7. (Electromagnetism)

An infinite straight wire carrying a current I is suspended parallel to the plane interface between vacuum and a medium with magnetic permeability $\mu \neq 1$, at a distance a from the interface. Calculate the force per unit length on the wire, and state whether it is attractive or repulsive.

Hint: for this problem, it is helpful to introduce a scalar potential. Also, it is helpful to take coordinates in the complex plane perpendicular to the wire.

Solution: *Solution by Jonah Hyman (jthyman@g.ucla.edu)*

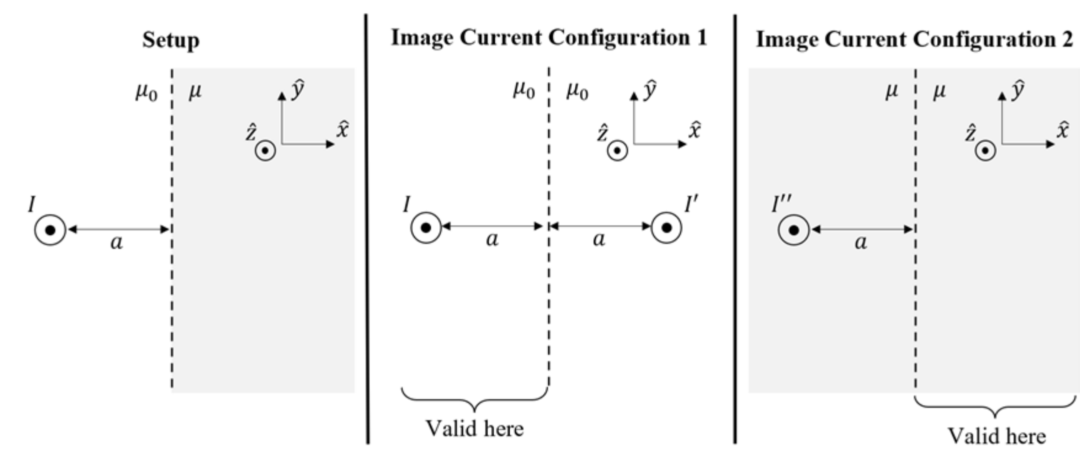
We will use SI units, and we will take μ_0 to be the magnetic permeability of the vacuum. (The problem statement seems to be working in units where $\mu_0 = 1$; if you want to work in these units, just replace μ_0 with 1 throughout.)

This is a rare case in which I will disregard the hint given by the problem. This problem can be solved using the magnetic scalar potential and complex numbers, but the extra math required by this approach doesn't translate into an easier or more efficient solution.

Instead, I will treat this as a traditional method of images problem. To calculate the force per unit length on the wire, we need to know the magnetic field due to the magnetic material in the area of the wire. By a variant of the uniqueness theorem (as applied to the magnetic scalar potential), if we can find a magnetic field that satisfies all the boundary conditions, that must be the correct magnetic field for this problem.

To start the problem, we must have a good guess for the magnetic field. The method of images allows us to guess that the magnetic field in one region is the same as that of an "image current" in the *other* region. (We can't place image currents in the region of interest, since that would change the differential equation for the magnetic field in the region we're trying to solve.) Therefore, we need two image current configurations: one valid in the vacuum region, and one valid in the magnetic material.

We now need to know the correct image current configurations. These configurations are analogous to the image charge configurations for a point charge and two dielectric materials, which you need to have memorized (or written on your formula sheet). They look like this:



Reminder: If an image current is present in a region, the image current configuration is not valid for that region.

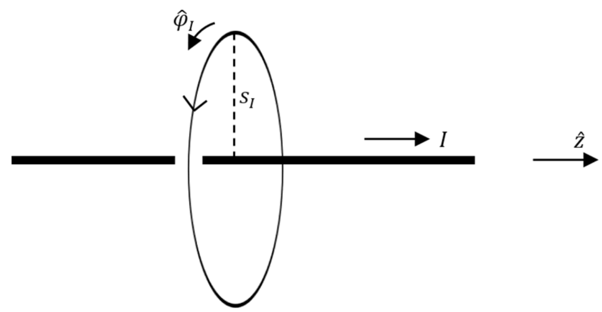
The next step is to find the magnetic fields in image current configurations 1 and 2. To start, let's find the **H** fields due to a single wire with steady current $I\,\hat{\mathbf{z}}$ in a linear isotropic homogeneous magnetic material with magnetic permeability μ, which fills all space. We will call this magnetic fields $\mathbf{H}_I$. The equations for $\mathbf{H}_I$ in this magnetostatic situation are

$$\nabla \cdot \mathbf{H}_I = \nabla \cdot \left(\frac{\mathbf{B}_I}{\mu}\right) = \frac{1}{\mu}\left(\nabla \cdot \mathbf{B}_I\right) = 0 \tag{39}$$

and

$$\nabla \times \mathbf{H}_I = \mathbf{J}_f \quad \text{where } \mathbf{J}_f \text{ is the free current density} \tag{40}$$

(39) and the azimuthal symmetry of the setup implies that $\mathbf{H}_I$ is in the $\hat{\varphi}_I$ direction and depends only on the distance to the wire, which we will call s_I. (40) allows us to use the integral version of Ampere's law with a counterclockwise-oriented circular loop of radius s_I, centered on and perpendicular to the wire:



Using Ampere's law for $\mathbf{H}_I$ (40), we get

$$\begin{aligned} H_I \cdot 2\pi s_I &= \int_{\text{loop}} \mathbf{H}_I \cdot d\boldsymbol{\ell} \\ &= \int_{\text{loop interior}} (\nabla \times \mathbf{H}_I) \cdot d\mathbf{a} \quad \text{by Stokes' theorem} \\ &= \int_{\text{loop interior}} \mathbf{J}_f \cdot d\mathbf{a} \quad \text{by Ampere's law (40)} \\ &= I \end{aligned} \tag{41}$$

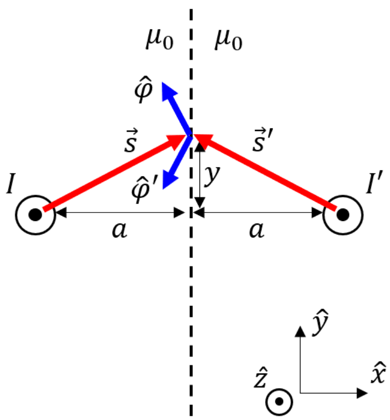
Therefore,

$$H_I = \frac{I}{2\pi s_I} \qquad \Longrightarrow \qquad \mathbf{H}_I = \frac{I}{2\pi s_I}\hat{\varphi}_I \tag{42}$$

With this information, we can calculate the $\mathbf{H}$ fields in image current configurations 1 and 2. Let s, s', and s'', be the distance to the wires I, I', and I'', respectively (the same nomenclature applies to $\hat{\varphi}$, $\hat{\varphi}'$, and $\hat{\varphi}''$). Using superposition, the $\mathbf{H}$ fields in current configurations 1 and 2 are

$$\mathbf{H}_1 = \frac{I}{2\pi s}\hat{\varphi} + \frac{I'}{2\pi s'}\hat{\varphi}' \quad \text{and} \quad \mathbf{H}_2 = \frac{I''}{2\pi s''}\hat{\varphi}'' \tag{43}$$

We need to impose the boundary conditions on the yz-plane, so we should shift to rectangular coordinates. Set the origin to be the point on the boundary that is on the same level as the wire, and consider a point with a specific y-coordinate y on the boundary. We want to find s and $\hat{\varphi}$ (as well as their primed equivalents) in rectangular coordinates for this point. Here is a diagram for image current configuration 1:



The vectors $\mathbf{s}$ and $\mathbf{s}'$ from each wire to the point on the boundary are

$$\mathbf{s} = a\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} \quad \text{and} \quad \mathbf{s}' = -a\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} \tag{44}$$

By the Pythagorean theorem, the lengths s and s' are

$$s = s' = \sqrt{a^2 + y^2} \quad \text{on the boundary} \tag{45}$$

Since $\mathbf{s}$ and $\hat{\varphi}$ are perpendicular, we must have $\mathbf{s} \cdot \hat{\varphi} = 0$. Therefore, since $\mathbf{s} = a\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$, $\hat{\varphi}$ must be proportional to $-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}$. The diagram tells us the correct direction of $\hat{\varphi}$; the last thing we need to do is divide the vector $-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}$ by its length to get the unit vector $\hat{\varphi}$:

$$\hat{\varphi} = \frac{-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}}{\sqrt{a^2 + y^2}} = \frac{-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}}{s} \quad \text{on the boundary} \tag{46}$$

By the same logic, we can write the unit vector $\hat{\varphi}'$, which is perpendicular to $\mathbf{s}'$:

$$\hat{\varphi}' = \frac{-y\,\hat{\mathbf{x}} - a\,\hat{\mathbf{y}}}{\sqrt{a^2 + y^2}} = \frac{-y\,\hat{\mathbf{x}} - a\,\hat{\mathbf{y}}}{s} \quad \text{on the boundary} \tag{47}$$

Image current I'' in image current configuration 2 is in the same place as image current I was in configuration 1. Therefore, $s'' = s$ and $\hat{\varphi}'' = s''$:

$$s'' = s = \sqrt{a^2 + y^2} \quad \text{and} \quad \hat{\varphi}'' = \frac{-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}}{s} \quad \text{on the boundary} \tag{48}$$

Plugging these results into (44), we can derive the $\mathbf{H}$ fields on the boundary for image current configurations 1 and 2:

$$\begin{aligned}
\mathbf{H}_1 &= \frac{I}{2\pi s}\hat{\varphi} + \frac{I'}{2\pi s'}\hat{\varphi}' \quad \text{by (44)} \\
&= \frac{I}{2\pi s}\left(\frac{-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}}{s}\right) + \frac{I'}{2\pi s}\left(\frac{-y\,\hat{\mathbf{x}} - a\,\hat{\mathbf{y}}}{s}\right) \\
&= \frac{1}{2\pi s^2}\left[-y(I + I')\,\hat{\mathbf{x}} + a(I - I')\,\hat{\mathbf{y}}\right] \\
\mathbf{H}_1 &= \frac{1}{2\pi(a^2 + y^2)}\left[-y(I + I')\,\hat{\mathbf{x}} + a(I - I')\,\hat{\mathbf{y}}\right] \quad \text{on the boundary}
\end{aligned} \tag{49}$$

$$\begin{aligned}\mathbf{H}_2 &= \frac{I''}{2\pi s''}\hat{\varphi}'' \quad \text{by (44)} \\ &= \frac{I''}{2\pi s}\left(\frac{-y\,\hat{\mathbf{x}} + a\,\hat{\mathbf{y}}}{s}\right) \\ &= \frac{1}{2\pi s^2}\left[-yI''\,\hat{\mathbf{x}} + aI''\,\hat{\mathbf{y}}\right] \\ \mathbf{H}_2 &= \frac{1}{2\pi(a^2+y^2)}\left[-yI''\,\hat{\mathbf{x}} + aI''\,\hat{\mathbf{y}}\right] \quad \text{on the boundary} \end{aligned} \tag{50}$$

We are now ready to impose the boundary conditions for $\mathbf{B}$ and $\mathbf{H}$. Recall that $\mathbf{H}_1$ comes from image current configuration 1, so it is valid to the left of the boundary (in the vacuum region). Similarly, $\mathbf{H}_2$ comes from image current configuration 2, so it is valid to the right of the boundary (in the magnetic material).

Here's the first boundary condition. Since $\nabla \cdot \mathbf{B} = 0$ always, the perpendicular component of $\mathbf{B}$ across the boundary is continuous:

$$B_{1,\perp} = B_{2,\perp} \tag{51}$$

In this case, the direction perpendicular to the boundary is the $\hat{\mathbf{x}}$-direction. Since $\mathbf{B} = \mu\mathbf{H}$ for a magnetic material, this implies that

$$\mu_0 H_{1,x} = \mu H_{2,x} \quad \text{on the boundary} \tag{52}$$

For the second boundary condition, since there is no free surface current on the boundary, $\nabla \times \mathbf{H} = 0$ near the boundary. Therefore, the component of $\mathbf{H}$ parallel to the boundary is continuous:

$$H_{1,\parallel} = H_{2,\parallel} \tag{53}$$

In this case, the direction parallel to the boundary is the $\hat{\mathbf{y}}$-direction. Therefore,

$$H_{1,y} = H_{2,y} \quad \text{on the boundary} \tag{54}$$

Applying the two boundary conditions (52) and (54) to the expressions for $\mathbf{H}_1$ and $\mathbf{H}_2$ on the boundary, which we derived earlier in (49) and (50), we get

$$\mu_0 H_{1,x} = \mu H_{2,x} \quad \Longrightarrow \quad \mu_0 \cdot \frac{1}{2\pi(a^2+y^2)}\left[-y(I+I')\right] = \mu \cdot \frac{1}{2\pi(a^2+y^2)}\left[-yI''\right]$$

$$\mu_0(I+I') = \mu I'' \tag{55}$$

$$H_{1,y} = H_{2,y} \quad \Longrightarrow \quad \frac{1}{2\pi(a^2+y^2)}\left[a(I-I')\right] = \frac{1}{2\pi(a^2+y^2)}\left[aI''\right]$$

$$I - I' = I'' \tag{56}$$

(55) and (56) are a system of two linear equations that we can use to solve for I' and I''. Multiplying (56) by μ and subtracting (55), we can solve for I':

$$\begin{array}{rcl} \mu(I-I') &=& \mu I'' \\ -\big(\mu_0(I+I') &=& \mu I''\big) \\ \hline I(\mu-\mu_0) - I'(\mu+\mu_0) &=& 0 \end{array}$$

$$I' = I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right) \tag{57}$$

If we want, we can then solve for I'':

$$\begin{aligned} I'' &= I - I' \quad \text{by (56)} \\ &= I - I\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right) \quad \text{by (57)} \\ &= I\left[1 - \frac{\mu-\mu_0}{\mu+\mu_0}\right] \\ I'' &= I\left(\frac{2\mu_0}{\mu+\mu_0}\right) \end{aligned} \tag{58}$$

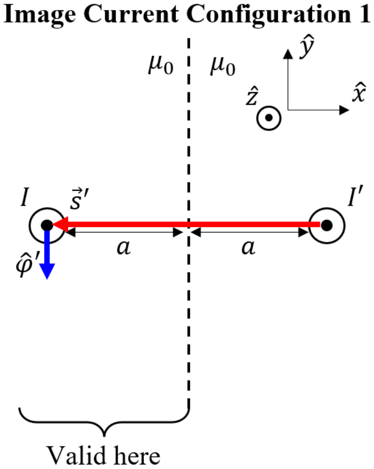
We have shown that there are values for I' and I'' for which $\mathbf{H}_1$ and $\mathbf{H}_2$ satisfy all the boundary conditions. By the uniqueness theorem, this must be the correct magnetic field for the given setup.

Our last step is to find the force per unit length on the wire I. To do this, we need to find the magnetic field *due to the magnetic material* at the location of the wire. The field $\mathbf{H}_1$ is valid at the location of the wire, and we calculated its general form in (44). The wire exerts no force on itself, so we ignore its magnetic field $\frac{I}{2\pi s}\hat{\varphi}$. What is left is

$$\mathbf{H}_{\text{on wire}} = \frac{I'}{2\pi s'}\hat{\varphi}' = \frac{I}{2\pi s'}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat{\varphi}' \quad \text{by (57)} \tag{59}$$

Recall that s' and $\hat{\varphi}'$ are measured relative to the image wire with current I', as shown in the diagram below:

Image Current Configuration 1



We therefore have

$$s' = 2a \quad \text{and} \quad \hat{\varphi}' = -\hat{\mathbf{y}} \quad \text{at the location of the wire} \tag{60}$$

Plugging this into (59), we get

$$\mathbf{H}_{\text{on wire}} = \frac{I}{2\pi(2a)}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)(-\hat{\mathbf{y}}) = -\frac{I}{4\pi a}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat{\mathbf{y}} \tag{61}$$

Since $\mathbf{B} = \mu_0\hat{\mathbf{H}}$ in a vacuum, this implies

$$\mathbf{B}_{\text{on wire}} = -\frac{\mu_0 I}{4\pi a}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat{\mathbf{y}} \tag{62}$$

From this magnetic field, we can derive the force on a portion of the wire with length $d\ell$ and moving charge dq using the Lorentz force law:

$$\begin{aligned} d\mathbf{F} &= dq\,\mathbf{v} \times \mathbf{B}_{\text{on wire}} \\ &= \lambda\,\mathbf{v} \times \mathbf{B}_{\text{on wire}}\, d\ell \quad \text{where } \lambda \text{ is the linear current density of moving charges} \\ d\mathbf{F} &= \mathbf{I} \times \mathbf{B}_{\text{on wire}}\, d\ell \quad \text{since } \mathbf{I} = \lambda \mathbf{v} \end{aligned} \tag{63}$$

Dividing by the unit of length $d\ell$, we find that the force per unit length of the wire is

$$\begin{aligned} \mathbf{f} &= \mathbf{I} \times \mathbf{B}_{\text{on wire}} \\ &= (I\,\hat{\mathbf{z}}) \times \left[-\frac{\mu_0 I}{4\pi a}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat{\mathbf{y}}\right] \quad \text{by (62)} \\ &= \frac{\mu_0 I^2}{4\pi a}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)(-\hat{\mathbf{z}} \times \hat{\mathbf{y}}) \end{aligned}$$

Since $-\hat{\mathbf{z}} \times \hat{\mathbf{y}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, we have

$$\boxed{\mathbf{f} = \frac{\mu_0 I^2}{4\pi a}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\hat{\mathbf{x}}} \tag{64}$$

Since $\hat{\mathbf{x}}$ points toward the magnetic material, the force is attractive if $\mu > \mu_0$ (i.e. if the material is paramagnetic). The force is repulsive if $\mu < \mu_0$ (i.e. if the material is diamagnetic).