43. (a) One year contains 3.15×10^7 sec. Consequently $t_{\text{earth}} = \gamma t_{\text{neutron}} = 10^3\ \gamma$ sec $= 3.15 \times 10^7$ sec $\Longrightarrow \gamma = 3.15 \times 10^4$. As seen from the earth, the neutron energy is 3.15×10^4 times the neutron rest mass, or about 30,000 BeV.

(b) Let variables in the neutron rest frame be primed, and those in the earth frame be unprimed. The angles made by the decay product with respect to the neutron velocity (designated as along the z-axis) are θ and θ' in the two systems. The relation

$$\tan\theta = \frac{1}{\gamma}\frac{u'\sin\theta'}{u'\cos\theta' + v},$$

where u' is the velocity of the decay product in question, is derived in the preceding problem. This relation can also be found from the Einstein addition law for velocities.

Both the neutrino and electron are ultrarelativistic in the primed frame. For the electron, $d(\tan\theta)/d\theta' = 0$ implies $\cos\theta' = -u'/v$. Then

$$\tan\theta_{\max} = \frac{u'/v}{\gamma_{\text{C.M.}}}\frac{1}{\sqrt{1-(u'/v)^2}} \approx \frac{\gamma_{\text{electron}}}{\gamma_{\text{C.M.}}}.$$

Now $\gamma_{\text{C.M.}} = 3.15\times10^4$, while $\gamma_{\max}$ (electron) ≈ 2.6. We see $\theta_{\max}\approx 10^{-4}$ rad.

(c) The method of (b) breaks down when $(u' = 1) > v$, since this would lead to $\cos\theta' < -1$. The largest angle θ is π; we obtain this for backward motion in the neutron rest frame.

(d) In the neutron frame, the neutrino has maximum energy

$$E_\nu \approx M_{\text{N}} - M_{\text{p}} - M_{\text{e}} = 0.8\ \text{MeV}.$$

And for a neutrino emitted backwards in the lab frame,

$$\begin{aligned} E^{\max} &= \gamma[E' + \mathbf{v}\cdot\mathbf{p}'] \\ &= E'\gamma(1-v) \approx E'\sqrt{\frac{1-v}{2}} \\ &\approx \frac{E'}{2\gamma_{\text{C.M.}}} \approx 12.7\ \text{eV}. \end{aligned}$$

Solution 1.9

 a) $V = -mga\cos\theta$

The kinetic energy, which separates into a term due to the bead's motion along the wire and a term due to the rotation of the bead with the wire, is

$$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(a\sin\theta)^2. \tag{10.80}$$

The Lagrangian is $L = T - V$. Using Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0, \tag{10.81}$$

we find that

$$a\ddot{\theta} + g\sin\theta - a\omega^2\cos\theta\sin\theta = 0. \tag{10.82}$$

At an equilibrium point $\ddot{\theta} = 0$, so $g = a\omega^2\cos\theta$, or $\omega^2 = g/a\cos\theta$. This equation has a solution for ω only if $\omega^2 \geq g/a$, so the critical angular velocity is

$$\omega_c = \sqrt{\frac{g}{a}}, \tag{10.83}$$

and the equilibrium angle is

$$\theta_0 = \cos^{-1}\left(\frac{g}{a\omega^2}\right). \tag{10.84}$$

b) If the mass makes small oscillations around the equilibrium point θ_0, then we can describe the motion in terms of a small parameter $\phi = \theta - \theta_0$. The equation of motion (10.82) becomes

$$a\ddot{\phi} + g\sin(\theta_0 + \phi) - a\omega^2\cos(\theta_0 + \phi)\sin(\theta_0 + \phi) = 0. \tag{10.85}$$

Using standard trigonometric identities, the small angle approximations $\sin\phi \approx \phi$ and $\cos\phi \approx 1$, and our solution for θ_0 (10.84), it is easy to show that

$$\ddot{\phi} + \omega^2\left(1 - \frac{g^2}{a^2\omega^4}\right)\phi = 0. \tag{10.86}$$

This has the general solution

$$\phi = A\cos\Omega t + B\sin\Omega t, \tag{10.87}$$

where

$$\Omega = \omega\sqrt{1 - \frac{g^2}{a^2\omega^4}}, \tag{10.88}$$

and A and B are arbitrary constants. The period of oscillation is $2\pi/\Omega$.

Solution 3.10. First we choose units such that $\hbar = 1$. In the orbital ground state, the orbital angular momentum is zero, so the relevant part of the Hamiltonian is

$$H = (\alpha \mathbf{S}_1 + \beta \mathbf{S}_2) \cdot \mathbf{B} + J \mathbf{S}_1 \cdot \mathbf{S}_2. \tag{12.131}$$

Let us choose the z-axis to be parallel to the uniform magnetic field, $\mathbf{B} = B\hat{\mathbf{z}}$. Then

$$H = (\alpha S_{1z} + \beta S_{2z})B + J \left[S_{1z} S_{2z} + \frac{1}{2}(S_1^+ S_2^- + S_1^- S_2^+) \right], \tag{12.132}$$

where $S^{\pm} = S_x \pm i S_y$. For two particles with spin, we usually describe the spin part of the wavefunction in either the basis of states given by $|S_1, S_2, S_{1z}, S_{2z}\rangle$, or $|S, S_z, S_1, S_2\rangle$, where we define $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. H is not diagonal in either of these bases, for general α, β, and J. However, since the proton and electron are spin 1/2 particles, we have only a small number of basis vectors and we can solve the problem by brute force. Let us choose the first basis suggested above, writing the basis vectors as

$$|\psi_1\rangle = |\uparrow\uparrow\rangle\,, \quad |\psi_2\rangle = |\uparrow\downarrow\rangle\,, \quad |\psi_3\rangle = |\downarrow\uparrow\rangle\,, \quad |\psi_4\rangle = |\downarrow\downarrow\rangle\,, \tag{12.133}$$

where the first arrow represents the S_{1z}, and the second arrow represents the S_{2z}.

If we form the 4×4 matrix $\langle \psi_i | H | \psi_j \rangle$ then, by definition, its eigenvalues are the energy eigenvalues and its eigenvectors are the eigenstates of the system. We find that the matrix elements are given by:

$$\begin{pmatrix} \frac{(\alpha+\beta)B}{2} + \frac{J}{4} & 0 & 0 & 0 \\ 0 & \frac{(\alpha-\beta)B}{2} - \frac{J}{4} & \frac{J}{2} & 0 \\ 0 & \frac{J}{2} & -\frac{(\alpha-\beta)B}{2} - \frac{J}{4} & 0 \\ 0 & 0 & 0 & -\frac{(\alpha+\beta)B}{2} + \frac{J}{4} \end{pmatrix}. \tag{12.134}$$

Let us denote the four eigenvectors of this matrix as ϕ_a, ϕ_b, ϕ_c and ϕ_d. Since H is in block-diagonal form, we can immediately write down two eigenvectors and their corresponding eigenvalues:

$$|\phi_a\rangle = |\uparrow\uparrow\rangle \text{ with } E_a = +\frac{B}{2}(\alpha+\beta) + \frac{J}{4}, \tag{12.135}$$

$$|\phi_b\rangle = |\downarrow\downarrow\rangle \text{ with } E_b = -\frac{B}{2}(\alpha+\beta) + \frac{J}{4}. \tag{12.136}$$

To find the other two eigenenergies and eigenstates we need to diagonalize the submatrix

$$\mathbf{A} = \begin{pmatrix} \frac{B}{2}(\alpha-\beta) - \frac{J}{4} & \frac{J}{2} \\ \frac{J}{2} & -\frac{B}{2}(\alpha-\beta) - \frac{J}{4} \end{pmatrix}. \tag{12.137}$$

The eigenvalues λ of $\mathbf{A}$ are given by the quadratic equation $\det[\mathbf{A} - \lambda\mathbf{I}] = 0$. Solving this equation for the two eigenvalues yields

$$E_c = -\frac{J}{4} + \frac{1}{2}k, \tag{12.138}$$

$$E_d = -\frac{J}{4} - \frac{1}{2}k, \tag{12.139}$$

where for simplicity we have defined

$$k = \sqrt{J^2 + B^2(\alpha-\beta)^2}. \tag{12.140}$$

We sketch the energy splittings as a function of B in Figure 12.3.

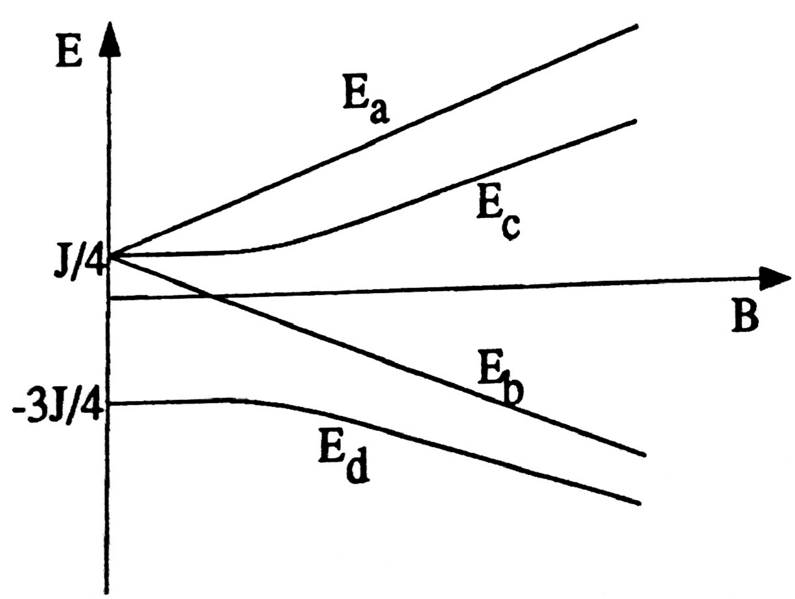

Figure 12.3.

b) Two of the eigenvectors, $\phi_a = |\uparrow\uparrow\rangle$ and $\phi_b = |\downarrow\downarrow\rangle|$ are given above. The other two are the normalized eigenvectors of the submatrix **A**:

$$|\phi_c\rangle = \frac{1}{\sqrt{N}}\left\{|\uparrow\downarrow\rangle + \frac{1}{J}[k - B(\alpha-\beta)]\,|\downarrow\uparrow\rangle\right\}, \qquad (12.141)$$

$$|\phi_d\rangle = \frac{1}{\sqrt{N}}\left\{\frac{1}{J}[-k + B(\alpha-\beta)]\,|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right\}, \qquad (12.142)$$

where

$$N = 1 + \frac{1}{J^2}\left(k - B(\alpha-\beta)\right)^2. \qquad (12.143)$$

We note that for $B = 0$, the eigenvectors of **A** reduce to the basis $|S, S_z, S_1, S_2\rangle$, and for $J = 0$ they reduce to the basis $|S_1, S_2, S_{1z}, S_{2z}\rangle$.

Solution 3.6. a) The Schrödinger equation for the two-component wavefunction Ψ of an electron at rest in a uniform magnetic field is

$$\frac{g_s \mu_B}{\hbar} \mathbf{S} \cdot \mathbf{B} \cdot \Psi = i\hbar \frac{\partial \Psi}{\partial t}. \tag{12.63}$$

In this equation, the Bohr magneton is $\mu_B = e\hbar/2m_e c$, m_e is the electron mass, and g is the electron's gyromagnetic ratio. The spin is $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$, where $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices, presented here for ease of reference:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.64}$$

If we write $\mathbf{B} = B_0\hat{\mathbf{z}}$ then the eigenstates are

$$\Psi_\uparrow(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\Omega t} \quad \text{and} \quad \Psi_\downarrow(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\Omega t}, \tag{12.65}$$

where $\Omega \equiv \mu_B B_0/\hbar$.

Initially the electron has its spin pointing in the x-direction. This means that at $t = 0$ the wavefunction $\Psi(t)$ must be an eigenstate of the σ_x matrix, namely

$$\Psi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{12.66}$$

Alternatively we can write this in terms of the eigenstates (12.65). Then for arbitrary time the wavefunction is given by

$$\Psi(t) = \frac{1}{\sqrt{2}}(\Psi_\uparrow(t) + \Psi_\downarrow(t)) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\Omega t} \\ e^{i\Omega t} \end{pmatrix}. \tag{12.67}$$

We can now calculate the probability of finding the spin in the x-direction at time t :

$$\begin{aligned} \langle S_x \rangle &= \frac{\hbar}{2}\langle \sigma_x \rangle = \frac{\hbar}{4}\begin{pmatrix} e^{i\Omega t} & e^{-i\Omega t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\Omega t} \\ e^{i\Omega t} \end{pmatrix} \\ &= \frac{\hbar}{2}\cos 2\Omega t. \end{aligned} \tag{12.68}$$

We can also find the other components of $\langle \mathbf{S}(t) \rangle$:

$$\langle S_y \rangle = \frac{\hbar}{2}\sin 2\Omega t \quad \text{and} \quad \langle S_z \rangle = 0. \tag{12.69}$$

So we see that the spin precesses around the magnetic field with an angular precession frequency of 2Ω.

b) When an additional time-dependent magnetic field $\mathbf{B}_1$ is applied, it is tempting to try to use perturbation theory. However the question makes no mention of $\mathbf{B}_1$ being "weak." Instead, we find an exact solution.

Our first step is to express the interaction term in a useful form:

$$\mathbf{B} \cdot \boldsymbol{\sigma} = \begin{pmatrix} B_0 & \frac{1}{2}B_1 e^{-i\omega t} \\ \frac{1}{2}B_1 e^{i\omega t} & -B_0 \end{pmatrix}. \tag{12.70}$$

Now we substitute this into the Schrödinger equation (12.63) and look for two solutions of the form

$$\Psi(t) = \begin{pmatrix} a\ e^{i\omega_a t} \\ b\ e^{i\omega_b t} \end{pmatrix}. \tag{12.71}$$

This form can be motivated as follows. We can see that the wavefunction cannot have a simple exponential time dependence, as the Schrödinger equation couples the two components in a nontrivial, time-dependent way. However the time dependence of the *interaction* is simply that of an exponential, and we can hope to find a solution which is some combination of exponentials of different frequencies. In fact this turns out to be the case, as we will see. If we insert our would-be wavefunction (12.71) into the Schrödinger equation (12.63), we obtain the following linear equations in a and b:

$$\mu_B\left(B_0 e^{i\omega_a t}a + \frac{1}{2}B_1 e^{i(\omega_b-\omega)t}b\right) = -\hbar\omega_a e^{i\omega_a t}a, \tag{12.72}$$

$$\mu_B\left(\frac{1}{2}B_1 e^{i(\omega_a+\omega)t}a - B_0 e^{i\omega_b t}b\right) = -\hbar\omega_b e^{i\omega_b t}b. \tag{12.73}$$

Our first condition is that within each equation, the time dependence of all the terms should be the same, which requires

$$\omega_b - \omega_a = \omega. \tag{12.74}$$

Before we derive the other conditions, we note that we can set $a = 1$ without loss of generality. Further, for sake of clarity let us define $\beta = \mu_B B_1/2\hbar$ After we cancel the common exponential time dependence, our equations now reduce to the simple form

$$-\omega_a = \Omega + \beta b, \tag{12.75}$$

$$-\omega_b = -\Omega + \frac{\beta}{b}. \tag{12.76}$$

If we combine these with equation (12.74), we obtain a quadratic equation for b:

$$\omega_b - \omega_a = \omega = 2\Omega + \beta\left(b - \frac{1}{b}\right), \tag{12.77}$$

or equivalently,

$$b^2 - b\left(\frac{\omega}{\beta} - \frac{2\Omega}{\beta}\right) - 1 = 0. \tag{12.78}$$

We can solve this to find the two possible values of b, and then, from equations (12.75) and (12.76), obtain the frequencies of the components:

$$b_\pm = \frac{\omega - 2\Omega}{2\beta} \pm \frac{\Delta}{\beta}, \tag{12.79}$$

$$\omega_a^{\pm} = -\frac{\omega}{2} \mp \Delta, \text{ and} \tag{12.80}$$

$$\omega_b^{\pm} = \frac{\omega}{2} \mp \Delta, \tag{12.81}$$

where we have defined

$$\Delta \equiv \sqrt{\beta^2 + \left(\frac{\omega}{2} - \Omega\right)^2}. \tag{12.82}$$

The unnormalized eigenvectors of the hamiltonian (12.63) with $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ are then

$$\begin{pmatrix} e^{i\omega_a^+ t} \\ b_+ e^{i\omega_b^+ t} \end{pmatrix} \text{ and } \begin{pmatrix} e^{i\omega_a^- t} \\ b_- e^{i\omega_b^- t} \end{pmatrix}. \tag{12.83}$$

Initially we have $\langle S_z \rangle = +\hbar/2$, or

$$\Psi(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{12.84}$$

At time t this will have evolved into a linear superposition of the eigenvectors (12.83):

$$\Psi(t) = p \begin{pmatrix} e^{i\omega_a^+ t} \\ b_+ e^{i\omega_b^+ t} \end{pmatrix} + q \begin{pmatrix} e^{i\omega_a^- t} \\ b_- e^{i\omega_b^- t} \end{pmatrix}, \tag{12.85}$$

for some constants p and q. Our initial condition (12.84) gives us $p+q = 1$, and $pb_+ + qb_- = 0$. (Note that since Ψ is normalized at $t = 0$, it remains normalized for all time.) These can be solved to give

$$p = -\frac{\beta b_-}{2\Delta}, \text{ and } q = \frac{\beta b_+}{2\Delta}. \tag{12.86}$$

After a time t, the probability that the electron is in a state with $\langle S_z \rangle = -\hbar/2$ is the modulus squared of the lower component of $\Psi(t)$,

$$\begin{aligned} P(t) &= \left| p\, b_+ e^{i\omega_b^+ t} + q\, b_- e^{i\omega_b^- t} \right|^2 = \left| \frac{\beta b_1 b_+}{2\Delta} \right|^2 \left| e^{i\Delta t} - e^{-i\Delta t} \right|^2 \\ &= \frac{\beta^2}{\Delta^2} \sin^2 \Delta t. \end{aligned} \tag{12.87}$$

If the oscillating magnetic field has an angular frequency $\omega = 2\Omega$ then it is "at resonance" and $P(t) = \sin^2 \beta t$. This precession of the expectation value of the spin with angular frequency 2β is called Rabi precession, and 2β is called the Rabi flopping frequency.

Solution:

a) The scattering amplitude in the first Born approximation is given by

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{2m(2\pi)^{3/2}}{4\pi\hbar^2}\int d^3y e^{-i(\vec{k}'-\vec{k})\vec{y}}V(\vec{y}) \tag{0.1}$$

We have

$$\vec{k} = \hat{e}_z k, \qquad \vec{k}' = k(\hat{e}^z\cos\theta + \hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi) \tag{0.2}$$

Plugging in the potential and evaluating the integral over y gives

$$f^{(1)}(\vec{k}',\vec{k}) = -\frac{2m(2\pi)^{3/2}\alpha}{4\pi\hbar^2}\left(e^{ikb\sin\theta\sin\phi} + e^{-ikb\sin\theta\sin\phi}\right) \tag{0.3}$$

$$= -\frac{m\alpha(2\pi)^{3/2}}{\hbar^2\pi}\cos(kb\sin\theta\sin\phi) \tag{0.4}$$

b) The differential cross section can be calculated from the scattering amplitude by

$$\frac{d\sigma}{d\Omega} = \mid f(\theta,\phi)\mid^2$$

$$= \frac{\alpha^2m^2 8\pi}{\hbar^4}\mid\cos(kb\sin\theta\sin\phi)\mid^2 \tag{0.5}$$

c) The total cross section is given by

$$\sigma_{tot} = \int d\Omega \mid f(\theta,\phi)\mid^2$$

$$= \frac{\alpha^2m^2 8\pi}{\hbar^4}\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\phi\mid\cos(kb\cos\theta\cos\phi)\mid^2$$

$$= \frac{\alpha^2m^2 8\pi^2}{\hbar^4}\int_0^\pi d\theta\sin\theta\left(1 + \pi J_0(2kb\sin\theta)\right)$$

$$= \frac{\alpha^2m^2 8\pi^2}{\hbar^4}\left(2 + \frac{\sin(2bk)}{bk}\right) \tag{0.6}$$

Q. 6

A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively, denoted by $|lm\rangle$. $\mathbf{L}$ is the angular momentum operator, whose components obey the usual commutation algebra. Prove that the expectation values involving L_x and L_y obey

$$\langle L_x \rangle = \langle L_y \rangle = 0\,, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{l(l+1) - m^2}{2}\hbar^2$$

in the eigenstate $|lm\rangle$.

Solution: Using $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$, we evaluate (henceforth setting $\hbar = 1$)

$$\begin{aligned}
\langle L_x^2 - L_y^2 \rangle =& \langle [L_x, L_z]^2 - [L_y, L_z]^2 \rangle = 2m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle - \langle L_x L_z^2 L_x - L_y L_z^2 L_y \rangle \\
=& 2m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle \\
& - \langle L_x L_z [L_z, L_x] + m L_x L_z L_x - [L_y, L_z] L_z L_y - m L_y L_z L_y \rangle \\
=& m\langle L_x L_z L_x - L_y L_z L_y \rangle - m^2 \langle L_x^2 - L_y^2 \rangle \\
=& m\langle L_x [L_z, L_x] + m L_x^2 - [L_y, L_z] L_y - m L_y^2 \rangle - m^2 \langle L_x^2 - L_y^2 \rangle = 0\,.
\end{aligned}$$

Thus (restoring $\hbar$)

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{\langle \mathbf{L}^2 - L_z^2 \rangle}{2} = \frac{l(l+1) - m^2}{2}\hbar^2\,.$$

The other identities are also easily obtained:

$$i\hbar \langle L_x \rangle = \langle [L_y, L_z] \rangle = m\langle L_y - L_y \rangle = 0\,,$$

and similarly for $\langle L_y \rangle$.

2. The matrix $\mathbf{M} = (M_x, M_y, M_z)$ represents an angular momentum matrix, because of the commutation rules. It is evident that the matrices do not represent irreducible representations; rather they represent *several* irreducible representations.

If a state with spin J is represented, M_x has $(2J + 1)$ eigenvalues, ranging in integral steps from $+J$ to $-J$, each appearing once. Hence there are no states of spin greater than 2, only one of spin 2, and eight of spin $\frac{3}{2}$. One of the 28 entries of ± 1 is accounted for by the $J = 2$ state; there are, therefore 27 representations of $J = 1$. Similarly, there are $(56 - 8) = 48$ representations of $J = \frac{1}{2}$ and 42 of $J = 0$.

Each eigenvalue of M^2 corresponding to spin J has value $J(J + 1)$; to each representation there are $(2J + 1)$ such values. We then construct the following table:

J	$J(J + 1)$	$(2J + 1)$	Number of representations	Number of entries in M^2
2	6	5	1	5
$\frac{3}{2}$	$\frac{15}{4}$	4	8	32
1	2	3	27	81
$\frac{1}{2}$	$\frac{3}{4}$	2	48	96
0	0	1	42	42

16. In the region $x > 0$, ψ obeys the same differential equation as the two-sided harmonic oscillator; however, the only acceptable solutions are those that vanish at the origin. Therefore, the eigenvalues are those of the ordinary harmonic oscillator belonging to wave functions of odd parity. Now the parity of the S.H.O. wave functions alternates with increasing n, starting with an even-parity ground state. Hence,

$$E = \frac{(4n + 3)\hbar\omega}{2} \qquad \text{with } n = 0, 1, \ldots$$