13. *Electricity and Magnetism* (Spring 2006)

Consider an infinitely long filamentary current (i.e., a δ-function) carrying a total current I along the z-direction.

(a) Find the magnetic vector potential at a radial distance r from the current filament.

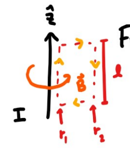
Now a non-relativistic particle of charge q and mass m is fired from a radial location d with velocity $\mathbf{v}$ pointing in the radial direction, away from the current filament.

(b) Evaluate the constants of the motion associated with the orbit of this particle.

(c) Deduce the maximum radial distance reached by the particle.

(d) What condition is required for the orbit size to be well-approximated by the usual Larmor radius expression?

(a)



Find $\vec{A} \rightarrow$ simplest approach $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$ or use $A = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r}')\,d^3r'}{|\vec{r}-\vec{r}'|}$ (booo)

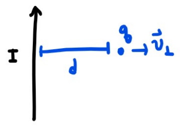
$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \int_S d\vec{s}\cdot\vec{B} = \int_C \vec{A}\cdot d\vec{l}$

$\frac{\mu_0 I}{2\pi}\int dr\,dz\left(\frac{1}{r}\right) = \int \vec{A}\cdot d\vec{l}$

$\frac{\mu_0 I l}{2\pi}\ln(r_2/r_1) = -A(r_2)l + A(r_1)l + \cancel{\int_{r_1}^{r_2} A\,dr} + \cancel{\int_{r_2}^{r_1} A\,dr}$

$\boxed{A(r) = \frac{\mu_0 I}{2\pi}\ln(r)}$

(b)



Evaluate the constants of motion associated w/ orbit

i.e. for some f, $\frac{df}{dt} = 0 = \frac{\partial f}{\partial t} + \{f, H\}$

EOM $m\dot{\vec{v}} = q\vec{v}\times\vec{B}$ w/ $\vec{v} = \vec{v}_\perp + \vec{v}_\parallel$ (rel to $\vec{B}$)

- $\delta W = \vec{F}\cdot\delta\vec{r} = q(\vec{v}\times\vec{B})\cdot\vec{v}\,\delta t = 0$
- the mag force is $\perp$ to $\vec{v}$ and thus does no work.
- kinetic energy T must be constant $\rightarrow$ $T = \frac{1}{2}mv^2$ so v is constant

let $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi} = B_0\left(\frac{d}{r}\right)\hat{\varphi}$ w/ $B_0 = \frac{\mu_0 I}{2\pi d}$

$= B_0\left(\frac{d}{r}\right)(-\sin\phi\,\hat{x}, \cos\phi\,\hat{y}, 0)$

$$\boxed{\begin{gathered} T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_\parallel^2 + v_\perp^2) = \text{const.} \\ v_\perp^2 = v_r^2 + v_z^2 = \text{const.} \\ v_\parallel^2 = v_\varphi^2 = v_{\varphi,0}^2 = 0 \end{gathered}}$$

(c) $\vec{v} = (v_r, v_\phi, v_z)$

$\dot{\vec{v}} = (a_r, a_\phi, a_z)$

define $v_c = \frac{qB_0 d}{m} = \frac{[m]}{[s]}$

$v_r(0) = v$

$v_\varphi(0) = v_z(0) = 0$

$m a_r = -qB_0\left(\frac{d}{r}\right)v_z$ ← $\hat{z}\times\hat{\varphi}$ $\Rightarrow$ $a_r = -\left(\frac{v_c}{r}\right)v_z$

$m a_\phi = 0$ $\qquad a_\varphi = 0$

$m a_z = qB_0\left(\frac{d}{r}\right)v_r$ ← $\hat{r}\times\hat{\varphi}$ $\qquad a_z = \left(\frac{v_c}{r}\right)v_r$

$\Rightarrow$

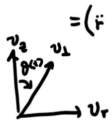
$\vec{r} = r\hat{e}_r + z\hat{e}_z$, $\vec{v} = \dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r + \dot{z}\hat{e}_z + \cancel{z\dot{\hat{e}}_z}$ 0

0 b/c $v_\varphi^2 = v_{\varphi,0}^2 = 0$

$\vec{v} = (\dot{r}, \cancel{r\omega}, \dot{z})$

$\dot{\vec{v}} = \dot{r}\dot{\hat{e}}_r + \ddot{r}\hat{e}_r + \cancel{(r\dot{\omega})}\hat{e}_\phi + \cancel{\frac{\partial}{\partial t}(r\omega)}\hat{e}_\phi + \ddot{z}\hat{e}_z$

$= (\ddot{r} - \cancel{r\omega^2}, \cancel{2\omega\dot{r} + \dot{\omega}r}, \ddot{z})$



$\frac{\partial\hat{e}_r}{\partial t} = (-\sin\phi, \cos\phi, 0)\frac{d\phi}{dt} = \omega\hat{e}_\phi$; $\frac{\partial\hat{e}_\phi}{\partial t} = (-\cos\phi, -\sin\phi, 0)\frac{\partial\phi}{\partial t} = -\omega\hat{e}_r$

EOM becomes:

$\Rightarrow$ $\dot{v}_r = -\left(\frac{v_c}{r}\right)v_z$

$\dot{v}_z = \left(\frac{v_c}{r}\right)v_r$

$v_r = \dot{r}$

$v_z = \dot{z}$

now, $v_\perp^2 = v_{\perp,0}^2 = v_{r,0}^2$ w/ $v_{r,0} \equiv v$

@ $t=0$ $\theta(0) = \pi/2$ w/ θ being angle b/t v_z and $v_\perp$

$v_r = v\sin\theta$ | $v_r(\pi/2) = v$ ✓

$v_z = v\cos\theta$ | $v_z(\pi/2) = 0$

$\dot{v}_z = -v\sin\theta\left(\frac{d\theta}{dt}\right) = \left(\frac{v_c}{r}\right)\left(\frac{dr}{dt}\right)$

$-\int_{\theta_0}^{\theta} v\sin\theta'\,d\theta' = \int_d^r \frac{v_c}{r'}\,dr'$

$v\cos\theta = v_c\ln(r/d) \Rightarrow r = d\exp\left[\frac{v}{v_c}\cos\theta\right]$

for max radial distance: $v_r = \dot{r} = 0 \longrightarrow v_z = \dot{z} = \pm v$

$v_r = 0$ when $\theta = 0, \pi$

$\boxed{r_{max} = d\exp\left[+v/v_c\right]}$

(d) $\omega_c = \frac{qB}{m} = \frac{1}{[t]}$ $\qquad [d] = [v]\cdot[t]$

$r_c = v_\perp/\omega_c$

we defined $v_c = \frac{qB_0}{m}d$

$\simeq \omega_c d$

$r_c = v_\perp/(v_c/d) = d\left(\frac{v_\perp}{v_c}\right) = \frac{mv_\perp}{qB_0}$

If $v_\perp/v_c \sim 1$, $r_c \simeq d$

and if $\cos\theta = 0$, $\theta = 0, \pi$

$$\boxed{r = \underbrace{d\exp\left[v/v_c\cos\theta\right]}_{\simeq r_c} \approx d \quad \text{for } v/v_c \sim 1 \text{ and } \theta = 0, \pi}$$