d(r) + 4\pi \varepsilon_0 A \, \delta(\vec r)$ where ρ_d is the non-singular distribution

$$\rho(\vec r) = \varepsilon_0 \vec\nabla \cdot \vec E = -\varepsilon_0 \vec\nabla \cdot (\vec\nabla \Phi) = -\varepsilon_0 \vec\nabla \cdot \left[f'(r) \frac{\hat r}{r} - f(r) \frac{\hat r}{r^2} \right]$$

$$= -\varepsilon_0 \left[\left(\vec\nabla f'(r) \cdot \frac{\hat r}{r} + f'(r)\, \vec\nabla \cdot \frac{\hat r}{r} \right) - \left(\vec\nabla f(r) \cdot \frac{\hat r}{r^2} + f(r)\, \vec\nabla \cdot \frac{\hat r}{r^2} \right) \right]$$

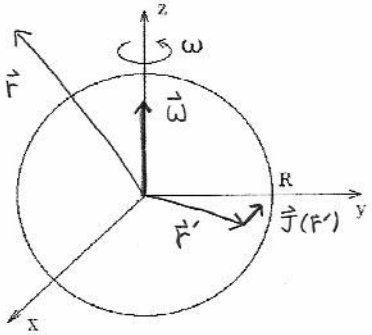
(since $\vec\nabla \cdot (\psi \vec a) = \vec\nabla \psi \cdot \vec a + \psi \vec\nabla \cdot \vec a$)

$$= -\varepsilon_0 \left[f''(r) \frac{1}{r} + \cancel{f'(r) \frac{1}{r^2}} - \cancel{f'(r) \frac{1}{r^2}} - f(r)^{A}\, 4\pi \delta(\vec r) \right]$$

$$= -\varepsilon_0 \frac{f''(r)}{r} + 4\pi \varepsilon_0 A \, \delta(\vec r)$$

14. *Electricity and Magnetism* (Fall 2006)

Consider a rotating sphere with radius R. A charge Q is distributed homogeneously over the sphere. The sphere rotates counter-clockwise around the z-axis with angular velocity ω. (See figure below.)



(a) Find the charge density ρ and the current density $\mathbf{j}$ in terms of delta functions. Show that $\nabla \cdot \mathbf{j} = 0$.

(b) Find the vector potential $\mathbf{A}(\mathbf{x})$ in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$).
Hint: To do the integral it is advantageous to choose $\mathbf{r} = r\mathbf{e}_z$ and choose $\boldsymbol{\omega}$ to be arbitrary.

(c) Calculate the magnetic field $\mathbf{B}$ from the vector potential $\mathbf{A}$.

a) $\rho(\vec{r}) = \dfrac{Q}{4\pi R^2}\delta(r-R)$ $\qquad \displaystyle\int_{\mathbb{R}^3} \rho(\vec{r})\,dv = \frac{Q}{4\pi R^2}\int \delta(r-R)\, r^2 dr\, d\Omega = \frac{Q}{4\pi R^2} R^2 4\pi = Q \;\checkmark$

$\vec{J}(\vec{r}) = \rho(\vec{r})\vec{v}(\vec{r})$ where $\vec{v}(\vec{r}) = v_\phi \hat{\phi} = s\omega\hat{\phi} = R\sin\theta\,\omega\,\hat{\phi}$

$= \rho(\vec{r}) R\omega \sin\theta\,\hat{\phi} = \dfrac{Q\omega}{4\pi R}\delta(r-R)\sin\theta\,\hat{\phi}$

$\vec{\nabla}\cdot\vec{J} = \dfrac{1}{r^2 S\theta}\left\langle \partial_r, \frac{1}{r}\partial_\theta, \frac{1}{rS\theta}\partial_\phi\right\rangle_s \cdot \left(r^2 S\theta \langle 0, 0, J_\phi\rangle_s\right) = 0$ since J_ϕ is independent of ϕ ✓

b) $\vec{A}(\vec{r}) = K_m \displaystyle\int_{\mathbb{R}^3} \frac{\vec{J}(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|} = K_m \frac{Q\omega}{4\pi R}\int \frac{\delta(r'-R)\sin\theta'\,\hat{\phi}'\, r'^2 dr'\, d\Omega'}{|\vec{r}-\vec{r}'|}$

$= K_m \dfrac{Q\omega}{4\pi R} R^2 \displaystyle\int \frac{\sin\theta' \langle -\sin\phi', \cos\phi', 0\rangle_d\, d\Omega'}{|\vec{r} - \vec{R}(\theta',\phi')|}$ $\qquad C\sin\theta\cos\phi = \frac{1}{2}[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)]$

$\qquad C\sin\theta\sin\phi = \frac{1}{2i}[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi)]$

$= K_m \dfrac{Q\omega R}{\cancel{4\pi}} \displaystyle\int \left\langle -\frac{1}{2iC}[Y_{11}(\theta',\phi') - Y_{1,-1}(\theta',\phi')], \frac{1}{2C}[Y_{11}(\theta',\phi') + Y_{1,-1}(\theta',\phi')], 0\right\rangle_d \sum_{lm} \frac{\cancel{4\pi}}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y^*_{lm}(\theta',\phi')\, d\Omega'$

$= K_m Q\omega R \dfrac{1}{2(1)+1}\dfrac{r_<}{r_>^2}\left\langle -\frac{1}{2iC}[Y_{11}(\theta,\phi) - Y_{1,-1}(\theta,\phi')], \frac{1}{2C}[Y_{11}(\theta,\phi) + Y_{1,-1}(\theta,\phi)], 0\right\rangle_d$ $\qquad r_<, r_> \in \{r, R\}$

14. Electricity and Magnetism (Fall 2006)

b) (continued)

$$= \frac{1}{3} K_m Q \omega R \frac{r_<}{r_>^2} \sin\theta \langle -\sin\phi, \cos\phi, 0 \rangle_d$$

$$= \frac{1}{3} K_m Q \omega R \frac{r_<}{r_>^2} \sin\theta \, \hat{\phi}$$

$$\vec{A}(\vec{r}) = \frac{1}{3} K_m Q \omega \begin{Bmatrix} r/R, & r<R \\ R^2/r^2, & r>R \end{Bmatrix} \sin\theta \, \hat{\phi}$$

$\vec{\nabla} \cdot \vec{A} = 0$ since $\vec{A} = A_\phi \hat{\phi}$ is not ϕ-dependent (like $\vec{J}$) ✓

c) $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}$

$$= \frac{1}{r^2 S\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & rS\theta\hat{\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ \cancel{A_r} & \cancel{rA_\theta} & rS\theta A_\phi \end{vmatrix} = \frac{1}{r^2 S\theta} \langle \partial_\theta [rS\theta A_\phi], r(-\partial_r [rS\theta A_\phi]), 0 \rangle_s$$

$$= \frac{1}{3} K_m Q \omega \frac{1}{r^2 S\theta} \left\langle \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix} \partial_\theta (S^2\theta), -r S^2\theta \, \partial_r \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix}, 0 \right\rangle_s$$

$$= \frac{1}{3} K_m Q \omega \frac{1}{r^2 \cancel{S\theta}} \left\langle \begin{Bmatrix} r^2/R \\ R^2/r \end{Bmatrix} 2 \cancel{S\theta} C\theta, \begin{Bmatrix} -2r^2/R \\ R^2/r \end{Bmatrix} S^2\theta, 0 \right\rangle_s$$

$$= \frac{1}{3} K_m Q \omega \begin{Bmatrix} 1/R, & r<R \\ R^2/r^3, & r>R \end{Bmatrix} \left\langle 2\cos\theta, \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} \sin\theta, 0 \right\rangle_s$$

Check: $\vec{\nabla} \cdot \vec{B} = \frac{1}{r^2 S\theta} \langle \partial_r, \frac{1}{r}\partial_\theta, \cancel{\frac{1}{rS\theta}\partial_\phi} \rangle_s \cdot \left(r^2 S\theta \langle B_r, B_\theta, 0 \rangle_s \right)$

$$= \frac{1}{3} K_m Q \omega \frac{1}{r^2 S\theta} \left[\partial_r \left(r^2 S\theta \begin{Bmatrix} 1/R \\ R^2/r^3 \end{Bmatrix} 2C\theta \right) + \frac{1}{r} \partial_\theta \left(r^2 S\theta \begin{Bmatrix} -2/R \\ R^2/r^3 \end{Bmatrix} S\theta \right) \right]$$

$$= \frac{1}{3} K_m Q \omega \frac{1}{r^2 S\theta} \left[\begin{Bmatrix} 2r/R \\ -R^2/r^2 \end{Bmatrix} 2S\theta C\theta + \begin{Bmatrix} -2r/R \\ R^2/r^2 \end{Bmatrix} 2S\theta C\theta \right]$$

$$= 0$$ ✓